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CLIFFORD ALGEBRAS AND MULTI-PARTICLE SPINORS

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To my mother Ozilia and my brothers Gilcilei and Gilcinei

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Abstract

Clifford, or geometric, algebras are introduced by presenting important particular cases. The introduction to the geometric algebra of the three-dimensional Euclidean space and the geometric algebra of spacetime shows how these algebras provide a synthetic and efficient way to describe geometric objects and rotations in three-dimensional Euclidean space and Minkowski spacetime, respectively. It is shown how the former algebra is included in the later, and how this algebra provides an elegant way to describe Lorentz transformations, the electromagnetic field and Maxwell's equations. The emergence of these algebras in the quantum mechanics of spin-1/2 particles is outlined, and a systematic study of Pauli and Dirac spinors is performed by transforming from the classical to the algebraic description of the spinors, which leads naturally to the operator definition of such spinors. These transformations are developed systematically for the first time in this work. At this point, the transformations are applied to obtain the corresponding versions of the Pauli and Dirac equations. The corresponding transformations for the adjoint spinors are also obtained and applied to express inner products and observables. This study concerning a single spin-1/2 particle is then extended to the context of systems of multiple spin-1/2 particles. In this new study, the Clifford algebra appropriate for description of non-relativistic multi-particle spinors is found to be identical to the so-called multi-particle spacetime algebra, introduced less formally in previous studies. Multi-particle algebraic and operator Pauli spinors are then defined for the first time, starting from the classical ones, in an analogous manner to the single-particle case. In order to properly define relativistic multi-particle spinors, the extension of the Dirac algebra from the usual complex algebra of Minkowski spacetime to a six-dimensional conformal space algebra is found to be necessary. In terms of this algebra, an extension of the algebra of operators to a Clifford algebra is performed, and multi-particle algebraic and operator Dirac spinors are defined for the first time, in terms of this extended algebra. Finally, the algebraic and operator versions of the Bethe-Salpeter equation are obtained. The different versions of spinors and their corresponding wave equations raise the possibility that the simpler operator versions could be more fundamental than the classical ones.

Resumo

As álgebras de Clifford, ou geométricas, são introduzidas através da apresentação de casos particulares importantes. A introdução à álgebra geométrica do espaço euclidiano tridimensional e à álgebra geométrica do espaço-tempo mostra como estas álgebras fornecem uma forma sintética e eficiente de descrever objetos geométricos e rotações no espaço euclidiano tridimensional e no espaço-tempo de Minkowski, respectivamente. Mostra-se como a primeira álgebra está inclusa na segunda, e como esta álgebra fornece uma forma elegante de descrever transformações de Lorentz, o campo eletromagnético e as equações de Maxwell. O surgimento dessas álgebras na mecânica quântica de partículas de spin 1/2 é esboçado, e um estudo sistemático dos espinores de Pauli e de Dirac é executado através da transformação da descrição clássica do espinor para a sua descrição algébrica, a qual conduz naturalmente à definição operatória desses espinores. Estas transformações são desenvolvidas sistematicamente pela primeira vez neste trabalho. Neste momento, as transformações são aplicadas para obter as versões correspondentes das equações de Pauli e de Dirac. As transformações correspondentes para os espinores adjuntos são também obtidas e aplicadas para expressar produtos internos e observáveis. Este estudo relacionado a uma única partícula de spin 1/2 é então estendido para o contexto de sistemas de múltiplas partículas de spin 1/2. Neste novo estudo, a álgebra de Clifford considerada adequada para a descrição de espinores não-relativísticos de múltiplas partículas é identificada com a chamada álgebra do espaço-tempo de múltiplas partículas, introduzida menos formalmente em estudos prévios. Espinores algébricos e operadores de Pauli de múltiplas partículas são então definidos pela primeira vez, a partir dos clássicos, de forma análoga ao caso de uma única partícula. A fim de definir adequadamente espinores relativísticos de múltiplas partículas, a extensão da álgebra de Dirac, partindo da álgebra complexa do espaço-tempo de Minkowski para uma álgebra do espaço conforme hexadimensional, é determinada necessária. Em termos desta álgebra, uma extensão da álgebra de operadores para uma álgebra de Clifford é executada, e espinores de Dirac de múltiplas partículas são então definidos pela primeira vez, em termos desta álgebra estendida. Por fim, as versões algébrica e operatória da equação de Bethe-Salpeter são obtidas. As diferentes versões dos espinores e suas equações de onda correspondentes levantam a possibilidade de que as versões operatórias, mais simples, possam ser mais fundamentais que as clássicas.

Contents

1 Introduction

Mathematics is fundamental for physics and vice versa. Geometry plays an important role in this reciprocal relation. Most fundamental physical phenomena occur in an arena modeled by some type of space. So an efficient approach to geometry and correlated areas can be important for the treatment of physical problems.

There are essentially two main approaches to geometry, a coordinate-based one and a coordinate-free one (DORAN; LASENBY, 2003). The first is chiefly based in descriptions using coordinate systems. In this approach geometric objects are treated by manipulating their components, and in some applications considerable emphasis is given in how components are transformed under a change of reference frame. The second tradition is primarily based on more direct descriptions of the geometric objects. The need for coordinates is very common in the ultimate stage of many realistic applications, but an adequate coordinate-free approach includes naturally a coordinate description. This work aims to apply *Clifford algebras*, also known as *geometric algebras*, in physics, given that such algebraic structures provide a modern and promising coordinate-free approach to geometry and physics.

As exposed by Vaz and da Rocha (2019) Clifford algebras appeared independently in mathematics and physics. The first appearance occurred in 1878 as a species of unification of the algebra of Grassmann and the algebra of quaternions of Hamilton by the English mathematician W. K. Clifford, who called it "geometric algebra". In physics, a Clifford algebra emerged naturally in 1927 in the context of the electron theory of Pauli, as the algebra of sigma matrices. In the following year, a Clifford algebra appeared again as the algebra of gamma matrices in the relativistic description of the electron by Dirac. Since then, many important results were achieved, in particular, new ways to describe spinors.

This history is much longer and richer than the previous brief paragraph can describe, but it recognizes that Clifford algebras appeared in physics as an intrinsic part of the description of spin-$\frac{1}{2}$ particles in quantum mechanics and that this can be an interesting subject to explore employing Clifford algebras. In fact, this is the subject most addressed by researchers. However, it is worth noting that almost all the studies concern single-particle states. This suggests that studying multi-particle states could yield new results.

This essentially defines the objective of this work. More specifically, this work aims to answer some basic questions about the description of multi-particle states that still seem to be unanswered.

In the next chapter, Clifford algebras are introduced in the form of important particular cases, which are referred to as geometric algebras. Other Clifford algebras are designated as such in the text, although the names "Clifford algebras" and "geometric algebras" could be considered as synonyms. In this chapter, the geometric algebra of the Euclidean plane is presented first, to pave the way to an introduction to the geometric algebra of the three-dimensional Euclidean space, which is presented immediately after. Then, after a brief introduction to the pseudo-Euclidean plane and its corresponding geometric algebra, Minkowski spacetime and its corresponding geometric algebra are introduced. At the end of each presentation, the manner in which the concepts introduced can be used to represent reflections and rotations is shown. Chapter 3 focuses on how the Clifford algebras introduced in the previous chapter can be used to describe relativistic physics. In this chapter, the geometric algebra of the three-dimensional Euclidean space is included in the geometric algebra of spacetime. Then, a brief description of relativistic observables is presented, and the representation of Lorentz transformations in terms of the geometric algebra of spacetime is given. Finally, the description of the electromagnetic field and Maxwell's equations in this context is presented. In chapter 4, the emergence of the Clifford algebras in quantum mechanics is outlined, and a systematic presentation, both in the non-relativistic and in the relativistic context, of the classical and of more modern ways of representing the wave functions through spinors is presented. This is essentially a study of the different ways to define Pauli and Dirac spinors. In each case, the relation between the different definitions are presented, and the corresponding wave equations (the Pauli and Dirac equations) are also presented and compared. In chapter 5, the study in the previous chapter, concerning a single spin-$\frac{1}{2}$ particle, is extended to the context of systems of multiple spin-$\frac{1}{2}$ particles. In this new study, the Clifford algebras suitable for description of non-relativistic and relativistic multi-particle spinors are identified. Multi-particle algebraic and operator spinors are then defined from the classical ones, both in the non-relativistic and in the relativistic context, and the definitions are compared in each context. Finally, the algebraic and operator versions of the Bethe-Salpeter equation are obtained. Chapter 6 contains the conclusions and final considerations.

2 Introduction to Clifford Algebras

Geometric, or Clifford, algebras are introduced in this chapter by presenting important particular cases. The basic references are two instructional articles by Vaz (1997, 2000) and the first five chapters from the textbook by Doran and Lasenby (2003). Additional information on quaternions was collected from an article by Lambek (1995). The Cartan-Dieudonné theorem, which is evoked in this chapter, is treated under a weak form in the textbook by Vaz and da Rocha (2019). Additional information about the Spin groups was also collected from the textbook by Vaz and da Rocha (2019).

2.1 The Geometric Algebra of the Euclidean Plane

2.1.1 Construction of the Structure

Consider the vector space $\mathbb{R}^2$, and let its vectors be denoted by Latin letters in boldface: $\mathbf{u}$, $\mathbf{v}$, etc. Let the canonical basis be denoted $\{\mathbf{e}_1, \mathbf{e}_2\} = \{(1,0),(0,1)\}$ (where the ordering of the basis is implied), in such a way that a vector is written, generally, $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$, etc. The interpretation for this space is the usual geometric interpretation: $\mathbb{R}^2$ corresponds to the plane, and its vectors represent oriented line segments in the plane. Consider the bilinear form $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that

$$g(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}, \quad \text{where} \quad i, j \in \{1, 2\}. \tag{2.1}$$

Taking $g(\mathbf{u}, \mathbf{v})$, where $\mathbf{u}$ and $\mathbf{v}$ are generic vectors, and then applying the bilinearity property, one verifies that $g(\mathbf{u}, \mathbf{v}) = g(\mathbf{v}, \mathbf{u})$, and that $g(\mathbf{u}, \mathbf{u}) \geq 0$, where $g(\mathbf{u}, \mathbf{u}) = 0$ if and only if $\mathbf{u} = \mathbf{o}$ ($\mathbf{o}$ is the null vector), that is, the bilinear form g above defined is symmetric and positive-definite, hence it corresponds to an inner product. Indeed, one has

$$\begin{aligned} g(\mathbf{u}, \mathbf{v}) &= g(u_1\mathbf{e}_1 + u_2\mathbf{e}_2, v_1\mathbf{e}_1 + v_2\mathbf{e}_2) \\ &= u_1v_1g(\mathbf{e}_1, \mathbf{e}_1) + u_1v_2g(\mathbf{e}_1, \mathbf{e}_2) + u_2v_1g(\mathbf{e}_2, \mathbf{e}_1) + u_2v_2g(\mathbf{e}_2, \mathbf{e}_2) \\ &= u_1v_1 + u_2v_2 = g(\mathbf{v}, \mathbf{u}). \end{aligned} \tag{2.2}$$

In particular,

$$g(\mathbf{u}, \mathbf{u}) = {u_1}^2 + {u_2}^2 \geq 0. \tag{2.3}$$

Note that this is the usual inner product associated to $\mathbb{R}^2$. Endowing $\mathbb{R}^2$ with such an inner product makes it a Euclidean space, called the *Euclidean plane.*

The *geometric algebra of the Euclidean plane* is determined by a space constructed from the Euclidean plane, endowed with another product, called the *geometric product*, which will be constructed in the following, as restrictions on its form are imposed. Such a product is denoted by juxtaposition, that is, $\mathbf{uv}$ denotes the geometric product of the vector $\mathbf{u}$ with the vector $\mathbf{v}$.

The first property imposed to the geometric product is

$$\mathbf{uu} = g(\mathbf{u}, \mathbf{u}), \tag{2.4}$$

for any vector $\mathbf{u}$ from $\mathbb{R}^2$, which can be written

$$\mathbf{u}^2 = |\mathbf{u}|^2, \tag{2.5}$$

where the notation $\mathbf{uu} = \mathbf{u}^2$ is introduced, and $|\cdot|$ corresponds to the norm induced by the inner product g, that is, the usual modulus. Writing $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, the above equation can be written in terms of components as

$$(u_1\mathbf{e}_1 + u_2\mathbf{e}_2)(u_1\mathbf{e}_1 + u_2\mathbf{e}_2) = {u_1}^2 + {u_2}^2. \tag{2.6}$$

Since bilinearity is a fundamental property for the product of an algebra, this property need be considered for the geometric product. In this way, by applying the bilinearity property for the geometric product in the above expression, one obtains

$${u_1}^2\,{\mathbf{e}_1}^2 + u_1u_2(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) + {u_2}^2\,{\mathbf{e}_2}^2 = {u_1}^2 + {u_2}^2. \tag{2.7}$$

For this equation to be satisfied, one must have

$${\mathbf{e}_1}^2 = 1, \quad {\mathbf{e}_2}^2 = 1, \quad \text{and} \quad \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1. \tag{2.8}$$

These relations determine the geometric product of the geometric algebra of the Euclidean plane in terms of the canonical basic vectors. Applying it to the computation of the geometric product of two arbitrary vectors $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ and $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$, furnishes

$$\begin{aligned}\mathbf{uv} &= (u_1\mathbf{e}_1 + u_2\mathbf{e}_2)(v_1\mathbf{e}_1 + v_2\mathbf{e}_2)\\ &= u_1v_1{\mathbf{e}_1}^2 + u_1v_2\mathbf{e}_1\mathbf{e}_2 + u_2v_1\mathbf{e}_2\mathbf{e}_1 + u_2v_2{\mathbf{e}_2}^2\\ &= (u_1v_1 + u_2v_2) + (u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2. \end{aligned} \tag{2.9}$$

Note that the first term on the right-hand side of the resulting equation is a scalar that corresponds to the inner product introduced earlier, better known as the scalar product. The second term, on the other hand, is neither a scalar nor a vector. For a scalar α and a vector $\mathbf{w}$ if follows that $\alpha\mathbf{w} = \mathbf{w}\alpha$, but in particular, allowing now the geometric product to be associative, one has

$$(\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_1 = -(\mathbf{e}_2\mathbf{e}_1)\mathbf{e}_1 = -\mathbf{e}_2(\mathbf{e}_1\mathbf{e}_1) = -\mathbf{e}_2 \tag{2.10}$$

and

$$\mathbf{e}_1(\mathbf{e}_1\mathbf{e}_2) = (\mathbf{e}_1\mathbf{e}_1)\mathbf{e}_2 = \mathbf{e}_2, \tag{2.11}$$

so that $\mathbf{e}_1\mathbf{e}_2$ is not a scalar. Since for any vector $\mathbf{w}$, one has $\mathbf{ww} = \mathbf{w}^2 = |\mathbf{w}|^2 \geq 0$, but

$$(\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_1\mathbf{e}_2) = -(\mathbf{e}_2\mathbf{e}_1)(\mathbf{e}_1\mathbf{e}_2) = -\mathbf{e}_2\big(\mathbf{e}_1(\mathbf{e}_1\mathbf{e}_2)\big) = -\mathbf{e}_2\big((\mathbf{e}_1\mathbf{e}_1)\mathbf{e}_2\big) = -\mathbf{e}_2\mathbf{e}_2 = -1, \tag{2.12}$$

$\mathbf{e}_1\mathbf{e}_2$ cannot be a vector from $\mathbb{R}^2$ either. The coefficient of $\mathbf{e}_1\mathbf{e}_2$ in the expression (2.9) suggests a geometrical interpretation for such an object. $|u_1v_2 - u_2v_1|$ corresponds to the area of the parallelogram determined by the vectors $\mathbf{u}$ and $\mathbf{v}$. While $\sqrt{|\mathbf{ww}|}$ corresponds to the length of an oriented line segment representing the vector $\mathbf{w}$, the quantity

$$\sqrt{\Big|\Big((u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2\Big)\Big((u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2\Big)\Big|} \tag{2.13}$$

corresponds to the area of the parallelogram determined by the vectors $\mathbf{u}$ and $\mathbf{v}$. This fact suggests that $\mathbf{e}_1\mathbf{e}_2$ is associated with an area in the plane, more specifically, with an area of unit magnitude. The multiplication of the object $\mathbf{e}_1\mathbf{e}_2$ by $(u_1v_2 - u_2v_1)$ associates it with an area of magnitude $|u_1v_2 - u_2v_1|$. According to the sign of the coefficient of $(u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2$, such an object has a kind of "orientation", analogous, in some sense, to the orientation of a vector (as an oriented line segment), lacking, for the moment, a pertinent meaning.

Based on the suggestion made in the previous paragraph, consider the association of the object $\mathbf{e}_1\mathbf{e}_2$ with the parallelogram/square determined by the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. One can think of the orientation of this square as being determined by the direction of the square, which is unique and corresponds to the direction of the plane, and by the sense of travel about the square boundary, which is uniquely associated with the order of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ (and their opposites) for taking the displacements needed to travel the square boundary (starting from the origin) in some sense, clockwise or counterclockwise. For example, the order of the geometric product $\mathbf{e}_1\mathbf{e}_2$ suggests associating to this object a square with the sense of travel about its boundary being counterclockwise, since, starting from the origin and taking the displacement given by $\mathbf{e}_1$ and then the displacement given by $\mathbf{e}_2$, and then taking the displacements $-\mathbf{e}_1$ and $-\mathbf{e}_2$, the square border is traversed in the counterclockwise sense (cf. figure 2.1). In the same way, $\mathbf{e}_2\mathbf{e}_1$ is associated with a sense of travel about its boundary being clockwise, the opposite sense of travel associated with $\mathbf{e}_1\mathbf{e}_2$, which is compatible with the fact that $\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$. This interpretation leads to the idea that the object $(u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2$ present in the expression for the geometric product $\mathbf{uv}$ corresponds to the oriented parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

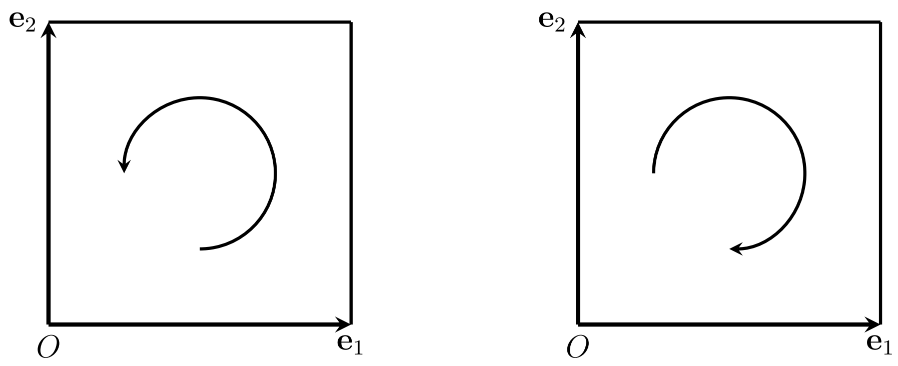

FIGURE 2.1 – The two oriented squares associated with the basic vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

The fact that one can write the geometric product $\mathbf{uv}$ as a sum of a symmetric part and an antisymmetric part relative to exchange between $\mathbf{u}$ and $\mathbf{v}$ as

$$\mathbf{uv} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu}) + \frac{1}{2}(\mathbf{uv} - \mathbf{vu}) \tag{2.14}$$

allows one to identify the symmetric part with the inner product $g(\mathbf{u}, \mathbf{v}) = u_1v_1 + u_2v_2$ and to express such a product in terms of the geometric product as

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu}). \tag{2.15}$$

The antisymmetric part of the product $\mathbf{uv}$ is defined as the *exterior product* or *wedge product* of the vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} \wedge \mathbf{v} = \frac{1}{2}(\mathbf{uv} - \mathbf{vu}) = -\mathbf{v} \wedge \mathbf{u}. \tag{2.16}$$

Given these definitions, the geometric product can be written as

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}. \tag{2.17}$$

Noting that

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2 \tag{2.18}$$

and that

$$(u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2 = \mathbf{u} \wedge \mathbf{v}, \tag{2.19}$$

one verifies that in general the exterior product $\mathbf{u}\wedge\mathbf{v}$ represents the oriented parallelogram determined by the vectors $\mathbf{u}$ and $\mathbf{v}$, in this order.

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$. Defining the sum of objects resulting from the exterior product as

$$\mathbf{u} \wedge \mathbf{v} + \mathbf{w} \wedge \mathbf{x} = (u_1v_2 + w_1x_2 - u_2v_1 - w_2x_1)\mathbf{e}_1\mathbf{e}_2, \tag{2.20}$$

one notes that such a sum furnishes an object of the same nature of the summed objects, and one can easily verify that

(i) $\mathbf{u} \wedge \mathbf{v} + \mathbf{w} \wedge \mathbf{x} = \mathbf{w} \wedge \mathbf{x} + \mathbf{u} \wedge \mathbf{v}$

and

(ii) $\mathbf{u} \wedge \mathbf{v} + (\mathbf{w} \wedge \mathbf{x} + \mathbf{y} \wedge \mathbf{z}) = (\mathbf{u} \wedge \mathbf{v} + \mathbf{w} \wedge \mathbf{x}) + \mathbf{y} \wedge \mathbf{z}$.

Additionally, one verifies that

(iii) $\mathbf{u} \wedge \mathbf{v} + \mathbf{w} \wedge \mathbf{w} = \mathbf{u} \wedge \mathbf{v} + \frac{1}{2}(\mathbf{ww} - \mathbf{ww}) = \mathbf{u} \wedge \mathbf{v}$,

that is, $\mathbf{u} \wedge \mathbf{u} = \mathbf{v} \wedge \mathbf{v} = \mathbf{w} \wedge \mathbf{w} = \cdots$ play the role of neutral element relative to the sum operation. Such an element is unique, and at the moment, it should be denoted $\mathbf{O}$. Its uniqueness it is verified by observing that, if there is $\mathbf{O}'$ which also satisfies

$$\mathbf{u} \wedge \mathbf{v} + \mathbf{O}' = \mathbf{u} \wedge \mathbf{v}, \tag{2.21}$$

it follows that

$$\mathbf{u} \wedge \mathbf{v} + \mathbf{O} = \mathbf{u} \wedge \mathbf{v} + \mathbf{O}', \tag{2.22}$$

which implies $\mathbf{O} = \mathbf{O}'$. A fourth basic property of the sum of elements of the form $\mathbf{u} \wedge \mathbf{v}$, relative to the existence of $\mathbf{O}$, is that (iv) for any $\mathbf{u} \wedge \mathbf{v}$ an "opposite element" is associated, in the sense that $\mathbf{u} \wedge \mathbf{v}$ summed to its opposite furnishes $\mathbf{O}$. Indeed, the property $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$ automatically identifies the opposite of $\mathbf{u} \wedge \mathbf{v}$ with $\mathbf{v} \wedge \mathbf{u}$:

$$\mathbf{u} \wedge \mathbf{v} + \mathbf{v} \wedge \mathbf{u} = \mathbf{O}. \tag{2.23}$$

One can also define naturally the multiplication of an object of the form $\mathbf{u} \wedge \mathbf{v}$ by a real scalar α through

$$\alpha(\mathbf{u} \wedge \mathbf{v}) = \big(\alpha(u_1 v_2 - u_2 v_1)\big)\mathbf{e}_1\mathbf{e}_2, \tag{2.24}$$

and verify without difficulty that such an operation, which furnishes an object of the same nature of the multiplied object, obeys the following properties (where β is also a real scalar):

(I) $\alpha\big(\beta(\mathbf{u} \wedge \mathbf{v})\big) = (\alpha\beta)(\mathbf{u} \wedge \mathbf{v})$;

(II) $\alpha(\mathbf{u} \wedge \mathbf{v} + \mathbf{w} \wedge \mathbf{x}) = \alpha(\mathbf{u} \wedge \mathbf{v}) + \alpha(\mathbf{w} \wedge \mathbf{x})$;

(III) $(\alpha + \beta)(\mathbf{u} \wedge \mathbf{v}) = \alpha(\mathbf{u} \wedge \mathbf{v}) + \beta(\mathbf{u} \wedge \mathbf{v})$;

(IV) $1(\mathbf{u} \wedge \mathbf{v}) = \mathbf{u} \wedge \mathbf{v}$.

The properties i, ii, iii, iv, I, II, III and IV show that the set of objects of the form $\mathbf{u} \wedge \mathbf{v}$ endowed with the operations of summation and multiplication by a real scalar as defined above determine the structure of a vector space over the field of real numbers. The vectors from this vector space are called 2-*vectors* or *bivectors*, since they are determined by the exterior product of two "usual" vectors. The vector space of bivectors is denoted by $\bigwedge^2(\mathbb{R}^2)$.

For the sake of future construction, the space of vectors from the Euclidean plane is denoted by $\bigwedge^1(\mathbb{R}^2)$, and the vectors themselves are called 1-*vectors*. In the same way, the vector space of real scalars is denoted by $\bigwedge^0(\mathbb{R}^2)$, and its vectors can be called 0-*vectors*.

Since the geometric product of two vectors in the plane results in the "sum" of two quantities of a different nature, a scalar and a bivector, this "sum" must not be a sum in the usual sense. In fact, the sum of two objects, each belonging to a different vector space, is a direct sum, which corresponds to a vector of the vector space resulting from the direct sum of the spaces to which the two distinct objects belong.

In order to construct a closed algebraic structure with respect to the geometric product, the vector space $\bigwedge(\mathbb{R}^2)$ is defined as the direct sum of the spaces of the form $\bigwedge^k(\mathbb{R}^2)$:

$$\bigwedge(\mathbb{R}^2) = \bigoplus_{k=0}^{2} \bigwedge^k(\mathbb{R}^2) = \bigwedge^0(\mathbb{R}^2) \oplus \bigwedge^1(\mathbb{R}^2) \oplus \bigwedge^2(\mathbb{R}^2). \tag{2.25}$$

The elements of this vector space are called *multivectors*. The null vector of this space, $0 + \mathbf{o} + \mathbf{O}$, can be simply denoted 0, which usually does not cause any problems. An arbitrary multivector from $\bigwedge(\mathbb{R}^2)$ is then of the form

$$A = a + (a_1\mathbf{e}_1 + a_2\mathbf{e}_2) + a_{12}\mathbf{e}_1\mathbf{e}_2, \tag{2.26}$$

where $a, a_1, a_2, a_{12} \in \mathbb{R}$.

Defining then the exterior product of a scalar α with a vector $\mathbf{u}$ of the Euclidean plane by $\alpha \wedge \mathbf{u} = \alpha\mathbf{u}$, and completing the extension of the exterior product for multivectors by considering it bilinear and associative, $\left(\bigwedge(\mathbb{R}^2), \wedge\right)$ is established as an associative algebra over the field of real numbers. Such an algebra is known as an *exterior algebra* or *Grassmann algebra* associated with $\mathbb{R}^2$.

Defining the geometric product of a scalar with a multivector as the multiplication of the multivector by the scalar, and extending the geometric product to any multivectors by considering the properties of bilinearity and associativity, it follows that the vector space $\bigwedge(\mathbb{R}^2)$ endowed with the geometric product generalized in this way determines an associative algebra over the field of real numbers. Such an algebra is called the *geometric algebra of the Euclidean plane* or *Clifford algebra of the Euclidean plane*, and is usually denoted by $\mathcal{C}\ell(\mathbb{R}^2, g)$, or $\mathcal{C}\ell_{2,0}(\mathbb{R})$, or simply $\mathcal{C}\ell_{2,0}$.

2.1.2 Projection, Graded Involution, Reversion, the Norm and the Inverse

Let A_k be an arbitrary k-vetor and $A = \sum_{k=0}^{2} A_k$ an arbitrary multivector. The *projection* of A over the vector subspace $\bigwedge^k(\mathbb{R}^2)$, also called the *$k$-vector part* of A, is defined by

$$\langle A \rangle_k = A_k. \tag{2.27}$$

As an example, consider the multivector $f = \frac{1}{2}(1 + \mathbf{e}_1)$, for which

$$\langle f \rangle_0 = \frac{1}{2}, \quad \langle f \rangle_1 = \frac{1}{2}\mathbf{e}_1, \quad \text{and} \quad \langle f \rangle_2 = 0. \tag{2.28}$$

In terms of the projection operation one can define the *graded involution* operation by

$$\widehat{A} = \sum_{k=0}^{2} (-1)^k \langle A \rangle_k. \tag{2.29}$$

For any k-vector A_k, the number k is called the *grade* of A_k. If a multivector A satisfies $\widehat{A} = A$, it is said to be an *even grade* multivector, and if it satisfies $\widehat{A} = -A$ it is said to be an *odd grade* multivector. Also in terms of the projection operation one can define the *reversion* operation, which is given by

$$\widetilde{A} = \sum_{k=0}^{2} (-1)^{\frac{1}{2}k(k-1)} \langle A \rangle_k. \tag{2.30}$$

The multivector $\widetilde{A}$ is said to be the *reverse* of A. The reversion operation has this name because it reverses the order of the geometric product of two vectors, that is,

$$\widetilde{(\mathbf{uv})} = \mathbf{vu}, \tag{2.31}$$

for any vectors $\mathbf{u}$ and $\mathbf{v}$. In summary, for A given by (2.26), in terms of the basic vectors, one has

$$\widehat{A} = a - (a_1\mathbf{e}_1 + a_2\mathbf{e}_2) + a_{12}\mathbf{e}_1\mathbf{e}_2 \quad \text{and} \quad \widetilde{A} = a + (a_1\mathbf{e}_1 + a_2\mathbf{e}_2) - a_{12}\mathbf{e}_1\mathbf{e}_2. \tag{2.32}$$

A major operational advantage of the geometric algebra framework is the possibility of defining the inverse for a vector with respect to the geometric product, and even for a generic multivector, under certain conditions. The geometric product of a non-null vector $\mathbf{u}$ with $\mathbf{u}/|\mathbf{u}|^2$ furnishes the number 1, the unity of the algebra. Thus, the *inverse* of a non-null vector $\mathbf{u}$ is defined by

$$\mathbf{u}^{-1} = \frac{\mathbf{u}}{|\mathbf{u}|^2}. \tag{2.33}$$

For the bivector $\mathbf{e}_1\mathbf{e}_2$, for example, one can define the inverse as $(\mathbf{e}_1\mathbf{e}_2)^{-1} = \mathbf{e}_2\mathbf{e}_1$. Indeed,

$$(\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_2\mathbf{e}_1) = \mathbf{e}_1\big(\mathbf{e}_2(\mathbf{e}_2\mathbf{e}_1)\big) = \mathbf{e}_1\big((\mathbf{e}_2\mathbf{e}_2)\mathbf{e}_1\big) = \mathbf{e}_1\mathbf{e}_1 = 1. \tag{2.34}$$

However, it is not possible to define the inverse for an arbitrary multivector. For example, the multivector $f = \frac{1}{2}(1 + \mathbf{e}_1)$ has no inverse.

Because of the associativity of the geometric product, the products $\mathbf{u}(\mathbf{vw})$ and $(\mathbf{uv})\mathbf{w}$ can both be simply written as $\mathbf{uvw}$. In this way, one can leave the associativity implicit in calculations and express the above calculation (equation (2.34)) more succinctly as follows:

$$(\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_2\mathbf{e}_1) = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_1\mathbf{e}_1 = 1. \tag{2.35}$$

For an arbitrary multivector A, one can define the *norm* of A as the real scalar $\|A\|$ such that

$$\|A\|^2 = \left\langle \widetilde{A}A \right\rangle_0 = \left\langle A\widetilde{A} \right\rangle_0. \tag{2.36}$$

Note that, for A given by (2.26), one has

$$\|A\|^2 = a^2 + {a_1}^2 + {a_2}^2 + {a_{12}}^2 \geq 0. \tag{2.37}$$

In this way, it follows that $|\mathbf{u}| = \|\mathbf{u}\|$, for any vector $\mathbf{u}$.

From the definition of norm of a multivector, it follows that, if

$$\left\langle \widetilde{A}A \right\rangle_0 = \widetilde{A}A > 0, \tag{2.38}$$

then

$$\|A\|^2 = \widetilde{A}A, \tag{2.39}$$

which implies that

$$\frac{1}{\|A\|^2}\widetilde{A}A = \left(\frac{\widetilde{A}}{\|A\|^2}\right)A = 1, \tag{2.40}$$

which in turn induces the identification of $\widetilde{A}/\|A\|^2$ with the inverse of A:

$$A^{-1} = \frac{\widetilde{A}}{\|A\|^2}. \tag{2.41}$$

But, it should be noted that the inverse of A is only defined if the condition given by (2.38) is satisfied.

2.1.3 Inequalities, Parallelism and Orthogonality

Given two non-null vectors $\mathbf{u}$ and $\mathbf{v}$, it follows that

$$\|\mathbf{u}\wedge\mathbf{v}\|^2 = (\widetilde{\mathbf{u}\wedge\mathbf{v}})(\mathbf{u}\wedge\mathbf{v}) = (\mathbf{v}\wedge\mathbf{u})(\mathbf{u}\wedge\mathbf{v}). \tag{2.42}$$

Since $\mathbf{uv} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v}$, if follows that $\mathbf{u}\wedge\mathbf{v} = \mathbf{uv} - \mathbf{u}\cdot\mathbf{v}$, which, considering the above equation, implies

$$\begin{aligned}
\|\mathbf{u}\wedge\mathbf{v}\|^2 &= (\mathbf{vu} - \mathbf{v}\cdot\mathbf{u})(\mathbf{uv} - \mathbf{u}\cdot\mathbf{v}) \\
&= \mathbf{vuuv} - \mathbf{vu}(\mathbf{u}\cdot\mathbf{v}) - (\mathbf{v}\cdot\mathbf{u})\mathbf{uv} + (\mathbf{v}\cdot\mathbf{u})(\mathbf{u}\cdot\mathbf{v}) \\
&= \mathbf{v}\|\mathbf{u}\|^2\mathbf{v} - \mathbf{vu}(\mathbf{u}\cdot\mathbf{v}) - \mathbf{uv}(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{v})^2 \\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{uv} + \mathbf{vu})(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{v})^2 \\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - 2(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{v})^2 \\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2.
\end{aligned} \tag{2.43}$$

Then, since $\|\mathbf{u}\wedge\mathbf{v}\|^2 \geq 0$, it follows that

$$(\mathbf{u}\cdot\mathbf{v})^2 \leq \|\mathbf{u}\|^2\|\mathbf{v}\|^2, \tag{2.44}$$

which is known as the Cauchy-Schwarz inequality. This result implies

$$-1 \leq \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1. \tag{2.45}$$

This expression allows one to define the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ as the number θ such that $0 \leq \theta \leq \pi$ and

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}. \tag{2.46}$$

From equation (2.43) one also has

$$\left(\frac{\|\mathbf{u} \wedge \mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)^2 = 1 - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)^2, \tag{2.47}$$

which, given the above expression for $\cos(\theta)$, implies

$$\left(\frac{\|\mathbf{u} \wedge \mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)^2 = 1 - \cos^2(\theta) = \sin^2(\theta). \tag{2.48}$$

Since $0 \leq \theta \leq \pi$, one has $\sin(\theta) \geq 0$, so from the above equation it follows that

$$\sin(\theta) = \frac{\|\mathbf{u} \wedge \mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|}. \tag{2.49}$$

The inequality (2.44) can also be used to obtain the triangular inequality. Indeed,

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) \leq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| = (\|\mathbf{u}\| + \|\mathbf{v}\|)^2, \tag{2.50}$$

that is,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|. \tag{2.51}$$

If the non-null vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel, that is, the angle θ between them is null, one has equivalently $\sin(\theta) = 0$, and, from relation (2.49), this is equivalent to $\|\mathbf{u} \wedge \mathbf{v}\| = 0$, which in turn is equivalent to $\mathbf{u} \wedge \mathbf{v} = 0$, that is, $\frac{1}{2}(\mathbf{uv} - \mathbf{vu}) = 0$. Then $\mathbf{uv} = \mathbf{vu}$ is also a condition for parallelism of the vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} \parallel \mathbf{v} \quad \Leftrightarrow \quad \mathbf{u} \wedge \mathbf{v} = 0 \quad \Leftrightarrow \quad \mathbf{uv} = \mathbf{vu}. \tag{2.52}$$

The same is true when $\theta = \pi$, when it is also said that the vectors $\mathbf{u}$ and $\mathbf{v}$ are anti-parallel. Whereas when $\theta = \pi/2$, that is, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, one has equivalently $\cos(\theta) = 0$, which, according to (2.46), is equivalent to $\mathbf{u} \cdot \mathbf{v} = 0$, that is, $\frac{1}{2}(\mathbf{uv} + \mathbf{vu}) = 0$. Then $\mathbf{uv} = -\mathbf{vu}$ is also a condition for orthogonality of the vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} \perp \mathbf{v} \quad \Leftrightarrow \quad \mathbf{u} \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad \mathbf{uv} = -\mathbf{vu}. \tag{2.53}$$

2.1.4 Reflections and Rotations

Consider again two non-null vectors $\mathbf{u}$ and $\mathbf{v}$ from the Euclidean plane. The component of $\mathbf{v}$ parallel to $\mathbf{u}$, or the projection of the vector $\mathbf{v}$ on the vector $\mathbf{u}$, is given by

$$\mathbf{v}_{\parallel} = \mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\mathbf{v} \cdot \frac{\mathbf{u}}{\|\mathbf{u}\|}\right) \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\|\mathbf{u}\|^2}(\mathbf{v} \cdot \mathbf{u})\mathbf{u}. \tag{2.54}$$

The component of $\mathbf{v}$ orthogonal to $\mathbf{u}$ is then

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel}. \tag{2.55}$$

Note from the relations of parallelism (2.52) and orthogonality (2.53) that $\mathbf{u}\mathbf{v}_{\parallel} = \mathbf{v}_{\parallel}\mathbf{u}$ and $\mathbf{u}\mathbf{v}_{\perp} = -\mathbf{v}_{\perp}\mathbf{u}$. Note then that the geometric product of $\mathbf{v}_{\parallel}$ by $\mathbf{u}$ furnishes

$$\mathbf{u}\mathbf{v}_{\parallel} = \frac{1}{\|\mathbf{u}\|^2}(\mathbf{v} \cdot \mathbf{u})\mathbf{u}^2, \tag{2.56}$$

that is,

$$\mathbf{u}\mathbf{v}_{\parallel} = \mathbf{v} \cdot \mathbf{u} = \frac{1}{2}(\mathbf{v}\mathbf{u} + \mathbf{u}\mathbf{v}). \tag{2.57}$$

But, the geometric product of this expression by $\mathbf{u}$ gives

$$\|\mathbf{u}\|^2\mathbf{v}_{\parallel} = \frac{1}{2}(\mathbf{u}\mathbf{v}\mathbf{u} + \|\mathbf{u}\|^2\mathbf{v}), \tag{2.58}$$

that is,

$$\mathbf{v}_{\parallel} = \frac{1}{2}\left(\frac{1}{\|\mathbf{u}\|^2}\mathbf{u}\mathbf{v}\mathbf{u} + \mathbf{v}\right), \tag{2.59}$$

which, considering the definition of the inverse, can be written as

$$\mathbf{v}_{\parallel} = \frac{1}{2}\left(\mathbf{v} + \mathbf{u}\mathbf{v}\mathbf{u}^{-1}\right). \tag{2.60}$$

This expression for $\mathbf{v}_{\parallel}$, considering the expression (2.55) for $\mathbf{v}_{\perp}$, allows one to write also

$$\mathbf{v}_{\perp} = \frac{1}{2}\left(\mathbf{v} - \mathbf{u}\mathbf{v}\mathbf{u}^{-1}\right). \tag{2.61}$$

Now, consider the linear transformation given by

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = \mathbf{v}_{\perp} - \mathbf{v}_{\parallel}, \tag{2.62}$$

or, equivalently, by

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = \mathbf{v} - 2\mathbf{v}_{\parallel}. \tag{2.63}$$

Such a linear transformation is knwon as the *reflection* of the vector $\mathbf{v}$ through the line

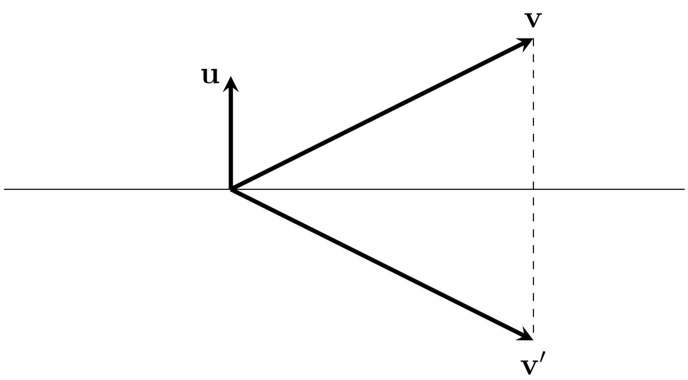

FIGURE 2.2 – Reflection of the vector $\mathbf{v}$ through the line with orthogonal vector $\mathbf{u}$.

with orthogonal vector $\mathbf{u}$. This transformation is illustrated in the figure 2.2.

Considering the expression (2.60) for $\mathbf{v}_{\|}$ one can express a reflection transformation of the vector $\mathbf{v}$ through the line with orthogonal vector $\mathbf{u}$ by

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = -\mathbf{u}\mathbf{v}\mathbf{u}^{-1}. \tag{2.64}$$

In particular, if the vector $\mathbf{u}$ is unitary, one has $\mathbf{u}\mathbf{u} = 1$, which implies $\mathbf{u}^{-1} = \mathbf{u}$, in such a way that, the reflection transformation can be expressed by

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = -\mathbf{u}\mathbf{v}\mathbf{u}, \quad \text{where} \quad \mathbf{u}^2 = 1. \tag{2.65}$$

A particular case of a statement known as *Cartan-Dieudonné theorem* concerns the possibility of expressing a rotation in terms of reflections. Specifically: "the composition of two reflections in the plane corresponds to a rotation". In this way, a rotation of the vector $\mathbf{v}$ can be expressed as

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = -\mathbf{u}_1(-\mathbf{u}_2\mathbf{v}\mathbf{u}_2)\mathbf{u}_1 = \mathbf{u}_1\mathbf{u}_2\mathbf{v}\mathbf{u}_2\mathbf{u}_1, \tag{2.66}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors. Then, one can express this rotation by

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = R\mathbf{v}R^{-1}, \tag{2.67}$$

where $R = \mathbf{u}_1\mathbf{u}_2$. The object R is called a *rotor*, because of the role it plays in describing a rotation. If θ is the angle between the unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, then

$$R = \mathbf{u}_1 \cdot \mathbf{u}_2 + \mathbf{u}_1 \wedge \mathbf{u}_2 = \cos(\theta) + \sin(\theta)B, \tag{2.68}$$

where B is a unit bivector. From this expression it follows that

$$R\widetilde{R} = \Big(\cos(\theta)+\sin(\theta)B\Big)\Big(\cos(\theta)-\sin(\theta)B\Big) = \cos^2(\theta)-\sin^2(\theta)B^2 = \cos^2(\theta)+\sin^2(\theta) = 1, \tag{2.69}$$

then

$$\widetilde{R} = R^{-1}, \tag{2.70}$$

and so a rotation can be written

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = R\mathbf{v}\widetilde{R}. \tag{2.71}$$

There are two possibilities for the unit bivector B present in the expression for R: $B = \mathbf{e}_1\mathbf{e}_2$ or $B = \mathbf{e}_2\mathbf{e}_1$. These two possibilities can be simulated considering the angle θ such that $0 \leq \theta \leq 2\pi$, in such way that, taking $B = \mathbf{e}_1\mathbf{e}_2$, one has, for $0 \leq \theta \leq \pi$, $\sin(\theta)B = \alpha\mathbf{e}_1\mathbf{e}_2$, where $\alpha \geq 0$, and for $\pi \leq \theta \leq 2\pi$, $\sin(\theta)B = -\alpha\mathbf{e}_1\mathbf{e}_2 = \alpha\mathbf{e}_2\mathbf{e}_1$. However, it turns out that the proper choice of the unit bivector for description of a counterclockwise rotation is $B = \mathbf{e}_2\mathbf{e}_1$. Indeed, for $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$ and $\mathbf{v}' = v_1{}'\mathbf{e}_1 + v_2{}'\mathbf{e}_2$ such that $\mathbf{v}' = R\mathbf{v}\widetilde{R}$, one has:

$$\begin{aligned}
\mathbf{v}' &= \Big(\cos(\theta) + \sin(\theta)\mathbf{e}_2\mathbf{e}_1\Big)(v_1\mathbf{e}_1 + v_2\mathbf{e}_2)\Big(\cos(\theta) - \sin(\theta)\mathbf{e}_2\mathbf{e}_1\Big) \\
&= \Big(\cos(\theta) + \sin(\theta)\mathbf{e}_2\mathbf{e}_1\Big)\Big(\Big(v_1\cos(\theta) - v_2\sin(\theta)\Big)\mathbf{e}_1 + \Big(v_2\cos(\theta) + v_1\sin(\theta)\Big)\mathbf{e}_2\Big) \\
&= \Big(v_1\Big(\cos^2(\theta)-\sin^2(\theta)\Big) - v_2\Big(2\sin(\theta)\cos(\theta)\Big)\Big)\mathbf{e}_1 + \Big(v_2\Big(\cos^2(\theta)-\sin^2(\theta)\Big) + v_1\Big(2\sin(\theta)\cos(\theta)\Big)\Big)\mathbf{e}_2 \\
&= \Big(v_1\cos(2\theta) - v_2\sin(2\theta)\Big)\mathbf{e}_1 + \Big(v_2\cos(2\theta) + v_1\sin(2\theta)\Big)\mathbf{e}_2.
\end{aligned} \tag{2.72}$$

The resultant equation can be expressed in matrix form as

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{2.73}$$

which serves as a confirmation that $\mathbf{v}' = R\mathbf{v}\widetilde{R}$ express a rotation of the vector $\mathbf{v}$ in the counterclockwise sense. But, note that it is a rotation by an angle 2θ. Therefore, to describe a rotation by an angle θ one must set

$$R = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\mathbf{e}_2\mathbf{e}_1. \tag{2.74}$$

The rotor R can be expressed in another way by defining the exponential of a generic multivector A by

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} = 1 + A + \frac{A^2}{2} + \cdots , \tag{2.75}$$

where $A^0 = 1$, $A^1 = A$, $A^2 = AA$, etc. In this way, using the expressions as power series for the sine and cosine functions in the above expression for R, and taking into account that $(\mathbf{e}_2\mathbf{e}_1)^2 = -1$, one has:

$$\begin{aligned} R = \cos(\theta/2) + \sin(\theta/2)\mathbf{e}_2\mathbf{e}_1 &= \sum_{n=0}^{\infty} \frac{(-1)^n(\theta/2)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n(\theta/2)^{2n+1}}{(2n+1)!}\mathbf{e}_2\mathbf{e}_1 \\ &= \sum_{n=0}^{\infty} \frac{(\mathbf{e}_2\mathbf{e}_1)^{2n}(\theta/2)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\mathbf{e}_2\mathbf{e}_1)^{2n+1}(\theta/2)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(\mathbf{e}_2\mathbf{e}_1\theta/2)^n}{n!}. \end{aligned} \tag{2.76}$$

Thus, one can write

$$R = \exp\left(\frac{1}{2}\theta\mathbf{e}_2\mathbf{e}_1\right). \tag{2.77}$$

Note that, to describe rotations in the clockwise sense one can simply allow the angle θ to be negative. Note also that R and $-R$ describe the same rotation:

$$(-R)\mathbf{v}\widetilde{(-R)} = R\mathbf{v}\widetilde{R}. \tag{2.78}$$

This fact can be understood by observing that the rotation of a vector by an angle ϕ in the counterclockwise sense has the same result as the rotation of this vector by the angle $2\pi-\phi$ in the clockwise sense. Indeed, if $R = \exp(\mathbf{e}_2\mathbf{e}_1\phi/2)$ and $R^* = \exp\big(\mathbf{e}_1\mathbf{e}_2(2\pi-\phi)/2\big)$, then

$$R^* = \exp\big(\mathbf{e}_1\mathbf{e}_2(2\pi-\phi)/2\big) = \exp(\mathbf{e}_1\mathbf{e}_2\pi)\exp(-\mathbf{e}_1\mathbf{e}_2\phi/2) = (-1)\exp(\mathbf{e}_2\mathbf{e}_1\phi/2) = -R. \tag{2.79}$$

2.1.5 The Even Subalgebra and the Complex Numbers

Let ${\mathcal{C}\ell_{2,0}}^+$ be the set formed by even grade multivectors from $\mathcal{C}\ell_{2,0}$, that is, the set of multivectors A satisfying $\widehat{A} = A$. If $A \in {\mathcal{C}\ell_{2,0}}^+$, then A is the sum of a scalar and a bivector:

$$A = a + a_{12}\mathbf{e}_1\mathbf{e}_2. \tag{2.80}$$

Given $A = a + a_{12}\mathbf{e}_1\mathbf{e}_2$ and $B = b + b_{12}\mathbf{e}_1\mathbf{e}_2$ from ${\mathcal{C}\ell_{2,0}}^+$, if follows that

$$AB = (a + a_{12}\mathbf{e}_1\mathbf{e}_2)(b + b_{12}\mathbf{e}_1\mathbf{e}_2) = (ab - a_{12}b_{12}) + (ab_{12} + a_{12}b)\mathbf{e}_1\mathbf{e}_2, \tag{2.81}$$

so that $AB \in \mathcal{C}\ell_{2,0}{}^{+}$. Thus, the vector subspace formed by multivectors from $\mathcal{C}\ell_{2,0}{}^{+}$ endowed with the geometric product has the properties of an algebra (closure relative to the product and bilinearity of the product), hence it corresponds to a subalgebra of $\mathcal{C}\ell_{2,0}$. This is called the *even subalgebra* of $\mathcal{C}\ell_{2,0}$, and it is denoted by $\mathcal{C}\ell_{2,0}{}^{+}$.

Note that the rotors introduced earlier are elements of the even subalgebra $\mathcal{C}\ell_{2,0}{}^{+}$, although not all elements of this algebra are rotors. But note that the elements of $\mathcal{C}\ell_{2,0}{}^{+}$ can be written in the form

$$\psi = \rho\cos(\phi) + \rho\sin(\phi)\mathbf{e}_1\mathbf{e}_2, \tag{2.82}$$

or, in terms of the exponential map,

$$\psi = \rho\exp(\phi\mathbf{e}_1\mathbf{e}_2), \tag{2.83}$$

where ρ and ϕ are real scalars. Therefore, an element of the even subalgebra can be written as a rotor multiplied by a scalar, and a rotor can be understood as an element of $\mathcal{C}\ell_{2,0}{}^{+}$ with unit norm. An even grade multivector ψ acting on a vector $\mathbf{u}$ through $\psi\mathbf{u}\widetilde{\psi}$ produces not only a rotation of the vector $\mathbf{u}$, but also a dilation (if $\rho > 1$) or a contraction (if $0 < \rho < 1$).

The proper expression for the even grade multivector ψ introduced above to produce a rotation by an angle ϕ in the counterclockwise sense and a dilation/contraction by a factor ρ through the transformation $\mathbf{u} \mapsto \psi\mathbf{u}\widetilde{\psi}$ is

$$\psi = \sqrt{\rho}R, \quad \text{where} \quad R = \exp\left(\frac{1}{2}\phi\mathbf{e}_2\mathbf{e}_1\right). \tag{2.84}$$

This is easily verified by evaluating the action of $\psi = \sqrt{\rho}R$ on a vector $\mathbf{u}$:

$$\psi\mathbf{u}\widetilde{\psi} = (\sqrt{\rho}R)\,\mathbf{u}\widetilde{(\sqrt{\rho}R)} = \rho R\mathbf{u}\widetilde{R}. \tag{2.85}$$

The expressions (2.82) and (2.83) for elements of $\mathcal{C}\ell_{2,0}{}^{+}$, together with the fact that $(\mathbf{e}_2\mathbf{e}_1)^2 = (\mathbf{e}_1\mathbf{e}_2)^2 = -1$, suggest a relation between the even subalgebra $\mathcal{C}\ell_{2,0}{}^{+}$ and the algebra of the complex numbers. Similarly to the complex numbers, the elements of $\mathcal{C}\ell_{2,0}{}^{+}$ can be written in the form

$$X = x_1 + x_2 I, \tag{2.86}$$

where $x_1, x_2 \in \mathbb{R}$ and $I = \mathbf{e}_1\mathbf{e}_2$, with $I^2 = -1$, and as can be observed from (2.81), the geometric product of elements of $\mathcal{C}\ell_{2,0}{}^{+}$ has the same form as the product of complex numbers: given $X = x_1 + x_2 I$ and $Y = y_1 + y_2 I$, one has

$$XY = (x_1 + x_2 I)(y_1 + y_2 I) = (x_1 y_1 - x_2 y_2) + (x_1 y_2 + x_2 y_1)I. \tag{2.87}$$

Thus, the even subalgebra $\mathcal{C}\ell_{2,0}{}^{+}$ is isomorphic to the algebra of the complex numbers, by means of the identification of $I = \mathbf{e}_1\mathbf{e}_2$ with the imaginary unit $i = \sqrt{-1}$ and the identification of the geometric product with the product of complex numbers.

2.2 The Geometric Algebra of the Three-Dimensional Euclidean Space

2.2.1 Construction of the Structure

Consider the vector space $\mathbb{R}^3$, and let its vectors be denoted by Latin letters in boldface: $\mathbf{u}$, $\mathbf{v}$, etc. Let the canonical basis be denoted $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$ (where the ordering of the basis is implied), in such a way that a vector is written, generally, $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$, etc. The interpretation for this space is the usual geometric interpretation: $\mathbb{R}^3$ corresponds to the three-dimensional physical space, and its vectors represent oriented line segments in that space.

Consider the symmetric bilinear form $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$g(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}, \quad \text{where} \quad i, j \in \{1, 2, 3\}, \tag{2.88}$$

which corresponds to the usual inner product defined for $\mathbb{R}^3$, also known as the scalar product. The vector space $\mathbb{R}^3$ endowed with such an inner product has the status of *three-dimensional Euclidean space.*

As in the case of the geometric algebra of the Euclidean plane, the *geometric algebra of the three-dimensional Euclidean space* is determined by a space constructed from the three-dimensional Euclidean space, endowed with the geometric product. As before, the construction of the structure is made gradually. Also as before, the geometric product is denoted by juxtaposition, which is usual in the study of geometric/Clifford algebras.

The fundamental property of the geometric product is given by

$$\mathbf{u}\mathbf{u} = g(\mathbf{u}, \mathbf{u}), \tag{2.89}$$

for any vector $\mathbf{u}$ from $\mathbb{R}^3$, which can be written

$$\mathbf{u}^2 = |\mathbf{u}|^2, \tag{2.90}$$

where $\mathbf{u}^2 = \mathbf{uu}$ and $|\cdot|$ is the norm induced by the inner product g, that is, the usual modulus of a vector from the Euclidean space. Writing $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, the above equation can be written in terms of components as

$$(u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3)(u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3) = {u_1}^2 + {u_2}^2 + {u_3}^3. \tag{2.91}$$

Imposing bilinearity to the geometric product in the above expression, one obtains

$$u_1^2\mathbf{e}_1^2+u_2^2\mathbf{e}_2^2+u_3^2\mathbf{e}_3^2+u_1u_2(\mathbf{e}_1\mathbf{e}_2+\mathbf{e}_2\mathbf{e}_1)+u_1u_3(\mathbf{e}_1\mathbf{e}_3+\mathbf{e}_3\mathbf{e}_1)+u_2u_3(\mathbf{e}_2\mathbf{e}_3+\mathbf{e}_3\mathbf{e}_2) = u_1^2+u_2^2+u_3^2. \tag{2.92}$$

For this equation to be satisfied, one must have

$$\mathbf{e}_i{}^2 = 1 \quad \text{and} \quad \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i, \quad \text{where} \quad i, j \in \{1, 2, 3\} \quad \text{and} \quad i \neq j. \tag{2.93}$$

These relations determine the geometric product of the geometric algebra of the three-dimensional Euclidean space in terms of the canonical basic vectors. Applying it to the computation of the geometric product of two arbitrary vectors $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ and $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$, one has

$$\begin{aligned}\mathbf{uv} &= (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3)(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3)\\ &= (u_1v_1 + u_2v_2 + u_3v_3)+\\ &\qquad + (u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2 + (u_1v_3 - u_3v_1)\mathbf{e}_1\mathbf{e}_3 + (u_2v_3 - u_3v_2)\mathbf{e}_2\mathbf{e}_3.\end{aligned} \tag{2.94}$$

As in the two-dimensional case, the first term on the left-hand side from the resulting equation is a scalar, which corresponds to the scalar product of the vectors $\mathbf{u}$ and $\mathbf{v}$. The other terms are neither scalars nor vectors, if one considers the associability of the geometric product, as in the first construction (the same counterexamples can be taken to demonstrate this). Such sum of terms are combinations of objects that in the two-dimensional case were interpreted as representing oriented parallelograms. In this case, the same interpretation can be used for each term in that combination. For example, the term $(u_1v_3 - u_3v_1)\mathbf{e}_1\mathbf{e}_3$ represents the oriented parallelogram determined by the vectors $(u_1\mathbf{e}_1 + u_3\mathbf{e}_3)$ and $(v_1\mathbf{e}_1 + v_3\mathbf{e}_3)$, which belong to the plane determined by the vectors $\mathbf{e}_1$ and $\mathbf{e}_3$. The sum of the terms in the form $(u_iv_j - u_jv_i)\mathbf{e}_i\mathbf{e}_j$, with $i \neq j$, must then represent a combination of the oriented parallelograms represented by them. Considering then each component of this combination in terms of the exterior product, by writing $(u_iv_j - u_jv_i)\mathbf{e}_i\mathbf{e}_j$ as $(u_iv_j - u_jv_i)\mathbf{e}_i \wedge \mathbf{e}_j$, and then considering a natural extension of the exterior product for the three-dimensional case, one obtains

$$(u_1v_2 - u_2v_1)\mathbf{e}_1 \wedge \mathbf{e}_2 + (u_1v_3 - u_3v_1)\mathbf{e}_1 \wedge \mathbf{e}_3 + (u_2v_3 - u_3v_2)\mathbf{e}_2 \wedge \mathbf{e}_3 = \\ = (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3) \wedge (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3). \quad (2.95)$$

Thus, the combination of terms in question is identified with the exterior product $\mathbf{u} \wedge \mathbf{v}$, which must represent the oriented parallelogram determined by the three-dimensional vectors $\mathbf{u}$ and $\mathbf{v}$. As in the two-dimensional case, one can verify that the set of objects in this form endowed with the operations of summation and multiplication by a real scalar has the structure of a vector space. This is the vector space of the bivectors of the three-dimensional Euclidean space, which is denoted by $\bigwedge^2(\mathbb{R}^3)$.

As before, the fact that one can write the geometric product $\mathbf{uv}$ as a sum of a symmetric part and an antisymmetric part relative to the exchange between $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{uv} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu}) + \frac{1}{2}(\mathbf{uv} - \mathbf{vu}), \quad (2.96)$$

allows one to identify the symmetric part with the inner/scalar product $g(\mathbf{u}, \mathbf{v}) = u_1v_1 + u_2v_2 + u_3v_3$ and to express such a product in terms of the geometric product as

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu}). \quad (2.97)$$

Again, the exterior product of the vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{u} \wedge \mathbf{v} = (u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2 + (u_1v_3 - u_3v_1)\mathbf{e}_1\mathbf{e}_3 + (u_2v_3 - u_3v_2)\mathbf{e}_2\mathbf{e}_3, \quad (2.98)$$

is identified with the antisymmetric part of the product $\mathbf{uv}$:

$$\mathbf{u} \wedge \mathbf{v} = \frac{1}{2}(\mathbf{uv} - \mathbf{vu}) = -\mathbf{v} \wedge \mathbf{u}. \quad (2.99)$$

These definitions allow one to write

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}. \quad (2.100)$$

In the two-dimensional case, by taking the geometric product of a vector of the plane with a bivector one obtains another vector of the plane, but in the three-dimensional case this does not always occur. For example, by taking the geometric product of the vector $\mathbf{e}_1$ with the bivector $\mathbf{e}_2\mathbf{e}_3$, one obtains $\mathbf{e}_1(\mathbf{e}_2\mathbf{e}_3)$, which, by the associativity of the geometric product, is equivalent to $(\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_3$, or simply $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. Analogously to the case of the oriented parallelograms in the plane, one can identify the object $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ with an oriented volume element of the three-dimensional Euclidean space. Its orientation can be defined by the order of the geometric product. Since there is no longer any different combination of geometric products involving the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, except for the order of the product, which determines the orientation of the volume element, any other volume must be represented by $\alpha\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, where α is a real scalar. Real linear combinations of objects of the form $\alpha\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ always result in objects of the same form, and it is easy to verify that the set of such objects endowed with the operations of summation and multiplication by a real scalar has the structure of a real vector space. Such a vector space is denoted by $\bigwedge^3(\mathbb{R}^3)$, and its vectors are called 3-*vectors*, or *trivectores*, or even *pseudoscalars*, since $\bigwedge^3(\mathbb{R}^3)$ is a one-dimensional vector space. The unit pseudoscalar $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is usually denoted by I. In general, given three non-null and linearly independent vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, it is found that $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is a trivector, which represent the oriented paralleliped determined by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The figure 2.3 illustrates the oriented volume determined by $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.

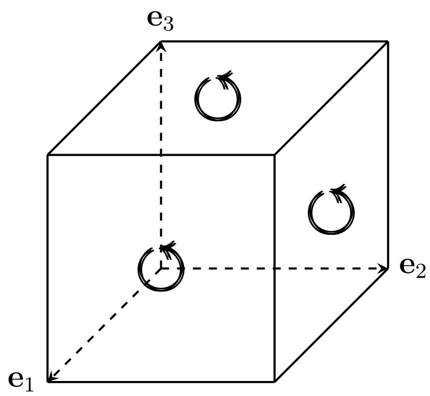

FIGURE 2.3 – The oriented volume associated with the unit pseudoscalar $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.

As in the two-dimensional case, in order to construct a closed algebraic structure with respect to the geometric product, the vector space $\bigwedge(\mathbb{R}^3)$ is defined as the direct sum of the spaces of the form $\bigwedge^k(\mathbb{R}^3)$:

$$\bigwedge(\mathbb{R}^3) = \bigoplus_{k=0}^{3} \bigwedge^k(\mathbb{R}^3) = \bigwedge^0(\mathbb{R}^3) \oplus \bigwedge^1(\mathbb{R}^3) \oplus \bigwedge^2(\mathbb{R}^3) \oplus \bigwedge^3(\mathbb{R}^3). \qquad (2.101)$$

Its elements are called *multivectors*. The null vector of this vector space is simply denoted by 0. An arbitrary multivector from $\bigwedge(\mathbb{R}^3)$ is of the form

$$A = a + (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) + (a_{12}\mathbf{e}_1\mathbf{e}_2 + a_{13}\mathbf{e}_1\mathbf{e}_3 + a_{23}\mathbf{e}_2\mathbf{e}_3) + a_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \qquad (2.102)$$

where $a, a_i, a_{ij}, a_{ijk} \in \mathbb{R}$, with $i, j, k \in \{1, 2, 3\}$.

Note that the geometric product of the unit pseudoscalar $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ with any vector is commutative, so that I commutes also with bivectors, and since it commutes with scalars and other pseudoscalars, it commutes with any multivector of the algebra.

Defining the exterior product of a scalar α with a vector $\mathbf{u}$ by $\alpha \wedge \mathbf{u} = \alpha\mathbf{u}$, and extending the exterior product for arbitrary multivectors by considering it, in addition to bilinear, associative, $\left(\bigwedge(\mathbb{R}^3), \wedge\right)$ determines an associative algebra over the field of real scalars, the *exterior algebra* or *Grassmann algebra* associated with $\mathbb{R}^3$.

Establishing then the geometric product of a scalar with a multivector from $\bigwedge(\mathbb{R}^3)$ as the multiplication of the multivector by the scalar, and extending the geometric product for arbitrary multivectors by bilinearity and associativity, the vector space $\bigwedge(\mathbb{R}^3)$ endowed with the geometric product becomes an associative algebra over the field of real scalars, the *geometric algebra of the three-dimensional Euclidean space* or the *Clifford algebra of the three-dimensional Euclidean space*, which can be denoted by $\mathcal{C}\ell(\mathbb{R}^3, g)$, or $\mathcal{C}\ell_{3,0}(\mathbb{R})$, or $\mathcal{C}\ell_{3,0}$.

2.2.2 Projection, Graded Involution, Reversion, the Norm and the Inverse

Given an arbitrary k-vector A_k, such that $A = \sum_{k=0}^{3} A_k$ is an arbitrary multivector from $\mathcal{C}\ell_{3,0}$, the operations of projection, graded involution and reversion are defined in a similar way to the two-dimensional case:

$$\langle A\rangle_k = A_k, \quad \widehat{A} = \sum_{k=0}^{3}(-1)^k \langle A\rangle_k, \quad \text{and} \quad \widetilde{A} = \sum_{k=0}^{3}(-1)^{\frac{1}{2}k(k-1)}\langle A\rangle_k. \tag{2.103}$$

In this way, for the arbitrary multivector $A = \sum_{k=0}^{3} A_k$, one has

$$\widehat{A} = A_0 - A_1 + A_2 - A_3 \quad \text{and} \quad \widetilde{A} = A_0 + A_1 - A_2 - A_3 \tag{2.104}$$

The operation of projection on the subspace of scalars is important, and is generally denoted in a more simplified way by omitting the subscript number zero:

$$\langle A\rangle_0 = \langle A\rangle. \tag{2.105}$$

By inspecting particular cases, one can conclude that the reversion of the geometric product of two multivectors corresponds to the geometric product in the opposite order of the reverses of the multivectors. That is, if A and B are two multivectors, then

$$\widetilde{(AB)} = \widetilde{B}\widetilde{A}. \tag{2.106}$$

From the associativity of the geometric product, this property extends to the geometric product of an arbitrary list of multivectors $A, B, \ldots, C$ as follows:

$$\widetilde{(AB\cdots C)} = \widetilde{C}\cdots\widetilde{B}\widetilde{A}. \tag{2.107}$$

Note also that, the reversion operation does not alter scalars, so that the reversion of the scalar part of any multivector is equivalent to the scalar part of the reverse of the multivector:

$$\langle A\rangle = \widetilde{\langle A\rangle} = \left\langle \widetilde{A}\right\rangle. \tag{2.108}$$

The above two relations imply an important property of the operation of projection on the subspace of scalars:

$$\langle AB\rangle = \langle BA\rangle. \tag{2.109}$$

In general, for the geometric product of any number of multivectors, the scalar part is invariant under cyclic permutations of the multivectors present in the product. For example, given the multivectors A, B and C, one has

$$\langle ABC\rangle = \langle BCA\rangle = \langle CAB\rangle. \tag{2.110}$$

The norm of a multivector A is defined in the same way as in the two-dimensional case:

$$\|A\|^2 = \left\langle \widetilde{A}A\right\rangle_0 = \left\langle A\widetilde{A}\right\rangle_0. \tag{2.111}$$

The inverse of a multivector is also defined in the same way as before,

$$A^{-1} = \frac{\widetilde{A}}{\|A\|^2}, \tag{2.112}$$

provided that

$$\|A\|^2 = \left\langle \widetilde{A}A\right\rangle_0 = \widetilde{A}A > 0. \tag{2.113}$$

2.2.3 Interior, Exterior and Commutator Products

Considering the expression for the scalar product as the symmetric part of the geometric product, equation (2.97), one can rewrite the geometric product of a vector $\mathbf{u}$ with the geometric product $\mathbf{v}_1\mathbf{v}_2$ of two arbitrary vectors as follows:

$$\begin{aligned}\mathbf{u}(\mathbf{v}_1\mathbf{v}_2) &= (\mathbf{u}\mathbf{v}_1)\mathbf{v}_2 \\ &= (2\mathbf{u}\cdot\mathbf{v}_1 - \mathbf{v}_1\mathbf{u})\mathbf{v}_2 \\ &= 2(\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_2 - \mathbf{v}_1(\mathbf{u}\mathbf{v}_2) \\ &= 2(\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_2 - \mathbf{v}_1(2\mathbf{u}\cdot\mathbf{v}_2 - \mathbf{v}_2\mathbf{u}) \\ &= 2(\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_2 - 2\mathbf{v}_1(\mathbf{u}\cdot\mathbf{v}_2) + (\mathbf{v}_1\mathbf{v}_2)\mathbf{u}.\end{aligned} \tag{2.114}$$

The resulting expression can be written

$$\frac{1}{2}\Big(\mathbf{u}(\mathbf{v}_1\mathbf{v}_2) - (\mathbf{v}_1\mathbf{v}_2)\mathbf{u}\Big) = (\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_2 - \mathbf{v}_1(\mathbf{u}\cdot\mathbf{v}_2). \tag{2.115}$$

The fact that the right-hand side of the above equation is a vector, implies that the left-hand side is also a vector. This motivates the definition of the *contraction from the left* of $\mathbf{v}_1\mathbf{v}_2$ by the vector $\mathbf{u}$, or the *interior product* of $\mathbf{u}$ with $\mathbf{v}_1\mathbf{v}_2$, as

$$\mathbf{u}\cdot(\mathbf{v}_1\mathbf{v}_2) = \frac{1}{2}\Big(\mathbf{u}(\mathbf{v}_1\mathbf{v}_2) - (\mathbf{v}_1\mathbf{v}_2)\mathbf{u}\Big) = (\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_2 - \mathbf{v}_1(\mathbf{u}\cdot\mathbf{v}_2). \tag{2.116}$$

Since $\mathbf{v}_1\mathbf{v}_2$ is a scalar plus a bivector, and consequently an even grade multivector, one can generally define the contraction from the left of an even grade multivector A_+ by the vector $\mathbf{u}$, or the interior product of $\mathbf{u}$ with A_+, as

$$\mathbf{u}\cdot A_+ = \frac{1}{2}(\mathbf{u}A_+ - A_+\mathbf{u}). \tag{2.117}$$

In the same way as above, one can rewrite the geometric product of a vector $\mathbf{u}$ with the geometric product $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$ of three arbitrary vectors as follows:

$$\begin{aligned}\mathbf{u}(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3) &= (\mathbf{u}\mathbf{v}_1)\mathbf{v}_2\mathbf{v}_3 \\ &= (2\mathbf{u}\cdot\mathbf{v}_1 - \mathbf{v}_1\mathbf{u})\mathbf{v}_2\mathbf{v}_3 \\ &= 2(\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_1(\mathbf{u}\mathbf{v}_2)\mathbf{v}_3 \\ &= 2(\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_1(2\mathbf{u}\cdot\mathbf{v}_2 - \mathbf{v}_2\mathbf{u})\mathbf{v}_3 \\ &= 2(\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_2\mathbf{v}_3 - 2\mathbf{v}_1(\mathbf{u}\cdot\mathbf{v}_2)\mathbf{v}_3 + \mathbf{v}_1\mathbf{v}_2(\mathbf{u}\mathbf{v}_3) \\ &= 2(\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_2\mathbf{v}_3 - 2\mathbf{v}_1(\mathbf{u}\cdot\mathbf{v}_2)\mathbf{v}_3 + \mathbf{v}_1\mathbf{v}_2(2\mathbf{u}\cdot\mathbf{v}_3 - \mathbf{v}_3\mathbf{u}) \\ &= 2(\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_2\mathbf{v}_3 - 2\mathbf{v}_1(\mathbf{u}\cdot\mathbf{v}_2)\mathbf{v}_3 + 2\mathbf{v}_1\mathbf{v}_2(\mathbf{u}\cdot\mathbf{v}_3) - (\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3)\mathbf{u}.\end{aligned} \tag{2.118}$$

The resulting expression can be written

$$\frac{1}{2}\Big(\mathbf{u}(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3) + (\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3)\mathbf{u}\Big) = (\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_1(\mathbf{u}\cdot\mathbf{v}_2)\mathbf{v}_3 + \mathbf{v}_1\mathbf{v}_2(\mathbf{u}\cdot\mathbf{v}_3). \tag{2.119}$$

The fact that the right-hand side of the above equation is an even grade multivector, implies that the left-hand side is also an even grade multivector. This fact, in addition to the fact that the product $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$ is necessarily an odd grade multivector, motivates the definition of the *contraction from the left* of $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$ by the vector $\mathbf{u}$, or the *interior product* of $\mathbf{u}$ with $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$, as

$$\mathbf{u}\cdot(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3) = \frac{1}{2}\Big(\mathbf{u}(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3) + (\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3)\mathbf{u}\Big) = (\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_2\mathbf{v}_3 - \mathbf{v}_1(\mathbf{u}\cdot\mathbf{v}_2)\mathbf{v}_3 + \mathbf{v}_1\mathbf{v}_2(\mathbf{u}\cdot\mathbf{v}_3). \tag{2.120}$$

In general, the contraction from the left of an odd grade multivector A_- by the vector $\mathbf{u}$, or the interior product of $\mathbf{u}$ with A_-, can be defined by

$$\mathbf{u}\cdot A_- = \frac{1}{2}(\mathbf{u}A_- + A_-\mathbf{u}). \tag{2.121}$$

The definitions (2.117) and (2.121) can be generalized by defining the contraction from the left of the multivector A by the vector $\mathbf{u}$, or the interior product of $\mathbf{u}$ with A, through the expression

$$\mathbf{u}\cdot A = \frac{1}{2}\left(\mathbf{u}A - \widehat{A}\mathbf{u}\right). \tag{2.122}$$

The analysis made so far to motivate the definition of contraction from the left can be repeated, with appropriate modifications, to define the *contraction from the right* of the multivector A by the vector $\mathbf{u}$, or the *interior product* of A with $\mathbf{u}$, as

$$A\cdot\mathbf{u} = \frac{1}{2}\left(A\mathbf{u} - \mathbf{u}\widehat{A}\right). \tag{2.123}$$

Note that the contraction of a k-vector by a vector (from the left or right), or the interior product of a vector with a k-vector (or the opposite), always produces a $(k-1)$-vector, which justifies the terminology.

From the expression for the exterior product, equation (2.99), note that $\mathbf{u}\wedge\mathbf{u} = 0$, for any vector $\mathbf{u}$. From this fact, and from the associativity and bilinearity of the exterior product, it follows that the exterior product of any set of linearly dependent vectors is null. (In particular, a set with more than three vectors is linearly dependent, and consequently the exterior product of these vectors is null.) Note also that any non-null exterior product $\mathbf{u}_1\wedge\mathbf{u}_2$ can be written as the geometric product of two orthogonal vectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{u}_1\wedge\mathbf{u}_2 = \mathbf{v}_1\mathbf{v}_2. \tag{2.124}$$

In this way, given a vector $\mathbf{u}$ orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, it follows that

$$\mathbf{u}\wedge\mathbf{u}_1\wedge\mathbf{u}_2 = \mathbf{u}\mathbf{v}_1\mathbf{v}_2. \tag{2.125}$$

As in the two-dimensional case, the geometric product of two orthogonal vectors is anti-commutative, so that $\mathbf{u}\mathbf{v}_1\mathbf{v}_2 = \mathbf{v}_1\mathbf{v}_2\mathbf{u}$, and the above equation allows one to write

$$\mathbf{u} \wedge (\mathbf{u}_1 \wedge \mathbf{u}_2) = \frac{1}{2}\Big(\mathbf{u}(\mathbf{v}_1\mathbf{v}_2) + (\mathbf{v}_1\mathbf{v}_2)\mathbf{u}\Big). \tag{2.126}$$

Since, by hypothesis, $\mathbf{u}_1 \wedge \mathbf{u}_2 = \mathbf{v}_1\mathbf{v}_2$, the above equation implies that

$$\mathbf{u} \wedge B = \frac{1}{2}(\mathbf{u}B + B\mathbf{u}), \tag{2.127}$$

for any bivector B. In general, the exterior product of a vector $\mathbf{u}$ with any even grade multivector A_+ (a scalar or a bivector, or a sum of both) is always commutative, such that one can write

$$\mathbf{u} \wedge A_+ = \frac{1}{2}(\mathbf{u}A_+ + A_+\mathbf{u}). \tag{2.128}$$

On the other hand, the exterior product of a vector $\mathbf{u}$ with any odd grade multivector A_- (a vector or a trivector, or a sum of both) is always anticommutative, in such way that, one can write

$$\mathbf{u} \wedge A_- = \frac{1}{2}(\mathbf{u}A_- - A_-\mathbf{u}). \tag{2.129}$$

The above two equations can be generalized for the exterior product of a vector $\mathbf{u}$ with any multivector A as follows:

$$\mathbf{u} \wedge A = \frac{1}{2}\left(\mathbf{u}A + \widehat{A}\mathbf{u}\right). \tag{2.130}$$

Similar observations, but in relation to the exterior product in opposite order, can be made to furnish:

$$A \wedge \mathbf{u} = \frac{1}{2}\left(A\mathbf{u} + \mathbf{u}\widehat{A}\right). \tag{2.131}$$

The summation of the equations (2.122) and (2.130) furnishes

$$\mathbf{u}A = \mathbf{u} \cdot A + \mathbf{u} \wedge A, \tag{2.132}$$

and the summation of the equations (2.123) and (2.131) furnishes

$$A\mathbf{u} = A \cdot \mathbf{u} + A \wedge \mathbf{u}. \tag{2.133}$$

These relations for the geometric product of a vector and a multivector in terms of the interior and exterior products are natural generalizations of the relation (2.100).

Note that, in general, neither the interior product nor the exterior product commute or anticommute. In general, one has

$$\mathbf{u} \cdot A = -\widehat{A} \cdot \mathbf{u} \quad \text{and} \quad \mathbf{u} \wedge A = \widehat{A} \wedge \mathbf{u}, \tag{2.134}$$

which can be obtained by observing that $\widehat{A} \cdot \mathbf{u} = \frac{1}{2}\left(\widehat{A}\mathbf{u} - \mathbf{u}A\right)$ and $\widehat{A} \wedge \mathbf{u} = \frac{1}{2}\left(\widehat{A}\mathbf{u} + \mathbf{u}A\right)$, which combined with (2.122) and (2.130), respectively, furnish $\mathbf{u} \cdot A + \widehat{A} \cdot \mathbf{u} = 0$ and $\mathbf{u} \wedge A - \widehat{A} \wedge \mathbf{u} = 0$.

It should be noted that, in general, the geometric product of two generic multivectors can not be written as the sum of a interior and a exterior product. This fact can be illustrated by considering the geometric product of bivectors. Let A and B be arbitrary bivectors. Consider the expression of A as the geometric product of two orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$:

$$A = \mathbf{u} \wedge \mathbf{v} = \mathbf{u}\mathbf{v}. \tag{2.135}$$

It follows that,

$$\begin{aligned} AB &= \mathbf{u}\mathbf{v}B \\ &= \mathbf{u}(\mathbf{v} \cdot B + \mathbf{v} \wedge B) \\ &= \mathbf{u} \cdot (\mathbf{v} \cdot B) + \mathbf{u} \cdot (\mathbf{v} \wedge B) + \mathbf{u} \wedge (\mathbf{v} \cdot B) + \mathbf{u} \wedge \mathbf{v} \wedge B \\ &= \mathbf{u} \cdot (\mathbf{v} \cdot B) + \mathbf{u} \cdot (\mathbf{v} \wedge B) + \mathbf{u} \wedge (\mathbf{v} \cdot B), \end{aligned} \tag{2.136}$$

where was considered the fact that $\mathbf{u} \wedge \mathbf{v} \wedge B = 0$, which follows from the fact that $\mathbf{u} \wedge \mathbf{v} \wedge B$ corresponds to the exterior product of four vectors, which are necessarily linearly dependent. The term $\mathbf{u} \cdot (\mathbf{v} \cdot B)$ in the resulting above equation is a scalar, since it is the result of two followed interior products with a vector applied on a bivector. The remain terms are bivectors, since both are the result of the combination of a interior and an exterior product with a vector applied on a bivector. The geometric product AB can then be written

$$AB = \langle AB \rangle_0 + \langle AB \rangle_2. \tag{2.137}$$

Now, note that such a product can be written as the sum of a symmetric part and an antisymmetric part in relation to the exchange of the bivectors:

$$AB = \frac{1}{2}(AB + BA) + \frac{1}{2}(AB - BA). \tag{2.138}$$

Since the symmetric part is invariant and the antisymmetric part changes the sign under the reversion operation, one identifies the symmetric part as the scalar part of the product and the antisymmetric part as the bivector part of the product:

$$\langle AB \rangle_0 = \frac{1}{2}(AB + BA) \quad \text{and} \quad \langle AB \rangle_2 = \frac{1}{2}(AB - BA). \tag{2.139}$$

The antisymmetric part of the geometric product AB of two bivectors, which corresponds to another bivector, is defined as the *commutator product* of the bivectors A and B, and is denoted by

$$A \times B = \frac{1}{2}(AB - BA). \tag{2.140}$$

The commutator product satisfies the *Jacobi identity*, that is,

$$A \times (B \times C) + C \times (A \times B) + B \times (C \times A) = 0, \tag{2.141}$$

for arbitrary bivectors A, B and C, which can be verified directly by using the definition of the commutator product.

2.2.4 Inequalities, Parallelism and Orthogonality

The Cauchy-Schwarz inequality in the form (2.44) is a general result concerning Euclidean spaces, so it holds in the case of the three-dimensional Euclidean space. Consequently, the angle between two vectors can be defined in the same way as in the case of the Euclidean plane, in terms of its cosine through (2.46), and the sine of such an angle can also be expressed by relation (2.49). Then, the conditions for parallelism and orthogonality of vectors, given by (2.52) and (2.53), are also the same as in the two-dimensional case. Another consequence of the preservation of the form of the Cauchy-Schwarz inequality is that the triangular inequality, given by (2.51), also has the same form.

Since a bivector represents an oriented area, one can also consider the conditions of parallelism and orthogonality between a vector and a bivector, and between bivectors. As seen in the above subsection, if a set of vectors $\{\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent, then $\mathbf{u} \wedge \mathbf{v}_1 \wedge \mathbf{v}_2 = 0$, which can be written

$$\mathbf{u} \wedge B = 0, \tag{2.142}$$

where $B = \mathbf{v}_1 \wedge \mathbf{v}_2$. Since linearly dependent vectors in the three-dimensional Euclidean space belong to the same plane, the above equation is a condition for parallelism of the vector $\mathbf{u}$ with the bivector B. On the other hand, if $\{\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2\}$ is a set of mutually orthogonal vectors, then (cf. equation (2.116))

$$\mathbf{u} \cdot (\mathbf{v}_1 \mathbf{v}_2) = (\mathbf{u} \cdot \mathbf{v}_1)\mathbf{v}_2 - \mathbf{v}_1(\mathbf{u} \cdot \mathbf{v}_2) = 0, \tag{2.143}$$

which implies that

$$\mathbf{u} \cdot B = 0 \tag{2.144}$$

is a condition for orthogonality of a vector $\mathbf{u}$ and a bivector B. As seen in the above subsection, given a bivector $A = \mathbf{u} \wedge \mathbf{v} = \mathbf{uv}$ and another bivector B, one has

$$AB = \langle AB \rangle + A \times B = \mathbf{u} \cdot (\mathbf{v} \cdot B) + \mathbf{u} \cdot (\mathbf{v} \wedge B) + \mathbf{u} \wedge (\mathbf{v} \cdot B). \tag{2.145}$$

If the bivector B is given by $B = \mathbf{w} \wedge \mathbf{x} = \mathbf{wx}$, the above expression furnishes

$$\begin{aligned} AB &= \mathbf{u} \cdot \Big(\mathbf{v} \cdot (\mathbf{wx})\Big) + \mathbf{u} \cdot \Big(\mathbf{v} \wedge (\mathbf{wx})\Big) + \mathbf{u} \wedge \Big(\mathbf{v} \cdot (\mathbf{wx})\Big) \\ &= \mathbf{u} \cdot \Big((\mathbf{v} \cdot \mathbf{w})\mathbf{x} - \mathbf{w}(\mathbf{v} \cdot \mathbf{x})\Big) + \mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w} \wedge \mathbf{x}) + \mathbf{u} \wedge \Big((\mathbf{v} \cdot \mathbf{w})\mathbf{x} - \mathbf{w}(\mathbf{v} \cdot \mathbf{x})\Big) \\ &= (\mathbf{u} \cdot \mathbf{x})(\mathbf{v} \cdot \mathbf{w}) - (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{x}) + \\ &\qquad + \mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w} \wedge \mathbf{x}) + (\mathbf{u} \wedge \mathbf{x})(\mathbf{v} \cdot \mathbf{w}) - (\mathbf{u} \wedge \mathbf{w})(\mathbf{v} \cdot \mathbf{x}), \end{aligned} \tag{2.146}$$

where one identifies

$$\langle AB \rangle = (\mathbf{u} \cdot \mathbf{x})(\mathbf{v} \cdot \mathbf{w}) - (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{x}) \tag{2.147}$$

and

$$A \times B = \mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w} \wedge \mathbf{x}) + (\mathbf{u} \wedge \mathbf{x})(\mathbf{v} \cdot \mathbf{w}) - (\mathbf{u} \wedge \mathbf{w})(\mathbf{v} \cdot \mathbf{x}). \tag{2.148}$$

If the bivectors A and B are associated to parallelograms/planes which are parallel, then one can always choose the vectors $\mathbf{w}$ and $\mathbf{x}$ in such way that one is parallel to $\mathbf{u}$ and orthogonal to $\mathbf{v}$, and the other is parallel to $\mathbf{v}$ and orthogonal to $\mathbf{u}$. In this case, the commutator product above is null, and the condition for parallelism of the bivectors A and B is

$$A \times B = 0. \tag{2.149}$$

On the other hand, if the bivectors A and B represent orthogonal parallelograms/planes, then either $\mathbf{u}$ or $\mathbf{v}$ is mutually orthogonal to $\mathbf{w}$ and $\mathbf{x}$. In this case, the scalar part of the product AB above is null, that is,

$$\langle AB \rangle = 0 \tag{2.150}$$

is the condition for orthogonality of the bivectors A and B.

2.2.5 Duality

The usual vector algebra, founded mainly on the cross product, emerged at the end of the 19th century as an attempt by J. W. Gibbs, and independently by O. Heaviside, to unify the structure of the Grassmann algebra with that of the quaternion algebra, as done by the then almost unknown Clifford algebras. The cross product is an anticommutative and non-associative product of vectors from the three-dimensional Euclidean space resulting in another vector of this space. This product is defined as follows. Given the vectors $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ and $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$, the cross product of $\mathbf{u}$ with $\mathbf{v}$ is the vector given by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{e}_1 + (u_3v_1 - u_1v_3)\mathbf{e}_2 + (u_1v_2 - u_2v_1)\mathbf{e}_3. \tag{2.151}$$

whose notation should not be confused with that of the commutator product. In this context, it is common to use the notations $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$ and $\mathbf{e}_3 = \mathbf{k}$, similar to that used in the context of the algebra of quaternions, and define the cross product as being such that

$$\left\{ \begin{array}{ccccc} \mathbf{i} \times \mathbf{j} & = & -\mathbf{j} \times \mathbf{i} & = & \mathbf{k} \\ \mathbf{j} \times \mathbf{k} & = & -\mathbf{k} \times \mathbf{j} & = & \mathbf{i} \\ \mathbf{k} \times \mathbf{i} & = & -\mathbf{i} \times \mathbf{k} & = & \mathbf{j} \end{array} \right\}, \tag{2.152}$$

where one can observe a complete analogy with the basic relations defining the product of quaternions (cf. relations (2.166) in the next subsection). Using the above relations, and bilinearity, one can easily obtain the relation (2.151) for the cross product of two arbitrary vectors. These definitions are shown to be inconsistent when it is noted that any vector transforms into its opposite under a spatial inversion transformation, but not the cross product as defined above. Indeed, an arbitrary vector $\mathbf{u}$ transforms in the way $\mathbf{u} \mapsto -\mathbf{u}$ under a spatial inversion transformation, whereas the cross product $\mathbf{u} \times \mathbf{v}$ transforms in the way $\mathbf{u} \times \mathbf{v} \mapsto (-\mathbf{u}) \times (-\mathbf{v}) = \mathbf{u} \times \mathbf{v}$ under a spatial inversion transformation. Thus, the cross product of two vectors does not exhibit a property satisfied by any usual vector. Historically, this fact has led the result of a cross product to be called a pseudovector. (In addition, the result of the scalar triple product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ changes sign under a spatial inversion, which is not satisfied by scalars — the result of such a product is then usually called a pseudoscalar). An inconsistency in the usual definition of cross product is then observed: the cross product does not result in a usual vector from $\mathbb{R}^3$, although the expression on the right-hand side of the equation (2.151) is clearly a vector. However, given the vectors $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ and $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$, one has

$$\mathbf{u} \wedge \mathbf{v} = (u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2 + (u_3v_1 - u_1v_3)\mathbf{e}_3\mathbf{e}_1 + (u_2v_3 - u_3v_2)\mathbf{e}_2\mathbf{e}_3, \tag{2.153}$$

which can be rewritten as

$$\mathbf{u} \wedge \mathbf{v} = (u_2v_3 - u_3v_2)I\mathbf{e}_1 + (u_3v_1 - u_1v_3)I\mathbf{e}_2 + (u_1v_2 - u_2v_1)I\mathbf{e}_3, \tag{2.154}$$

where $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the unit pseudoscalar. This allows one to write

$$-(\mathbf{u} \wedge \mathbf{v})I = (u_2v_3 - u_3v_2)\mathbf{e}_1 + (u_3v_1 - u_1v_3)\mathbf{e}_2 + (u_1v_2 - u_2v_1)\mathbf{e}_3, \tag{2.155}$$

that is, one can rewrite the cross product as

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{u} \wedge \mathbf{v})I. \tag{2.156}$$

Note that the right-hand side of the above equation behaves like a vector under a spatial inversion. Indeed, since $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ transforms in the way $I \mapsto -I$ under a spatial inversion, $-(\mathbf{u} \wedge \mathbf{v})I$ transforms in the way

$$-(\mathbf{u} \wedge \mathbf{v})I \;\mapsto\; -\Big((-\mathbf{u}) \wedge (-\mathbf{v})\Big)(-I) = (\mathbf{u} \wedge \mathbf{v})I \tag{2.157}$$

under a spatial inversion. The operation $(\mathbf{u} \wedge \mathbf{v}) \mapsto -(\mathbf{u} \wedge \mathbf{v})I$ then transforms a bivector into a vector. This operation is found to be an isomorphism between the space of vectors and the space of bivectors, which is a special case of the *Hodge isomorphism* or the *Hodge duality*, given by the *Hodge star operator*:

$$\star(\mathbf{u} \wedge \mathbf{v}) = -(\mathbf{u} \wedge \mathbf{v})I. \tag{2.158}$$

For vectors, the Hodge duality is given by

$$\star\mathbf{u} = \mathbf{u}I, \tag{2.159}$$

which in fact is found to be a bivector. It should be noted that $\star(\mathbf{u} \wedge \mathbf{v})$ corresponds to a vector orthogonal to the plane described by $\mathbf{u} \wedge \mathbf{v}$, and that $\star\mathbf{u}$ corresponds to a bivector describing the plane orthogonal to the vector $\mathbf{u}$.

Since $\mathbf{u} \times \mathbf{v}$ transforms like $\mathbf{u} \wedge \mathbf{v}$ under a spatial inversion transformation, it is more natural to associate physical quantities usually defined in terms of the cross product by a bivector. For example, the angular momentum vector, which does not change sign under a spatial inversion and is usually called a pseudovector, can be defined more naturally as the bivector $L = \mathbf{r} \wedge \mathbf{p}$, also because it is a quantity that is naturally related to areas, and not to lengths. This definition is in agreement with the description of angular momentum as an antisymmetric tensor $L_{ij} = -L_{ji}$, since the corresponding bivector quantity can be written

$$L = L_{12}\mathbf{e}_1\mathbf{e}_2 + L_{31}\mathbf{e}_3\mathbf{e}_1 + L_{23}\mathbf{e}_2\mathbf{e}_3 = \sum_{i<j \text{ and } i,j\in\{1,2,3\}} L_{ij}\mathbf{e}_i\mathbf{e}_j. \tag{2.160}$$

In this way, the angular momentum vector $\boldsymbol{\ell}$ is described as the *Hodge dual* of the angular momentum bivector: $\boldsymbol{\ell} = \star L = \star(\mathbf{r} \wedge \mathbf{p})$.

In the context of the geometric algebra of the three-dimensional Euclidean space, the Hodge isomorphim, or Hodge duality, between $\bigwedge^k(\mathbb{R}^3)$ and $\bigwedge^{(3-k)}(\mathbb{R}^3)$, for $k \in \{0, 1, 2, 3\}$, establishes the correspondence between a k-vector A_k and a $(3-k)$-vector $\star A_k$ through

$$\star A_k = \widetilde{A}_k I, \tag{2.161}$$

where $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the unit pseudoscalar. The $(3-k)$-vector $\star A_k$ is called the *Hodge dual* of the k-vector A_k. Thus, the Hodge dual of a scalar is a pseudoscalar, and vice versa, and the Hodge dual of a vector is a bivector (which is alternatively called a *pseudovector*), and vice versa. In particular, one has the relations in the following table.

$\star 1 = I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$
$\star\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_3$
$\star\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_1$
$\star\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2$
$\star(\mathbf{e}_1\mathbf{e}_2) = \mathbf{e}_3$
$\star(\mathbf{e}_3\mathbf{e}_1) = \mathbf{e}_2$
$\star(\mathbf{e}_2\mathbf{e}_3) = \mathbf{e}_1$
$\star I = \star(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = 1$

TABLE 2.1 – Hodge duals of the basic multivectors from $\mathcal{C}\ell_{3,0}$.

2.2.6 The Even Subalgebra and the Algebra of Quaternions

After the demonstration of the fundamental theorem of algebra, which guarantees that a polynomial equation of degree n has n not necessarily distinct complex solutions, there seemed to be no further need to introduce new types of numbers. It was with a different motivation that W. R. Hamilton conceived of the quaternions. Hamilton was looking for numbers of the form $a + bi + cj$, where $a, b, c \in \mathbb{R}$ and $i^2 = j^2 = -1$, which should play the same role in three-dimensional space as complex numbers did in the plane. Influenced by the complex identity

$$(a + bi)(a - bi) = a^2 + b^2, \tag{2.162}$$

Hamilton observed that

$$(a + bi + cj)(a - bi - cj) = a^2 + b^2 + c^2 - (ij + ji)bc. \tag{2.163}$$

Then, in 1843, after years of study, he had the sudden idea of giving up the commutative law of multiplication, and considered ij as a third square root of -1, $ij = k$, in such way that

$$i^2 = j^2 = k^2 = ijk = -1, \tag{2.164}$$

according to which, $ij = -ji$ and

$$(a + bi + cj)(a - bi - cj) = a^2 + b^2 + c^2. \tag{2.165}$$

In general, as a consequence of equations (2.164), it follows that

$$\left\{ \begin{array}{ccccc} ij & = & -ji & = & k \\ jk & = & -kj & = & i \\ ki & = & -ik & = & j \end{array} \right\}, \tag{2.166}$$

according to which,

$$(a + bi + cj + dk)(a - bi - cj - dk) = a^2 + b^2 + c^2 + d^2, \tag{2.167}$$

where $a, b, c, d \in \mathbb{R}$. Numbers of the form $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$ and i, j and k are such that the equations (2.164) hold, are called *quaternions*. The set formed by the quaternions is denoted by $\mathbb{H}$, as a tribute to Hamilton. Quaternions are combined through the operations of sum and product according to the usual laws of arithmetic (commutativity, associativity, existence of the neutral element, existence of symmetric elements, and distributivity of the product with respect to the sum), just like real and complex numbers, except for the commutativity law of the product. Moreover, it is possible to multiply a quaternion by a real number. Such operations always generate other quaternions, which characterizes the closure property of $\mathbb{H}$ with respect to these operations. Thus, the set of quaternions endowed with the operations of summation and product of quaternions with the aforementioned properties forms a non-commutative field, or a division ring. It is also verified that $\mathbb{H}$ endowed with the operations of summation of quaternions and multiplication of a quaternion by a real scalar determines a vector space over the field of real scalars. This vector space, in turn, endowed with the quaternion product determines an algebra over the field of real scalars. It turns out that this algebra is equivalent to the even subalgebra of the geometric algebra $\mathcal{C}\ell_{3,0}$, as can be seen in the following.

Let ${\mathcal{C}\ell_{3,0}}^+$ be the set formed by even grade multivectors from $\mathcal{C}\ell_{3,0}$, that is, the set of multivectors A satisfying $\widehat{A} = A$:

$${\mathcal{C}\ell_{3,0}}^+ = \left\{ A \;\middle|\; A \in \mathcal{C}\ell_{3,0} \;\text{ and }\; \widehat{A} = A \right\}. \tag{2.168}$$

If $A \in {\mathcal{C}\ell_{3,0}}^+$, then A is the sum of a scalar and a bivector, and it can be written

$$A = a + a_{12}\mathbf{e}_1\mathbf{e}_2 + a_{31}\mathbf{e}_3\mathbf{e}_1 + a_{23}\mathbf{e}_2\mathbf{e}_3. \tag{2.169}$$

An even grade multivector can be expressed without reference to any basis as

$$M = \alpha + B, \tag{2.170}$$

where α is a scalar and B is a bivector. So, given the even grade multivectors $M_1 = \alpha_1 + B_1$ and $M_2 = \alpha_2 + B_2$, it follows that

$$\begin{aligned} M_1 M_2 &= (\alpha_1 + B_1)(\alpha_2 + B_2) \\ &= \alpha_1\alpha_2 + \alpha_1 B_2 + \alpha_2 B_1 + B_1 B_2 \\ &= \Big(\alpha_1\alpha_2 + \langle B_1 B_2\rangle\Big) + \Big(\alpha_1 B_2 + \alpha_2 B_1 + B_1 \times B_2\Big). \end{aligned} \tag{2.171}$$

Thus, the geometric product of two even grade multivectos is an even grade multivector, in such way that vector subspace formed by multivectors from ${\mathcal{C}\ell_{3,0}}^+$ endowed with the geometric product is a subalgebra of $\mathcal{C}\ell_{3,0}$. This subalgebra is known as the *even subalgebra* of $\mathcal{C}\ell_{3,0}$, which can also be denoted by ${\mathcal{C}\ell_{3,0}}^+$.

Consider now the following notation: $\mathbf{I} = \mathbf{e}_3\mathbf{e}_2$, $\mathbf{J} = \mathbf{e}_1\mathbf{e}_3$ and $\mathbf{K} = \mathbf{e}_2\mathbf{e}_1$. In this way, an element of ${\mathcal{C}\ell_{3,0}}^+$ can be written in the form

$$A = a + b\mathbf{I} + c\mathbf{J} + d\mathbf{K}, \tag{2.172}$$

and one can note without difficulty that the bivectors $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$ satisfy:

$$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = \mathbf{IJK} = -1. \tag{2.173}$$

Direct comparison of these expressions with the relations (2.164) allows one to conclude that the even subalgebra ${\mathcal{C}\ell_{3,0}}^+$ is isomorphic to the algebra of quaternions through the identification of the bivectors $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$ with the unit quaternions i, j and k, respectively, and through the identification of the geometric product with the product of quaternions. Since

$$\mathbf{I} = \mathbf{e}_3\mathbf{e}_2 = - \star \mathbf{e}_1, \quad \mathbf{J} = \mathbf{e}_1\mathbf{e}_3 = - \star \mathbf{e}_2 \quad \text{and} \quad \mathbf{K} = \mathbf{e}_2\mathbf{e}_1 = - \star \mathbf{e}_3, \tag{2.174}$$

it follows that (i, j, k) identifies with the Hodge duals $-\star(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, not with $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ as the usual vector algebra suggests. Thus, it is observed that the synthesis of the Grassmann algebra (for $\mathbb{R}^3$) with the Hamilton's quaternion algebra is adequately realized by Clifford's geometric algebra of three-dimensional Euclidean space.

2.2.7 Reflections and Rotations

Similarly to the two-dimensional case, reflection transformations are introduced in order to study rotations in the three-dimensional Euclidean space. Some steps are identical to those already presented, but are repeated for completeness. A particular case of the Cartan-Dieudonné theorem is evoked again, without any demonstration.

Let $\mathbf{u}$ and $\mathbf{v}$ be two non-null vectors from $\mathbb{R}^3$. The component of $\mathbf{v}$ parallel to $\mathbf{u}$, or the projection of the vector $\mathbf{v}$ on the vector $\mathbf{u}$, is given by

$$\mathbf{v}_{\|} = \mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\mathbf{v} \cdot \frac{\mathbf{u}}{\|\mathbf{u}\|}\right) \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\|\mathbf{u}\|^2}(\mathbf{v} \cdot \mathbf{u})\mathbf{u}. \tag{2.175}$$

The component of $\mathbf{v}$ orthogonal to $\mathbf{u}$ is then

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\|}. \tag{2.176}$$

From the known relations for parallelism and orthogonality of vectors, it follows that $\mathbf{u}\mathbf{v}_{\|} = \mathbf{v}_{\|}\mathbf{u}$ and $\mathbf{u}\mathbf{v}_{\perp} = -\mathbf{v}_{\perp}\mathbf{u}$. Note then that the geometric product of $\mathbf{v}_{\|}$ by $\mathbf{u}$ furnishes

$$\mathbf{u}\mathbf{v}_{\|} = \frac{1}{\|\mathbf{u}\|^2}(\mathbf{v} \cdot \mathbf{u})\mathbf{u}^2, \tag{2.177}$$

that is,

$$\mathbf{u}\mathbf{v}_{\|} = \mathbf{v} \cdot \mathbf{u} = \frac{1}{2}(\mathbf{v}\mathbf{u} + \mathbf{u}\mathbf{v}). \tag{2.178}$$

But, the geometric product of this expression by $\mathbf{u}$ gives

$$\|\mathbf{u}\|^2\mathbf{v}_{\|} = \frac{1}{2}(\mathbf{u}\mathbf{v}\mathbf{u} + \|\mathbf{u}\|^2\mathbf{v}), \tag{2.179}$$

that is,

$$\mathbf{v}_{\|} = \frac{1}{2}\left(\frac{1}{\|\mathbf{u}\|^2}\mathbf{u}\mathbf{v}\mathbf{u} + \mathbf{v}\right), \tag{2.180}$$

or,

$$\mathbf{v}_{\|} = \frac{1}{2}\left(\mathbf{v} + \mathbf{u}\mathbf{v}\mathbf{u}^{-1}\right). \tag{2.181}$$

This expression allows one to write also

$$\mathbf{v}_{\perp} = \frac{1}{2}\left(\mathbf{v} - \mathbf{u}\mathbf{v}\mathbf{u}^{-1}\right). \tag{2.182}$$

Consider then the linear transformation given by

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = \mathbf{v}_{\perp} - \mathbf{v}_{\|}, \tag{2.183}$$

or by

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = \mathbf{v} - 2\mathbf{v}_{\|}. \tag{2.184}$$

Such a linear transformation corresponds to the *reflection* of the vector $\mathbf{v}$ through the plane with orthogonal vector $\mathbf{u}$. Considering the expression above for $\mathbf{v}_{\|}$, such a transformation can be written

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = -\mathbf{u}\mathbf{v}\mathbf{u}^{-1}, \tag{2.185}$$

or, for the case in which the vector $\mathbf{u}$ is unitary,

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = -\mathbf{u}\mathbf{v}\mathbf{u}. \tag{2.186}$$

As a particular case of the Cartan-Dieudonné theorem, it is found that two reflections describe a rotation in the three-dimensional Euclidean space. In this way, a rotation of the vector $\mathbf{v}$ can be expressed as

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = -\mathbf{u}_1(-\mathbf{u}_2\mathbf{v}\mathbf{u}_2)\mathbf{u}_1 = \mathbf{u}_1\mathbf{u}_2\mathbf{v}\mathbf{u}_2\mathbf{u}_1, \tag{2.187}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors. Then, one can express a rotation by

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = R\mathbf{v}R^{-1}, \tag{2.188}$$

where the object $R = \mathbf{u}_1\mathbf{u}_2$, corresponding to an even grade multivector, is called a *rotor*. If θ is the angle between the unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, then the rotor $R = \mathbf{u}_1\mathbf{u}_2$ can be written

$$R = \mathbf{u}_1 \cdot \mathbf{u}_2 + \mathbf{u}_1 \wedge \mathbf{u}_2 = \cos(\theta) + \sin(\theta)B, \tag{2.189}$$

where B is a unit bivector. From this expression it follows that

$$R\widetilde{R} = \Big(\cos(\theta)+\sin(\theta)B\Big)\Big(\cos(\theta)-\sin(\theta)B\Big) = \cos^2(\theta)-\sin^2(\theta)B^2 = \cos^2(\theta)+\sin^2(\theta) = 1, \tag{2.190}$$

so that

$$\widetilde{R} = R^{-1}, \tag{2.191}$$

and thus a rotation can be written

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = R\mathbf{v}\widetilde{R}. \tag{2.192}$$

If $\mathbf{n}$ is a unit vector orthogonal to the bivector B in the expression for R, there are two possibilities to consider R: by taking $B = \star\mathbf{n} = I\mathbf{n}$ or by taking $B = -\star\mathbf{n} = -I\mathbf{n}$. These two possibilities can be simulated considering the angle θ between the unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that $0 \leq \theta \leq 2\pi$, in such way that, choosing $B = \star\mathbf{n} = I\mathbf{n}$, one has, for $0 \leq \theta \leq \pi$,

$\sin(\theta)B = \alpha I\mathbf{n}$, where $\alpha \geq 0$, and for $\pi \leq \theta \leq 2\pi$, $\sin(\theta)B = -\alpha I\mathbf{n}$. However, it turns out that the proper choice of the unit bivector for description of a counterclockwise rotation, that is, following the right-hand convention, is $B = -I\mathbf{n}$. Indeed, writing $\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$, where $\mathbf{v}_{\|}$ is the component of $\mathbf{v}$ parallel to $\mathbf{n}$ and $\mathbf{v}_{\perp}$ is the component of $\mathbf{v}$ orthogonal to $\mathbf{n}$, and $\mathbf{v}' = R\mathbf{v}\widetilde{R}$, one has:

$$\begin{aligned}
\mathbf{v}' &= \Big(\cos(\theta) - \sin(\theta)I\mathbf{n}\Big)(\mathbf{v}_{\|} + \mathbf{v}_{\perp})\Big(\cos(\theta) + \sin(\theta)I\mathbf{n}\Big) \\
&= \Big(\cos(\theta) - \sin(\theta)I\mathbf{n}\Big)\Big(\Big(\cos(\theta) + \sin(\theta)I\mathbf{n}\Big)\mathbf{v}_{\|} + \Big(\cos(\theta) - \sin(\theta)I\mathbf{n}\Big)\mathbf{v}_{\perp}\Big) \\
&= \Big(\cos^2(\theta) + \sin^2(\theta)\Big)\mathbf{v}_{\|} + \Big(\cos^2(\theta) - \sin^2(\theta) - 2\sin(\theta)\cos(\theta)I\mathbf{n}\Big)\mathbf{v}_{\perp} \\
&= \mathbf{v}_{\|} + \Big(\cos(2\theta) - \sin(2\theta)I\mathbf{n}\Big)\mathbf{v}_{\perp} \\
&= \mathbf{v}_{\|} + \Big(\cos(2\theta)\mathbf{v}_{\perp} - \sin(2\theta)I(\mathbf{n}\wedge\mathbf{v}_{\perp})\Big) \\
&= \mathbf{v}_{\|} + \Big(\cos(2\theta)\mathbf{v}_{\perp} + \sin(2\theta)\star(\mathbf{n}\wedge\mathbf{v}_{\perp})\Big). \qquad (2.193)
\end{aligned}$$

This in fact describes a rotation of the vector $\mathbf{v}$ through the plane orthogonal to the vector $\mathbf{n}$ in the counterclockwise sense, since the component $\mathbf{v}_{\|}$ is unchanged and the component $\mathbf{v}_{\perp}$ transforms into $\cos(2\theta)\mathbf{v}_{\perp} + \sin(2\theta)\star(\mathbf{n}\wedge\mathbf{v}_{\perp})$, which is $\mathbf{v}_{\perp}$ rotated by the angle 2θ in the counterclockwise sense of the plane determined by $\mathbf{n}$ (following the right-hand convention). Since the rotation described is by an angle 2θ, the corresponding rotation by an angle θ is given by

$$\mathbf{v} \mapsto \mathbf{v}' = R\mathbf{v}\widetilde{R}, \qquad (2.194)$$

where

$$R = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right) I\mathbf{n}. \qquad (2.195)$$

This rotor can be written in terms of the exponential map (in the same way as made to reach to the expression (2.76), since $B = -I\mathbf{n}$ is a unit bivector, like $\mathbf{e}_2\mathbf{e}_1$) as follows:

$$R = \exp\left(-\frac{1}{2}\theta I\mathbf{n}\right). \qquad (2.196)$$

The rotation transformation considered above can be extended for any multivector A as follows:

$$A \mapsto A' = RA\widetilde{R} = RAR^{-1}. \qquad (2.197)$$

A bivector $A = \mathbf{u}_1 \wedge \mathbf{u}_2 = \mathbf{u}_1\mathbf{u}_2$, for example, transforms under a rotation in the way

$$A \mapsto A' = RA\widetilde{R} = R\mathbf{u}_1\mathbf{u}_2\widetilde{R} = R\mathbf{u}_1\widetilde{R}R\mathbf{u}_2\widetilde{R} = \mathbf{u}_1{}'\mathbf{u}_2{}', \qquad (2.198)$$

in such way that A' describes the plane given by A rotated in the direction of the plane given by $I\mathbf{n}$ in the counterclockwise sense (i.e. following the right-hand convention relative the normal vector $\mathbf{n}$).

Note that the set of the rotors of $\mathcal{C}\ell_{3,0}$ can be characterized as

$$\left\{ R \;\middle|\; R \in \mathcal{C}\ell_{3,0}{}^{+} \text{ and } \widetilde{R}R = R\widetilde{R} = 1 \right\}, \tag{2.199}$$

that is, the set of even grade multivectors of $\mathcal{C}\ell_{3,0}$ with unit norm. Note then that, the set of rotors endowed with the geometric product has the structure of a group. Indeed, given the rotors R_1 e R_2, it follows that $\widetilde{(R_1R_2)}(R_1R_2) = \widetilde{R_2}\widetilde{R_1}R_1R_2 = \widetilde{R}_2R_2 = 1$, that is, R_1R_2 is also a rotor, hence (i) the set of rotors is closed with relation to the geometric product; in addition, (ii) the geometric product is known to be associative, (iii) there exists an neutral element with relation to the geometric product (the number 1), and, (iv) for any rotor R there exists the inverse, given by $R^{-1} = \widetilde{R}$. This group is denoted by Spin(3), and a rotor of $\mathcal{C}\ell_{3,0}$ can be characterized as an element of this group.

It should be noted that, as in the two-dimensional case, both R and $-R$ describe the same rotation:

$$(-R)\mathbf{v}\widetilde{(-R)} = R\mathbf{v}\widetilde{R}. \tag{2.200}$$

This can be understood by observing that the rotation by an angle ϕ in a given plane, in the counterclockwise sense, has the same result as the rotation by the angle $2\pi - \phi$ in the same plane, but in the clockwise sense. Indeed, given the rotors $R = \exp(-I\mathbf{n}\phi/2)$ and $R^* = \exp\big(I\mathbf{n}(2\pi - \phi)/2\big)$, it follows that

$$R^* = \exp\big(I\mathbf{n}(2\pi - \phi)/2\big) = \exp(I\mathbf{n}\pi)\exp(-I\mathbf{n}\phi/2) = (-1)R = -R. \tag{2.201}$$

The fact that R and $-R$ describe the same rotation implies in a two-to-one correspondence between the group Spin(3) and the group SO(3) (i.e. there are two rotors equivalent to a same special orthogonal transformation in the three-dimensional Euclidean space), in the same way that there is a two-to-one correspondence between SU(2) and SO(3), when it is said that SU(2) is a double covering of SO(3). In the same way, it is said that Spin(3) is a double covering of SO(3), and it is found that Spin(3) is isomorphic to SU(2).

Similarly to the two-dimensional case, an arbitrary element of the even subalgebra $\mathcal{C}\ell_{3,0}{}^{+}$ can be written in the form

$$\psi = \sqrt{\rho}R, \tag{2.202}$$

where ρ is a real scalar and R is a rotor. In this way, the transformation

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = \psi\mathbf{v}\widetilde{\psi} = \rho R\mathbf{v}\widetilde{R} \tag{2.203}$$

corresponds to a rotation given by the rotor R and a dilation/contraction (if $0 < \rho < 1$ or $\rho > 1$, respectively) of the vector $\mathbf{v}$.

2.3 The Geometric Algebra of Minkowski Spacetime

The concept of a pseudo-Euclidean space is introduced in this section by presenting the two-dimensional case, accompanied by comparisons with the Euclidean plane. Next, the geometric algebra for the pseudo-Euclidean plane is presented. Then, after an introduction to Minkowski spacetime, the corresponding geometric algebra is introduced and some of its basic properties studied.

2.3.1 Pseudo-Euclidean Spaces

Consider the vector space $\mathbb{R}^2$, and let its vectors be denoted by Latin letters in boldface: $\mathbf{u}$, $\mathbf{v}$, etc. Let the canonical basis be denoted $\{\mathbf{e}_1, \mathbf{e}_2\} = \{(1,0),(0,1)\}$ (where the ordering of the basis is implied), in such way that a vector is written, generally, $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$, etc. The interpretation for this space is the usual geometric interpretation: $\mathbb{R}^2$ corresponds to the plane, and its vectors represent oriented line segments in the plane.

Consider the symmetric bilinear form $h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$h(\mathbf{e}_1, \mathbf{e}_1) = -h(\mathbf{e}_2, \mathbf{e}_2) = 1 \quad \text{and} \quad h(\mathbf{e}_1, \mathbf{e}_2) = h(\mathbf{e}_2, \mathbf{e}_1) = 0. \tag{2.204}$$

The calculation of $h(\mathbf{u}, \mathbf{v})$, for two arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$, furnishes

$$\begin{aligned} h(\mathbf{u}, \mathbf{v}) &= h(u_1\mathbf{e}_1 + u_2\mathbf{e}_2, v_1\mathbf{e}_1 + v_2\mathbf{e}_2) \\ &= u_1v_1h(\mathbf{e}_1, \mathbf{e}_1) + u_1v_2h(\mathbf{e}_1, \mathbf{e}_2) + u_2v_1h(\mathbf{e}_2, \mathbf{e}_1) + u_2v_2h(\mathbf{e}_2, \mathbf{e}_2) \\ &= u_1v_1 - u_2v_2. \end{aligned} \tag{2.205}$$

In particular,

$$h(\mathbf{u}, \mathbf{u}) = {u_1}^2 - {u_2}^2. \tag{2.206}$$

Although h is symmetric, the above equation allows one to observe that it is not positive-definite, that is, $h(\mathbf{u}, \mathbf{u})$ can assume any real value, including zero, without $\mathbf{u}$ necessarily being the null vector. Nevertheless, $\mathbb{R}^2$ endowed with the symmetric bilinear form h it is important in mathematics and physics, determining a particular case of a *pseudo-Euclidean space*, the *pseudo-Euclidean plane*. For this space, one can define the "norm" induced by the symmetric bilinear form h by

$$\|\mathbf{u}\|_h{}^2 = h(\mathbf{u}, \mathbf{u}) = {u_1}^2 - {u_2}^2. \tag{2.207}$$

which will be called *pseudo-norm* (because it is not in fact a norm). In order to present some aspects of the pseudo-Euclidean plane, its analogues in the Euclidean plane will be recalled first.

Let g be the symmetric bilinear form given by (2.1), and let its induced norm be denoted by $\|\cdot\|_g$. Given the vector $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ of the Euclidean plane, the equation

$$\|\mathbf{x}\|_g{}^2 = g(\mathbf{x}, \mathbf{x}) = r^2, \tag{2.208}$$

which can be written in terms of components as

$$x_1{}^2 + x_2{}^2 = r^2 \quad \text{or} \quad \left(\frac{x_1}{r}\right)^2 + \left(\frac{x_2}{r}\right)^2 = 1, \tag{2.209}$$

describes a circle of radius $|r|$ centered at the origin. Such a circle can be parameterized by the angle θ that the position vector $\mathbf{x}$ on the circle makes with the axis of abscissas, as follows:

$$x_1 = r\cos(\theta) \quad \text{and} \quad x_2 = r\sin(\theta). \tag{2.210}$$

Then, from the equation of the circle one obtains the fundamental identity

$$\cos^2(\theta) + \sin^2(\theta) = 1. \tag{2.211}$$

The above mentioned parameterization allows one to express the vector $\mathbf{x}$ by

$$\mathbf{x} = r\big(\cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2\big). \tag{2.212}$$

With respect to the sine and cosine functions, it is appropriate to mention Euler's formula,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \tag{2.213}$$

which allows the cosine and sine functions to be written in terms of complex exponentials as follows:

$$\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \quad \text{and} \quad \sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right). \tag{2.214}$$

As a final remark, recall that another vector $\mathbf{x}' = x_1{}'\mathbf{e}_1 + x_2{}'\mathbf{e}_2$ on the circle, obtained from the vector $\mathbf{x}$ through a rotation by an angle $\Delta\theta$, is such that

$$\begin{pmatrix} x_1{}' \\ x_2{}' \end{pmatrix} = \begin{pmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.215}$$

Now, analogous aspects in the case of the pseudo-Euclidean plane will be considered.

Given the vector $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ of the pseudo-Euclidean plane, the equation

$$\|\mathbf{x}\|_h{}^2 = h(\mathbf{x}, \mathbf{x}) = \text{constant} \tag{2.216}$$

can describe different geometric shapes depending on whether the constant that appears in it is (I) positive, (II) negative or (III) null. Consider first the case I, where the equation can be written in the form

$$\|\mathbf{x}\|_h{}^2 = h(\mathbf{x}, \mathbf{x}) = r^2, \tag{2.217}$$

for some non-null real number r, or more explicitly, in terms of the components of $\mathbf{x}$:

$$x_1{}^2 - x_2{}^2 = r^2 \quad \text{or} \quad \left(\frac{x_1}{r}\right)^2 - \left(\frac{x_2}{r}\right)^2 = 1. \tag{2.218}$$

This is the equation of the equilateral hyperbola with vertices at $(-r, 0)$ and $(r, 0)$. In the case II, the equation takes the form

$$\|\mathbf{x}\|_h{}^2 = h(\mathbf{x}, \mathbf{x}) = -r^2, \tag{2.219}$$

which in terms of the components of the vector $\mathbf{x}$, can be written as

$$x_1{}^2 - x_2{}^2 = -r^2 \quad \text{or} \quad \left(\frac{x_2}{r}\right)^2 - \left(\frac{x_1}{r}\right)^2 = 1. \tag{2.220}$$

This is the equation of the equilateral hyperbola with vertices at $(0, -r)$ e $(0, r)$. About case III, one has the equation

$$\|\mathbf{x}\|_h{}^2 = h(\mathbf{x}, \mathbf{x}) = 0, \tag{2.221}$$

which in terms of the components of the vector $\mathbf{x}$ is written as

$$x_1{}^2 - x_2{}^2 = 0 \quad \text{or} \quad x_1 = \pm x_2. \tag{2.222}$$

These equations describe the asymptotes of the hyperbolas considered above. Figure 2.4 shows the graphs of the geometric figures considered in each case.

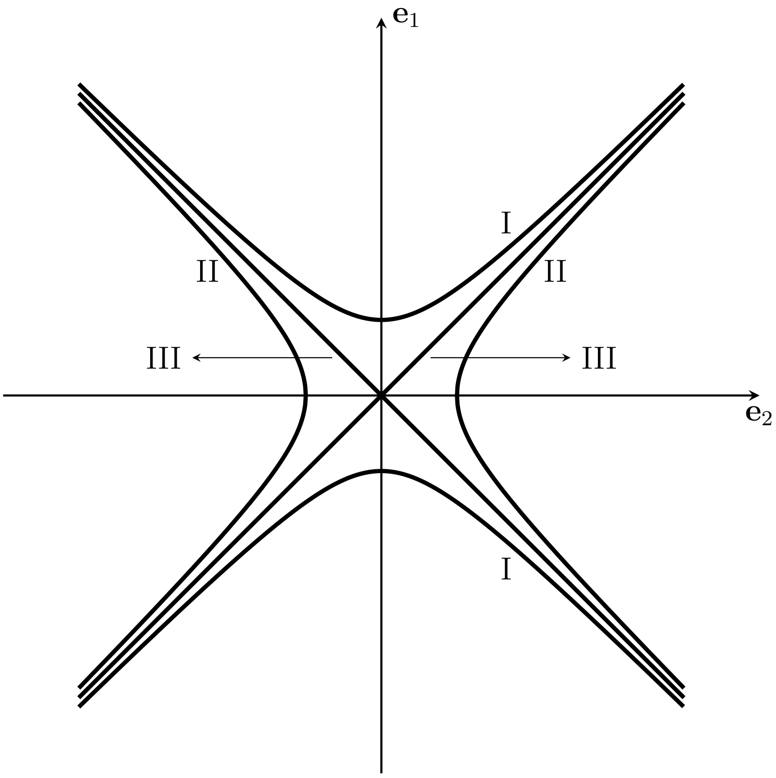

FIGURE 2.4 – Curves considered in cases I, II and III in the text (the axis of abscissas is taken vertically and the axis of ordinates horizontally, placing the "other side" of the plane in perspective).

Just as a circle can be parameterized by an angle, a hyperbola branch can be parameterized by a quantity called a *hyperbolic angle*, which does not consist of an angle in the usual sense. The hyperbolic angle α can be understood as the argument of the hyperbolic cosine and sine functions, which can be defined respectively by

$$\cosh(\alpha) = \frac{1}{2}\left(e^{\alpha} + e^{-\alpha}\right) \quad \text{and} \quad \sinh(\alpha) = \frac{1}{2}\left(e^{\alpha} - e^{-\alpha}\right). \tag{2.223}$$

Note that such functions satisfy

$$e^{\alpha} = \cosh(\alpha) + \sinh(\alpha) \tag{2.224}$$

and

$$\cosh^2(\alpha) - \sinh^2(\alpha) = 1. \tag{2.225}$$

The above three relations has as its analogues in the case of "circular geometry" the relations (2.214), (2.213) and (2.211), respectively. The parameterization of the upper branch of the hyperbola I is given by means of the relations

$$\cosh(\alpha) = \frac{v_1}{r} \quad \text{and} \quad \sinh(\alpha) = \frac{v_2}{r}, \tag{2.226}$$

which allow one to write a generic position vector $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$ on the branch of hyperbola as

$$\mathbf{v} = r\big(\cosh(\alpha)\mathbf{e}_1 + \sinh(\alpha)\mathbf{e}_2\big). \tag{2.227}$$

Another position vector $\mathbf{v}' = v_1{}'\mathbf{e}_1 + v_2{}'\mathbf{e}_2$ on the considered branch of hyperbola can be obtained from $\mathbf{v}$ through a *hyperbolic rotation* through a certain hyperbolic angle $\Delta\alpha$, according to

$$\begin{pmatrix} v_1{}' \\ v_2{}' \end{pmatrix} = \begin{pmatrix} \cosh(\Delta\alpha) & \sinh(\Delta\alpha) \\ \sinh(\Delta\alpha) & \cosh(\Delta\alpha) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.228}$$

Such a hyperbolic rotation, for a positive hyperbolic angle, is illustrated in the figure 2.5.

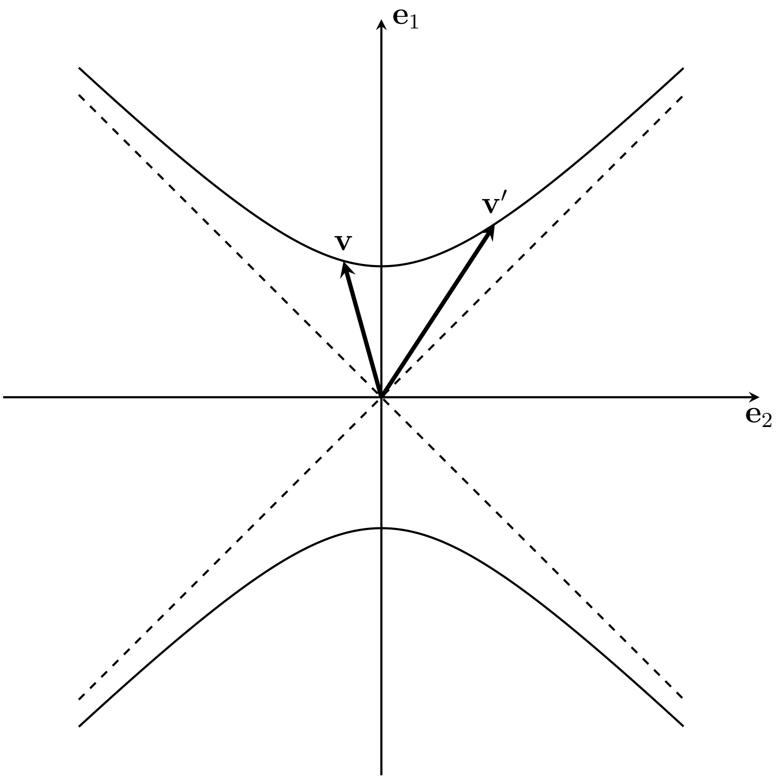

FIGURE 2.5 – A hyperbolic rotation of the vector $\mathbf{v}$ on the upper branch of a hyperbola of type I, by a positive hyperbolic angle, resulting in the vector $\mathbf{v}'$.

The fact that the symmetric bilinear form h is not positive-definite implies that there exists not just one, but an infinity of vectors $\mathbf{u}$ such that $h(\mathbf{u},\mathbf{u}) = 0$, all of which are *null vectors*, but not in the sense that $g(\mathbf{u},\mathbf{u}) = 0$, but rather with respect to the form h. According to the value of $h(\mathbf{u},\mathbf{u})$, it is also possible to classify a vector $\mathbf{u}$ into two other types: (1) $\mathbf{u}$ such that $h(\mathbf{u},\mathbf{u}) > 0$, and (2) $\mathbf{u}$ such that $h(\mathbf{u},\mathbf{u}) < 0$. According to the definition of the form h (cf. equation (2.204)) the basic vector $\mathbf{e}_1$ is of type 1, and the basic vector $\mathbf{e}_2$ is of type 2. This feature implies that the two components of a vector in the pseudo-Euclidean plane have a distinct nature, such that each of the two subspaces resulting from an orthogonal decomposition have a distinct nature with respect to the form h. In view of this fact, the pseudo-Euclidean plane is denoted by $\mathbb{R}^{1,1}$. It should be noted that a generic pseudo-Euclidean space is construct in a similar way to the pseudo-Euclidean plane, being denoted by $\mathbb{R}^{m,n}$, in such way that its canonical basis contains m vectors of type 1 and n vectors of type 2. In agreement with the physical context, vectors of type 1 are called *time-like* vectors, vectors of type 2 are called *space-like* vectors, and null vectors are also called *light-like* vectors.

2.3.2 The Geometric Algebra of the Pseudo-Euclidean Plane

As in the constructions made earlier, the *geometric algebra of the pseudo-Euclidean plane* is determined by the multivector space constructed from $\mathbb{R}^{1,1}$ endowed with the geometric product. In this case, the fundamental property of the geometric product is given by

$$\mathbf{u}\mathbf{u} = h(\mathbf{u}, \mathbf{u}), \tag{2.229}$$

for any vector $\mathbf{u}$ from $\mathbb{R}^{1,1}$, or, in terms of the pseudo-norm and using the notation $\mathbf{u}^2 = \mathbf{u}\mathbf{u}$,

$$\mathbf{u}^2 = \|\mathbf{u}\|_h{}^2. \tag{2.230}$$

This expression can be written in terms of components as follows:

$$(u_1\mathbf{e}_1 + u_2\mathbf{e}_2)(u_1\mathbf{e}_1 + u_2\mathbf{e}_2) = {u_1}^2 - {u_2}^2. \tag{2.231}$$

Imposing bilinearity to the geometric product, one can write

$${u_1}^2{\mathbf{e}_1}^2 + u_1u_2(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) + {u_2}^2{\mathbf{e}_2}^2 = {u_1}^2 - {u_2}^2, \tag{2.232}$$

which implies

$${\mathbf{e}_1}^2 = 1, \quad {\mathbf{e}_2}^2 = -1, \quad \text{and} \quad \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1. \tag{2.233}$$

These are the basic relations for calculation of the geometric product in the geometric algebra of the pseudo-Euclidean plane in terms of the canonical basic vectors. Applying it to the calculation of the geometric product of two arbitrary vectors $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ and $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$, one obtains

$$\begin{aligned}\mathbf{u}\mathbf{v} &= (u_1\mathbf{e}_1 + u_2\mathbf{e}_2)(v_1\mathbf{e}_1 + v_2\mathbf{e}_2) \\ &= u_1v_1{\mathbf{e}_1}^2 + u_1v_2\mathbf{e}_1\mathbf{e}_2 + u_2v_1\mathbf{e}_2\mathbf{e}_1 + u_2v_2{\mathbf{e}_2}^2 \\ &= (u_1v_1 - u_2v_2) + (u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2,\end{aligned} \tag{2.234}$$

which corresponds to the sum of a symmetric part, $(u_1v_1 - u_2v_2)$, and an antisymmetric part, $(u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2$, with relation to the exchange of $\mathbf{u}$ and $\mathbf{v}$. Since the geometric product of two vectors can be uniquely written in the form

$$\mathbf{uv} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu}) + \frac{1}{2}(\mathbf{uv} - \mathbf{vu}), \tag{2.235}$$

where the first term is symmetric and the second antisymmetric under the exchange of $\mathbf{u}$ and $\mathbf{v}$, one can write

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}, \tag{2.236}$$

where are defined the *scalar product* and the *exterior product*, respectively, by

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu}) = u_1v_1 - u_2v_2 = h(\mathbf{u}, \mathbf{v}) \tag{2.237}$$

and

$$\mathbf{u} \wedge \mathbf{v} = \frac{1}{2}(\mathbf{uv} - \mathbf{vu}) = (u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2. \tag{2.238}$$

As in the case of the Euclidean plane, the objects of the form $\mathbf{u} \wedge \mathbf{v}$, such as $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2$, are defined as bivectors and interpreted as oriented parallelograms. This is independent of the metric properties of the space, determined in this case by the form h. Although in the pseudo-Euclidean case $(\mathbf{e}_1\mathbf{e}_2)^2 = 1$, whereas in the Euclidean case $(\mathbf{e}_1\mathbf{e}_2)^2 = -1$, which leads to different metric relations for the pseudo-Euclidean case, there is no change in the underlying multivector structure of the algebra under consideration, and consequently the underlying exterior algebra is the same as for the geometric algebra of the Euclidean plane. In agreement to the notation $\mathbb{R}^{1,1}$ for the pseudo-Euclidean plane, the vector space of real scalars can now be denoted by $\bigwedge^0(\mathbb{R}^{1,1})$, the vector space of vectors of the pseudo-Euclidean plane can also be denoted by $\bigwedge^1(\mathbb{R}^{1,1})$, and the vector space of bivectors can now be denoted by $\bigwedge^2(\mathbb{R}^{1,1})$. Thus, one can define the vector space

$$\bigwedge\left(\mathbb{R}^{1,1}\right) = \bigoplus_{k=0}^{2} \bigwedge^k\left(\mathbb{R}^{1,1}\right) = \bigwedge^0\left(\mathbb{R}^{1,1}\right) \oplus \bigwedge^1\left(\mathbb{R}^{1,1}\right) \oplus \bigwedge^2\left(\mathbb{R}^{1,1}\right), \tag{2.239}$$

whose elements, also called multivectors, can be written in terms of the basic vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ under the form

$$A = a + (a_1\mathbf{e}_1 + a_2\mathbf{e}_2) + a_{12}\mathbf{e}_1\mathbf{e}_2. \tag{2.240}$$

Defining then the geometric product of a scalar with a multivector as the multiplication of the multivector by the scalar, and extending the geometric product to any multivectors by bilinearity and associativity, it follows that the vector space $\bigwedge(\mathbb{R}^{1,1})$ endowed with the geometric product determines an associative algebra over the field of real scalars, the *geometric algebra of the pseudo-Euclidean plane*, or the *Clifford algebra of the pseudo-Euclidean plane*, which can be denoted by $\mathcal{C}\ell(\mathbb{R}^{1,1}, h)$, or $\mathcal{C}\ell_{1,1}(\mathbb{R})$, or simply $\mathcal{C}\ell_{1,1}$.

Projection, Graded Involution, Reversion, the Norm and the Inverse

The operations of projection, graded involution and reversion are defined in the same way as for $\mathcal{C}\ell_{2,0}$ (cf. subsection 2.1.2). The pseudo-norm of a multivector A from $\mathcal{C}\ell_{1,1}$ is appropriately defined by

$$\|A\|_h{}^2 = \left\langle \widetilde{A}A \right\rangle = \left\langle A\widetilde{A} \right\rangle, \tag{2.241}$$

and the inverse can be defined by

$$A^{-1} = \frac{\widetilde{A}}{\|A\|_h{}^2}, \tag{2.242}$$

provided that

$$\|A\|_h{}^2 = \left\langle \widetilde{A}A \right\rangle = \widetilde{A}A \neq 0. \tag{2.243}$$

Note the difference in the conditions for the existence of the inverse (cf. relations (2.38)).

Inequalities, Parallelism and Orthogonality

For a bivector $B = a_{12}\mathbf{e}_1\mathbf{e}_2$ from $\mathcal{C}\ell_{1,1}$, one has

$$\|B\|_h{}^2 = \left\langle B\widetilde{B} \right\rangle = \langle (a_{12}\mathbf{e}_1\mathbf{e}_2)(a_{12}\mathbf{e}_2\mathbf{e}_1)\rangle = \left\langle a_{12}{}^2\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 \right\rangle = -a_{12}{}^2 \leq 0. \tag{2.244}$$

But note that the same calculation to obtain the equation (2.43), in the case of the Euclidean plane, can be made to find

$$\|\mathbf{u}\wedge\mathbf{v}\|_h{}^2 = \|\mathbf{u}\|_h{}^2\|\mathbf{v}\|_h{}^2 - (\mathbf{u}\cdot\mathbf{v})^2, \tag{2.245}$$

for two arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$ of the pseudo-Euclidean plane, so that $\|\mathbf{u}\wedge\mathbf{v}\|_h{}^2 \leq 0$ implies

$$(\mathbf{u}\cdot\mathbf{v})^2 \geq \|\mathbf{u}\|_h{}^2\|\mathbf{v}\|_h{}^2, \tag{2.246}$$

which is the analog of the Cauchy-Schwarz inequality for vectors of the pseudo-Euclidean plane (note the difference in relation to the original inequality, given by (2.44)). In this way, if $\mathbf{u}$ and $\mathbf{v}$ are time-like vectors with time-like coordinate of the same sign (which is expressed in special relativity by saying that both vectors are directed either to the future or to the past), one has

$$\mathbf{u}\cdot\mathbf{v} \geq \|\mathbf{u}\|_h\|\mathbf{v}\|_h, \tag{2.247}$$

so that

$$\|\mathbf{u}+\mathbf{v}\|_h{}^2 = \|\mathbf{u}\|_h{}^2 + \|\mathbf{v}\|_h{}^2 + 2(\mathbf{u}\cdot\mathbf{v}) \geq \|\mathbf{u}\|_h{}^2 + \|\mathbf{v}\|_h{}^2 + 2\|\mathbf{u}\|_h\|\mathbf{v}\|_h = (\|\mathbf{u}\|_h + \|\mathbf{v}\|_h)^2, \tag{2.248}$$

which implies

$$\|\mathbf{u}+\mathbf{v}\|_h \geq \|\mathbf{u}\|_h + \|\mathbf{v}\|_h. \tag{2.249}$$

This is the triangular inequality for time-like vectors of the pseudo-Euclidean plane whose time-like coordinate has the same sign (note the difference in relation to (2.51)). The hyperbolic angle α between these two vectors is such that

$$\cosh(\alpha) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|_h\|\mathbf{v}\|_h} \tag{2.250}$$

and

$$\sinh(\alpha) = \frac{\sqrt{-\|\mathbf{u}\wedge\mathbf{v}\|_h{}^2}}{\|\mathbf{u}\|_h\|\mathbf{v}\|_h}. \tag{2.251}$$

This can be justified by writing

$$\mathbf{u} = u(\cosh(\beta)\mathbf{e}_1 + \sinh(\beta)\mathbf{e}_2) \quad \text{and} \quad \mathbf{v} = v(\cosh(\gamma)\mathbf{e}_1 + \sinh(\gamma)\mathbf{e}_2), \tag{2.252}$$

so that

$$\mathbf{u}\cdot\mathbf{v} = uv\cosh(\beta)\cosh(\gamma) - uv\sinh(\beta)\sinh(\gamma) = uv\cosh(\gamma-\beta) \tag{2.253}$$

and

$$\mathbf{u}\wedge\mathbf{v} = uv\cosh(\beta)\sinh(\gamma)\mathbf{e}_1\mathbf{e}_2 - uv\sinh(\beta)\cosh(\gamma)\mathbf{e}_1\mathbf{e}_2 = uv\sinh(\gamma-\beta)\mathbf{e}_1\mathbf{e}_2. \tag{2.254}$$

Note then that the conditions for parallelism and orthogonality for such vectors are the same as for vectors of the Euclidean plane, (2.52) and (2.53), where the products must be reconsidered according to the pseudo-Euclidean case.

Reflections and Rotations

The fact that the form of the conditions for parallelism and orthogonality are preserved in the pseudo-Euclidean case, although the scalar product (hence the geometric product) is different, implies that the expression for the reflection transformation has the same form. Thus, the reflection of a vector $\mathbf{v}$ through a line with orthogonal vector $\mathbf{u}$ is also given by

$$\mathbf{v} \mapsto \mathbf{v}' = -\mathbf{u}\mathbf{v}\mathbf{u}^{-1}. \tag{2.255}$$

However, there are two cases to consider: $\mathbf{u}^2 = 1$, which implies $\mathbf{v}' = -\mathbf{u}\mathbf{v}\mathbf{u}$, and $\mathbf{u}^2 = -1$, which implies $\mathbf{v}' = \mathbf{u}\mathbf{v}\mathbf{u}$. In the first case the vector $\mathbf{u}$ is time-like and the reflection is through a space-like line, that is, a line with space-like parallel vector. In this case, since

$\mathbf{u}^2 = \|\mathbf{u}\|_h{}^2 = 1$, one can write

$$\mathbf{u} = \cosh(\beta)\mathbf{e}_1 + \sinh(\beta)\mathbf{e}_2, \tag{2.256}$$

and, with $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$, the reflected vector $\mathbf{v}' = -\mathbf{uvu}$ is given by

$$\begin{aligned}
\mathbf{v}' &= -\mathbf{uvu} \\
&= -\Big(\cosh(\beta)\mathbf{e}_1 + \sinh(\beta)\mathbf{e}_2\Big)(v_1\mathbf{e}_1 + v_2\mathbf{e}_2)\Big(\cosh(\beta)\mathbf{e}_1 + \sinh(\beta)\mathbf{e}_2\Big) \\
&= -\Big(\Big(\cosh^2(\beta)v_1 + \sinh^2(\beta)v_1 - 2\sinh(\beta)\cosh(\beta)v_2\Big)\mathbf{e}_1 + \\
&\qquad + \Big(2\sinh(\beta)\cosh(\beta)v_1 - \cosh^2(\beta)v_2 - \sinh^2(\beta)v_2\Big)\mathbf{e}_2\Big) \\
&= \Big(\sinh(2\beta)v_2 - \cosh(2\beta)v_1\Big)\mathbf{e}_1 + \Big(\cosh(2\beta)v_2 - \sinh(2\beta)v_1\Big)\mathbf{e}_2.
\end{aligned} \tag{2.257}$$

Note that, in particular, if $\mathbf{u} = \mathbf{e}_1$, then $\beta = 0$, hence

$$\mathbf{v}' = -v_1\mathbf{e}_1 + v_2\mathbf{e}_2, \tag{2.258}$$

which corresponds to the vector $\mathbf{v}$ with the time-like component inverted. In the case that $\mathbf{u}$ is space-like, the reflection of $\mathbf{v}$ is through a time-like line, that is, a line with time-like parallel vector. In this case, where $\mathbf{u}^2 = \|\mathbf{u}\|_h{}^2 = -1$, one can write

$$\mathbf{u} = \sinh(\beta)\mathbf{e}_1 + \cosh(\beta)\mathbf{e}_2. \tag{2.259}$$

The reflected vector $\mathbf{v}' = \mathbf{uvu}$ is then given by

$$\begin{aligned}
\mathbf{v}' &= \mathbf{uvu} \\
&= \Big(\sinh(\beta)\mathbf{e}_1 + \cosh(\beta)\mathbf{e}_2\Big)(v_1\mathbf{e}_1 + v_2\mathbf{e}_2)\Big(\sinh(\beta)\mathbf{e}_1 + \cosh(\beta)\mathbf{e}_2\Big) \\
&= \Big(\Big(\sinh^2(\beta)v_1 + \cosh^2(\beta)v_1 - 2\sinh(\beta)\cosh(\beta)v_2\Big)\mathbf{e}_1 + \\
&\qquad + \Big(2\sinh(\beta)\cosh(\beta)v_1 - \cosh^2(\beta)v_2 - \sinh^2(\beta)v_2\Big)\mathbf{e}_2\Big) \\
&= \Big(\cosh(2\beta)v_1 - \sinh(2\beta)v_2\Big)\mathbf{e}_1 + \Big(\sinh(2\beta)v_1 - \cosh(2\beta)v_2\Big)\mathbf{e}_2,
\end{aligned} \tag{2.260}$$

In particular, if $\mathbf{u} = \mathbf{e}_2$, then $\beta = 0$, hence

$$\mathbf{v}' = v_1\mathbf{e}_1 - v_2\mathbf{e}_2, \tag{2.261}$$

which corresponds to the vector $\mathbf{v}$ with the space-like component inverted.

Similarly to rotations in the Euclidean plane, a hyperbolic rotation of a vector from the pseudo-Euclidean plane can be described as a composition of two reflections. It is found that a hyperbolic rotation is given by

$$\mathbf{v} \ \mapsto \ \mathbf{v}' = L\mathbf{v}L^{-1} = L\mathbf{v}\widetilde{L}, \tag{2.262}$$

where $L = \mathbf{u}_1\mathbf{u}_2$, being $\mathbf{u}_1$ and $\mathbf{u}_2$ vectors such that ${\mathbf{u}_1}^2 = {\mathbf{u}_2}^2 = 1$. The object L is called a *rotor* and can be written as

$$L = \mathbf{u}_1\mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_2 + \mathbf{u}_1 \wedge \mathbf{u}_2, \tag{2.263}$$

so that, if α is the hyperbolic angle between $\mathbf{u}_1$ and $\mathbf{u}_2$, a direct calculation furnishes (cf. equations (2.253) and (2.254))

$$L = \cosh(\alpha) + \sinh(\alpha)\mathbf{e}_1\mathbf{e}_2. \tag{2.264}$$

Analogously to the case of rotations in the Euclidean plane, it is observed that this rotor describes a hyperbolic rotation in the sense of decreasing hyperbolic angle, in such way that the choice of the bivector $\mathbf{e}_2\mathbf{e}_1$ in place of $\mathbf{e}_1\mathbf{e}_2$ is the appropriated one for the rotor L to describe a hyperbolic rotation in the sense of increasing hyperbolic angle. Indeed, given the rotor

$$L = \cosh(\alpha) + \sinh(\alpha)\mathbf{e}_2\mathbf{e}_1 \tag{2.265}$$

and the vector $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$, the rotated vector $\mathbf{v}' = L\mathbf{v}\widetilde{L}$ is given by

$$\begin{aligned}
\mathbf{v}' &= \Big(\cosh(\alpha) + \sinh(\alpha)\mathbf{e}_2\mathbf{e}_1\Big)(v_1\mathbf{e}_1 + v_2\mathbf{e}_2)\Big(\cosh(\alpha) - \sinh(\alpha)\mathbf{e}_2\mathbf{e}_1\Big) \\
&= \Big(\Big(\cosh^2(\alpha) + \sinh^2(\alpha)\Big)v_1 + \Big(2\sinh(\alpha)\cosh(\alpha)\Big)v_2\Big)\mathbf{e}_1 + \\
&\qquad\qquad + \Big(\Big(2\sinh(\alpha)\cosh(\alpha)\Big)v_1 + \Big(\cosh^2(\alpha) + \sinh^2(\alpha)\Big)v_2\Big)\mathbf{e}_2 \\
&= \Big(\cosh(2\alpha)v_1 + \sinh(2\alpha)v_2\Big)\mathbf{e}_1 + \Big(\sinh(2\alpha)v_1 + \cosh(2\alpha)v_2\Big)\mathbf{e}_2.
\end{aligned} \tag{2.266}$$

This result can be expressed in matrix form as

$$\begin{pmatrix} {v_1}' \\ {v_2}' \end{pmatrix} = \begin{pmatrix} \cosh(2\alpha) & \sinh(2\alpha) \\ \sinh(2\alpha) & \cosh(2\alpha) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{2.267}$$

which in fact represents a hyperbolic rotation of the vector $\mathbf{v}$ by 2α, in the sense of increasing hyperbolic angle. In this way, a hyperbolic rotation by a hyperbolic angle α in the sense of increasing hyperbolic angle is described by the rotor

$$L = \cosh\left(\frac{\alpha}{2}\right) + \sinh\left(\frac{\alpha}{2}\right)\mathbf{e}_2\mathbf{e}_1. \tag{2.268}$$

Using the expressions as power series for the hyperbolic cosine and sine functions in the above expression, and taking into account that $(\mathbf{e}_2\mathbf{e}_1)^{2n} = 1$ and $(\mathbf{e}_2\mathbf{e}_1)^{2n+1} = \mathbf{e}_2\mathbf{e}_1$ for any non-negative integer n, one obtains

$$\begin{aligned} L &= \sum_{n=0}^{\infty} \frac{(\alpha/2)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\alpha/2)^{2n+1}}{(2n+1)!}\mathbf{e}_2\mathbf{e}_1 \\ &= \sum_{n=0}^{\infty} \frac{(\mathbf{e}_2\mathbf{e}_1\alpha/2)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\mathbf{e}_2\mathbf{e}_1\alpha/2)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(\mathbf{e}_2\mathbf{e}_1\alpha/2)^{n}}{n!}, \end{aligned} \tag{2.269}$$

which can be written in terms of the exponential map as

$$L = \exp\left(\frac{1}{2}\alpha\mathbf{e}_2\mathbf{e}_1\right). \tag{2.270}$$

In order to better illustrate a hyperbolic rotation, consider the rotated basis $\{\mathbf{e}_1{}', \mathbf{e}_2{}'\}$, given by $\mathbf{e}_i{}' = L\mathbf{e}_i\widetilde{L}$, for $i \in \{1, 2\}$, where L is the rotor given by the above expression. It follows that

$$\mathbf{e}_1{}' = L\mathbf{e}_1\widetilde{L} = \cosh(\alpha)\mathbf{e}_1 + \sinh(\alpha)\mathbf{e}_2 \tag{2.271}$$

and

$$\mathbf{e}_2{}' = L\mathbf{e}_2\widetilde{L} = \sinh(\alpha)\mathbf{e}_1 + \cosh(\alpha)\mathbf{e}_2. \tag{2.272}$$

For a given α, an analysis of the orientation of the rotated basic vectors above, consisting in the analysis of the functional behavior of their components, falls into the analysis of the behavior of the hyperbolic sine and cosine functions, which allows one to observe that a hyperbolic rotation by α of the basic vectors is performed as illustrated in figure 2.6, if $\alpha > 0$, and is performed as illustrated in figure 2.7, if $\alpha < 0$. Indeed: $\alpha > 0$ implies $\cosh(\alpha) > 1$ and $\sinh(\alpha) > 0$; $\alpha < 0$ implies $\cosh(\alpha) > 1$ and $\sinh(\alpha) < 0$; furthermore, $\cosh(\alpha) > \sinh(\alpha)$ for any α, and $\lim_{\alpha\to\infty} |\cosh(\alpha) - \sinh(\alpha)| = 0$.

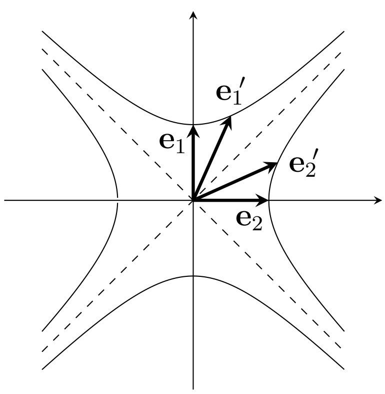

FIGURE 2.6 – Hyperbolic rotation of the basic vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ by the same positive hyperbolic angle.

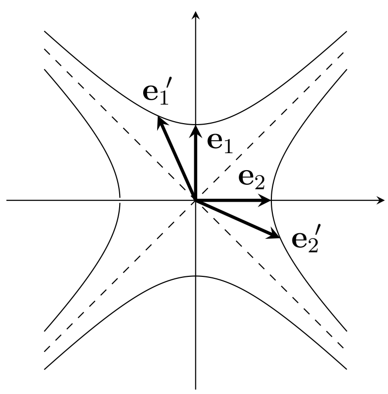

FIGURE 2.7 – Hyperbolic rotation of the basic vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ by the same negative hyperbolic angle.

2.3.3 The Minkowski Spacetime

A set S, whose elements are called *points*, is said to be an *affine space* if there exists a map $\varphi : S \times S \to V$, for some finite-dimensional real vector space V, such that:

(1) For any point P from S and vector v from V, there is a unique point Q from S such that $\varphi(P, Q) = v$;

(2) $\varphi(P, Q) + \varphi(Q, R) = \varphi(P, R)$, for any points P, Q and R from S.

In this context, a useful notation for the map φ is given by $\varphi(P, Q) = \overrightarrow{PQ}$. Note, from the above properties, that $\overrightarrow{PP} = o$ (where o is the null vector from V) and that $\overrightarrow{PQ} = -\overrightarrow{QP}$, for any points P and Q from S. The *dimension* of the affine space S is defined as the dimension of the vector space V: $\dim(S) = \dim(V)$. An affine space of dimension 1 is called a *line*, an affine space of dimension 2 is called a *plane*, etc. Given a point P from S, the set of vectors $T_P = \left\{ \overrightarrow{PQ} \;\middle|\; Q \in S \right\}$ from V can be combined to form a real vector space, which is found to be isomorphic to V. Intuitively this means that there is no preferred point in an affine space, so that any point can be taken as the origin of a reference system.

Let V be a real vector space of dimension n. This vector space can be studied in terms of the vectors from $\mathbb{R}^n$, since $V \simeq \mathbb{R}^n$. If S is an affine space of dimension n, a *reference frame* of S is a pair (O, β), where O is a point from S and $\beta = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a basis of $\mathbb{R}^n$. If $\beta = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the canonical basis of $\mathbb{R}^n$, then (O, β) is called a *canonical reference frame* of S. If a point P from S is such that the vector $\overrightarrow{OP}$ is given in terms of the basis $\beta = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ of $\mathbb{R}^n$ by

$$\overrightarrow{OP} = x_1\mathbf{u}_1 + \cdots + x_n\mathbf{u}_n, \tag{2.273}$$

then $x_1, \dots, x_n$, that is, the coordinates of the vector $\overrightarrow{OP}$ with relation to the basis β, are called the *coordinates* of P with relation to the reference frame (O, β).

Consider the vector space $\mathbb{R}^4$ and let its canonical basis be denoted by $\beta = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, in such way that a generic vector can be expressed by

$$\mathbf{x} = \sum_{\mu=0}^{3} x_\mu \mathbf{e}_\mu = x_0\mathbf{e}_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3. \tag{2.274}$$

Consider then the symmetric bilinear form $h : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ given by

$$h(\mathbf{e}_0, \mathbf{e}_0) = -h(\mathbf{e}_i, \mathbf{e}_i) = 1, \quad \text{where} \quad i \in \{1, 2, 3\}, \tag{2.275}$$

and

$$h(\mathbf{e}_\mu, \mathbf{e}_\nu) = 0, \quad \text{where} \quad \mu, \nu \in \{0, 1, 2, 3\} \quad \text{and} \quad \mu \neq \nu. \tag{2.276}$$

The vector space $\mathbb{R}^4$ endowed with the symmetric bilinear form h is a pseudo-Euclidean space, as discussed earlier, and it is denoted by $\mathbb{R}^{1,3}$. This space is called *Minkowski vector space*. The pseudo-norm induced by the form h is given by

$$\|\mathbf{x}\|_h{}^2 = x_0{}^2 - x_1{}^2 - x_2{}^2 - x_3{}^2, \tag{2.277}$$

or,

$$\|\mathbf{x}\|_h{}^2 = x_0{}^2 - \sum_{i=1}^{3} x_i{}^2. \tag{2.278}$$

As in the case of the pseudo-Euclidean plane, any vector $\mathbf{x}$ from $\mathbb{R}^{1,3}$ can be classified according to its pseudo-norm: it is time-like if $\|\mathbf{x}\|_h{}^2 > 0$, it is space-like if $\|\mathbf{x}\|_h{}^2 < 0$, and it is a null vector or a light-like vector if $\|\mathbf{x}\|_h{}^2 = 0$.

The affine space associated to $\mathbb{R}^{1,3}$ is called the *Minkowski spacetime*, and its points are called *events*. The term *spacetime* is frequently used as synonym of Minkowski spacetime. Given a canonical reference frame (O, β), the event of reference O is called *origin*, and the coordinates of an arbitrary event P with relation to (O, β) are represented by $(x_0, x_1, x_2, x_3) = (ct, x, y, z)$, so that the vector

$$\overrightarrow{OP} = x_0\mathbf{e}_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = ct\mathbf{e}_0 + x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \tag{2.279}$$

is the representative of the event P in $\mathbb{R}^{1,3}$, with relation to (O, β). The coordinates $x_1 = x$, $x_2 = y$ and $x_3 = z$ are the rectangular coordinates of the event P relative to (O, β), which localize the event in the three-dimensional Euclidean space. The coordinate $x_0 = ct$ is the temporal coordinate of the event P relative to (O, β), being c the speed of light in vacuum and t the time of occurrence of the event with relation to (O, β). Given two events A and B, given respectively by (ct_A, x_A, y_A, z_A) and (ct_B, x_B, y_B, z_B), the *interval* between A and B is a generalization of the concept of distance, which is given by

$$\left\|\overrightarrow{AB}\right\|_h^2 = h\left(\overrightarrow{AB}, \overrightarrow{AB}\right) = c^2(t_B - t_A)^2 - (x_B - x_A)^2 - (y_B - y_A)^2 - (z_B - z_A)^2. \tag{2.280}$$

Adopted a canonical reference frame and given an arbitrary event represented by the vector $\mathbf{x} = \sum_{\mu=0}^{3} x_\mu \mathbf{e}_\mu$, the equation

$$\|\mathbf{x}\|_h{}^2 = x_0{}^2 - x_1{}^2 - x_2{}^2 - x_3{}^2 = 0 \tag{2.281}$$

determines a geometric object which is called the *light cone*. For such an object, there is no graphic representation, since it is a four-dimensional object. However, if a spatial coordinate is ignored, which corresponds to take the intersection of the light cone with a hyperplane given by $x_i = 0$ ($i \in \{1, 2, 3\}$), the light cone can be represented by a usual three-dimensional cone. If two spatial coordinates are disregarded, the resulting space corresponds to the pseudo-Euclidean plane, and the light cone is reduced to the assymptotes of the hyperbolas in the figure 2.4. The equations $\|\mathbf{x}\|_h{}^2 = \pm r^2$, where r is a real constant, determine "hyper-hyperboloids". The light cone can be decomposed into two parts, one given by the equation $x_0 = \sqrt{x_1{}^2 + x_2{}^2 + x_3{}^2}$, which is called *future light cone*, and another given by the equation $x_0 = -\sqrt{x_1{}^2 + x_2{}^2 + x_3{}^2}$, which is called *past light cone*. The regions of the Minkowski space given by the inequalities $x_0 > \sqrt{x_1{}^2 + x_2{}^2 + x_3{}^2}$ and $x_0 < -\sqrt{x_1{}^2 + x_2{}^2 + x_3{}^2}$ are respectively called *future* and *past*, and the region given by $x_0{}^2 < x_1{}^2 + x_2{}^2 + x_3{}^2$ is called *present*.

Adopting a canonical reference frame, consider a curve λ in spacetime parameterized by a real variable α, which is represented by

$$\lambda : \quad \alpha \in \mathbb{R} \quad \mapsto \quad \mathbf{x} = \mathbf{x}(\alpha) \in \mathbb{R}^{1,3}. \tag{2.282}$$

A vector tangent to the curve λ in a generic point is given by

$$\mathbf{v} = \mathbf{v}(\alpha) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\alpha}. \tag{2.283}$$

The curve λ can be classified according to the classification of the vector $\mathbf{v}$, as being time-like or space-like or light-like. The trajectory of a particle in spacetime is called the *world line* of the particle. A particle with non-null mass has a time-like world line, and light has a light-like world line (this is why the null vectors of spacetime are called light-like). Consider now a smooth time-like curve given by

$$\lambda : \quad \alpha \in [\alpha_0, \alpha_1] \quad \mapsto \quad \mathbf{x} = \mathbf{x}(\alpha) \in \mathbb{R}^{1,3}, \tag{2.284}$$

where $[\alpha_0, \alpha_1]$ is a real interval. The *length* L of λ is given by

$$L = \int_{\alpha_0}^{\alpha_1} \sqrt{h(\mathbf{v}(\alpha), \mathbf{v}(\alpha))}\mathrm{d}\alpha. \tag{2.285}$$

The time-like curve λ can be parameterized by its length ℓ, such that $0 \leq \ell \leq L$, by writing ℓ as function of α as

$$\ell = \ell(\alpha) = \int_{\alpha_0}^{\alpha} \sqrt{h(\mathbf{v}(\alpha'), \mathbf{v}(\alpha'))}\mathrm{d}\alpha', \tag{2.286}$$

and then inverting this equation in order to write α as function of ℓ. According to the fundamental theorem of calculus, the derivative of $\ell(\alpha)$ is given by

$$\frac{\mathrm{d}\ell}{\mathrm{d}\alpha} = \frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{\alpha_0}^{\alpha} \sqrt{h(\mathbf{v}(\alpha'), \mathbf{v}(\alpha'))}\mathrm{d}\alpha' = \sqrt{h(\mathbf{v}(\alpha), \mathbf{v}(\alpha))}. \tag{2.287}$$

According to this result, another vector tangent to the curve λ is given by

$$\hat{\mathbf{v}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\ell} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}\ell} = \mathbf{v}\frac{1}{\sqrt{h(\mathbf{v}, \mathbf{v})}}, \tag{2.288}$$

which is a unit vector. In this way, a smooth time-like curve parameterized by its length has unit tangent vector. In practice, one writes $\ell = c\tau$, where the parameter τ, which has dimension of time, is called the *proper time* associated to the curve.

An *observer* is defined by a time-like curve parameterized by its proper time and oriented towards the future, in the sense that its unit tangent vector has positive time coordinate. In this way, an observer corresponds to the world line of a particle. If an observer is given by a straight line, it is said to be an *inertial observer*; in the corresponding case of a particle, it is said to be in uniform motion. Since a straight line is determined by a point and a parallel vector, one can define an inertial observer by a future-oriented unit time-like vector, with the origin implied as the reference point.

An observer naturally "splits" spacetime into two "parts", "time" and "space". If an observer has unit tangent vector $\hat{\mathbf{v}}$, this fact is formally described by

$$\mathbb{R}^{1,3} = T \oplus E, \tag{2.289}$$

where T is the vector subspace of $\mathbb{R}^{1,3}$ generated by the unit tangent vector $\hat{\mathbf{v}}$, and E is the orthogonal complement of T, that is, the vector subspace generated by any set of vectors mutually orthogonal to $\hat{\mathbf{v}}$. Another observer also "splits" the spacetime into "time" and "space", although in a distinct way, according to its unit tangent vector.

2.3.4 The Geometric Algebra of Minkowski Spacetime

In practice, events in Minkowski spacetime are considered with relation to a reference frame, so that the study of phenomena in spacetime is generally made in terms of the Minkowski vector space, $\mathbb{R}^{1,3}$. In this respect, in order to study geometry and physics of spacetime, the geometric algebra of Minkowski vector space can be constructed as follows.

The fundamental property of the geometric product is given by

$$\mathbf{u}^2 = h(\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|_h{}^2, \tag{2.290}$$

for any vector $\mathbf{u}$ from $\mathbb{R}^{1,3}$, or, in terms of components relative to the canonical basis,

$$(u_0\mathbf{e}_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3)(u_0\mathbf{e}_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3) = {u_0}^2 - {u_1}^2 - {u_2}^2 - {u_3}^2. \quad (2.291)$$

Considering the bilinearity of the geometric product, one can write

$$\begin{aligned} &{u_0}^2{\mathbf{e}_0}^2 + {u_1}^2{\mathbf{e}_1}^2 + {u_2}^2{\mathbf{e}_2}^2 + {u_3}^2{\mathbf{e}_3}^2 + \\ &\quad + u_0u_1(\mathbf{e}_0\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_0) + u_0u_2(\mathbf{e}_0\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_0) + u_0u_3(\mathbf{e}_0\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_0) + \\ &\qquad + u_1u_2(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) + u_1u_3(\mathbf{e}_1\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1) + u_2u_3(\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2) = \\ &\qquad\qquad = {u_0}^2 - {u_1}^2 - {u_2}^2 - {u_3}^2. \end{aligned} \quad (2.292)$$

For this equation to be satisfied, one must have

$$ {\mathbf{e}_0}^2 = 1 \quad \text{and} \quad {\mathbf{e}_i}^2 = -1, \quad \text{where} \quad i \in \{1, 2, 3\}, \quad (2.293)$$

and

$$\mathbf{e}_\mu\mathbf{e}_\nu = -\mathbf{e}_\mu\mathbf{e}_\nu, \quad \text{where} \quad \mu, \nu \in \{0, 1, 2, 3\} \quad \text{and} \quad \mu \neq \nu. \quad (2.294)$$

These are the basic relations for calculation of the geometric product of the geometric algebra of the Minkowski vector space in terms of the canonical basic vectors. Applying it to the calculation of the geometric product of two arbitrary vectors $\mathbf{u} = \sum_{\mu=0}^{3} u_\mu\mathbf{e}_\mu$ and $\mathbf{v} = \sum_{\mu=0}^{3} v_\mu\mathbf{e}_\mu$, one obtains

$$\begin{aligned} \mathbf{uv} &= (u_0\mathbf{e}_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3)(v_0\mathbf{e}_0 + v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \\ &= (u_0v_0 - u_1v_1 - u_2v_2 - u_3v_3) + \\ &\quad + (u_0v_1 - u_1v_0)\mathbf{e}_0\mathbf{e}_1 + (u_0v_2 - u_2v_0)\mathbf{e}_0\mathbf{e}_2 + (u_0v_3 - u_3v_0)\mathbf{e}_0\mathbf{e}_3 + \\ &\qquad + (u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2 + (u_1v_3 - u_3v_1)\mathbf{e}_1\mathbf{e}_3 + (u_2v_3 - u_3v_2)\mathbf{e}_2\mathbf{e}_3, \end{aligned} \quad (2.295)$$

which corresponds to the sum of a symmetric part and an antisymmetric part under the exchange of the vectors $\mathbf{u}$ and $\mathbf{v}$. Since the geometric product of two vectors can be uniquely written in the form

$$\mathbf{uv} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu}) + \frac{1}{2}(\mathbf{uv} - \mathbf{vu}), \quad (2.296)$$

where the first term is symmetric and the second term antisymmetric under the exchange of $\mathbf{u}$ and $\mathbf{v}$, one can write

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}, \quad (2.297)$$

where are defined the *scalar product* and the *exterior product*, respectively, by

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu}) = u_0v_0 - u_1v_1 - u_2v_2 - u_3v_3 = h(\mathbf{u}, \mathbf{v}) \quad (2.298)$$

and

$$\begin{aligned}\mathbf{u}\wedge\mathbf{v} &= \frac{1}{2}(\mathbf{uv}-\mathbf{vu}) \\ &= (u_0v_1-u_1v_0)\mathbf{e}_0\mathbf{e}_1+(u_0v_2-u_2v_0)\mathbf{e}_0\mathbf{e}_2+(u_0v_3-u_3v_0)\mathbf{e}_0\mathbf{e}_3+ \\ &\quad +(u_1v_2-u_2v_1)\mathbf{e}_1\mathbf{e}_2+(u_1v_3-u_3v_1)\mathbf{e}_1\mathbf{e}_3+(u_2v_3-u_3v_2)\mathbf{e}_2\mathbf{e}_3. \end{aligned} \tag{2.299}$$

As in the cases considered before, the objects of the form $\mathbf{u}\wedge\mathbf{v}$, such as $\mathbf{e}_0\wedge\mathbf{e}_1=\mathbf{e}_0\mathbf{e}_1$, $\mathbf{e}_1\wedge\mathbf{e}_2=\mathbf{e}_1\mathbf{e}_2$, etc., define bivectors, which form their own vector space and are interpreted as oriented parallelograms in a four-dimensional space (this interpretation is independent on the metric properties of $\mathbb{R}^{1,3}$, determined by the form h). The exterior product can then be taken successively, by considering it to be associative, in order to produce higher dimensional objects. In this manner, by considering the possible combinations of the basic vectors $\mathbf{e}_\mu$ ($\mu\in\{0,1,2,3\}$) to form a higher dimensional exterior product with a certain number of vectors, it is found that there is, up to a sign, in addition to 6 basic bivectors (i.e. $\mathbf{e}_0\mathbf{e}_1,\mathbf{e}_0\mathbf{e}_2,\mathbf{e}_0\mathbf{e}_3,\mathbf{e}_1\mathbf{e}_2,\mathbf{e}_1\mathbf{e}_3,\mathbf{e}_2\mathbf{e}_3$), 4 basic trivectors (i.e. $\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2,\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3,\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$), and 1 *quadrivector* (i.e. $\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$). The vector space of real scalars can now be denoted by $\bigwedge^0(\mathbb{R}^{1,3})$ and the vector space of vectors can be denoted by $\bigwedge^1(\mathbb{R}^{1,3})$. The bivectors form a 6-dimensional vector space, which is denoted by $\bigwedge^2(\mathbb{R}^{1,3})$, the trivectors form a 4-dimensional vector space denoted by $\bigwedge^3(\mathbb{R}^{1,3})$, and the quadrivectors form an one-dimensional vector space denoted by $\bigwedge^4(\mathbb{R}^{1,3})$. The vector spaces of the form $\bigwedge^k(\mathbb{R}^{1,3})$ can then be combined through a direct sum to form the multivector space

$$\bigwedge\left(\mathbb{R}^{1,3}\right)=\bigoplus_{k=0}^{4}\bigwedge\nolimits^k\left(\mathbb{R}^{1,3}\right), \tag{2.300}$$

whose elements are called multivectors. Defining then the geometric product of a scalar with a multivector as the multiplication of the multivector by the scalar, and extending the geometric product to arbitrary multivectors by bilinearity and associativity, it follows that the vector space $\bigwedge(\mathbb{R}^{1,3})$ endowed with the geometric product determines an associative algebra over the field of real scalars, the *geometric algebra of Minkowski spacetime*, or the *Clifford algebra of Minkowski spacetime*, which is denoted by $\mathcal{C}\ell(\mathbb{R}^{1,3},h)$, or $\mathcal{C}\ell_{1,3}(\mathbb{R})$, or $\mathcal{C}\ell_{1,3}$.

Projection, Graded Involution, Reversion, the Norm and the Inverse

The operations of projection, graded involution and reversion are defined for multivectors of the geometric algebra of spacetime in the same way as for the algebras introduced earlier, but with the maximum grade of the multivectors taken as 4:

$$\langle A\rangle_k = A_k, \quad \widehat{A} = \sum_{k=0}^{4}(-1)^k\langle A\rangle_k, \quad \text{and} \quad \widetilde{A} = \sum_{k=0}^{4}(-1)^{\frac{1}{2}k(k-1)}\langle A\rangle_k. \tag{2.301}$$

In this way, for an arbitrary multivector $A = \sum_{k=0}^{4} A_k$, one has

$$\widehat{A} = A_0 - A_1 + A_2 - A_3 + A_4 \quad \text{and} \quad \widetilde{A} = A_0 + A_1 - A_2 - A_3 + A_4. \tag{2.302}$$

The properties of these operations already presented remain valid, since they are inherent to the multivector structure. In particular, one has

$$\langle AB\rangle = \langle BA\rangle, \tag{2.303}$$

for arbitrary multivectors A and B, which implies in the invariance of the scalar part of a geometric product under cyclic permutations of the multivectors in the product. The fact that the reversion of a geometric product of multivectors corresponds to the geometric product in the opposite order of the reverses of the multivectors also holds,

$$\widetilde{(AB\cdots C)} = \widetilde{C}\cdots\widetilde{B}\widetilde{A}. \tag{2.304}$$

Differently from the geometric algebras already considered, there is no standard way to define a pseudo-norm for an arbitrary multivector A from $\mathcal{C}\ell_{1,3}$, since $A\widetilde{A}$ is not a scalar, in general. However, the pattern of the previous definitions of norm/pseudo-norm fits naturally for most multivectors of $\mathcal{C}\ell_{1,3}$, and does not represent any complication for future constructions. Then, the pseudo-norm of a multivector A from $\mathcal{C}\ell_{1,3}$ can be defined by

$$\|A\|_h{}^2 = \left\langle \widetilde{A}A \right\rangle = \left\langle A\widetilde{A} \right\rangle. \tag{2.305}$$

Note that $A\widetilde{A}$ is an even grade multivector and that it is equal to its reverse, so that it is a scalar plus a pseudoscalar. It follows that $A\widetilde{A}$ has a multiplicative inverse, provided it is different from zero. Indeed, by writing $A\widetilde{A} = \rho e^{I\beta}$, where $\rho, \beta \in \mathbb{R}$ and $\rho > 0$, one can identify $\left(A\widetilde{A}\right)^{-1} = \rho^{-1}e^{-I\beta}$ as the inverse of $A\widetilde{A} = \rho e^{I\beta}$. In this way, the definition of pseudo-norm is not necessary for defining the inverse, which can be defined by

$$A^{-1} = \widetilde{A}\left(A\widetilde{A}\right)^{-1}, \tag{2.306}$$

provided that

$$A\widetilde{A} \neq 0. \tag{2.307}$$

Interior, Exterior and Commutator Products

A similar argumentation to that used in the case of the geometric algebra of the three-dimensional Euclidean space (cf. subsection 2.2.3) leads to the definition of *contraction from the left* of the multivector A by the vector $\mathbf{u}$, or *interior product* of the vector $\mathbf{u}$ with the multivector A,

$$\mathbf{u} \cdot A = \frac{1}{2}\left(\mathbf{u}A - \widehat{A}\mathbf{u}\right). \tag{2.308}$$

As in that subsection, it is found that the exterior product of a vector $\mathbf{u}$ with a multivector A can be expressed as

$$\mathbf{u} \wedge A = \frac{1}{2}\left(\mathbf{u}A + \widehat{A}\mathbf{u}\right). \tag{2.309}$$

In terms of these two operations the geometric product of $\mathbf{u}$ with A can be written

$$\mathbf{u}A = \mathbf{u} \cdot A + \mathbf{u} \wedge A. \tag{2.310}$$

In the same way, the *contraction from the right* of the multivector A by the vector $\mathbf{u}$, or the *interior product* of the vector $\mathbf{u}$ with the multivector A, is defined by

$$A \cdot \mathbf{u} = \frac{1}{2}\left(A\mathbf{u} - \mathbf{u}\widehat{A}\right). \tag{2.311}$$

The exterior product of A with $\mathbf{u}$ can also be written

$$A \wedge \mathbf{u} = \frac{1}{2}\left(A\mathbf{u} + \mathbf{u}\widehat{A}\right). \tag{2.312}$$

In this way, the geometric product of the multivector A with the vector $\mathbf{u}$ can be expressed as

$$A\mathbf{u} = A \cdot \mathbf{u} + A \wedge \mathbf{u}. \tag{2.313}$$

As before, it is found that, in general, the interior and exterior products do not commute or anticommute, but satisfy

$$\mathbf{u} \cdot A = -\widehat{A} \cdot \mathbf{u} \quad \text{and} \quad \mathbf{u} \wedge A = \widehat{A} \wedge \mathbf{u}. \tag{2.314}$$

If A and B are bivectors, writing $A = \mathbf{u}\wedge\mathbf{v} = \mathbf{u}\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are vectors satisfying $\mathbf{u} \cdot \mathbf{v} = 0$, it follows that

$$\begin{aligned} AB &= \mathbf{u}\mathbf{v}B \\ &= \mathbf{u}(\mathbf{v} \cdot B + \mathbf{v} \wedge B) \\ &= \mathbf{u} \cdot (\mathbf{v} \cdot B) + \mathbf{u} \cdot (\mathbf{v} \wedge B) + \mathbf{u} \wedge (\mathbf{v} \cdot B) + \mathbf{u} \wedge \mathbf{v} \wedge B \\ &= \mathbf{u} \cdot (\mathbf{v} \cdot B) + \mathbf{u} \cdot (\mathbf{v} \wedge B) + \mathbf{u} \wedge (\mathbf{v} \cdot B) + A \wedge B. \end{aligned} \tag{2.315}$$

The term $\mathbf{u}\cdot(\mathbf{v}\cdot B)$ in the resulting expression is a scalar, since it is the result of two followed interior products with a vector applied on a bivector. The term $A \wedge B$ is a quadrivector, since it is the exterior product of two bivectors. The remain terms are bivectors, since both are the result of the combination of an interior and an exterior product with a vector applied on a bivector. The geometric product AB can then be written

$$AB = \langle AB\rangle_0 + \langle AB\rangle_2 + \langle AB\rangle_4. \tag{2.316}$$

But, note that such a product can be written as the sum of a symmetric part and an antisymmetric part in relation to the exchange of the bivectors:

$$AB = \frac{1}{2}(AB + BA) + \frac{1}{2}(AB - BA). \tag{2.317}$$

Since the symmetric part is invariant and the antisymmetric part changes the sign under the reversion operation, one identifies the symmetric part as the scalar part plus the quadrivector part and the antisymmetric part as the bivector part:

$$\langle AB\rangle_0 + \langle AB\rangle_4 = \frac{1}{2}(AB + BA) \quad \text{and} \quad \langle AB\rangle_2 = \frac{1}{2}(AB - BA). \tag{2.318}$$

In general, the antisymmetric part of the geometric product of two arbitrary multivectors A and B is defined as the *commutator product* of A and B, which is denoted by

$$A \times B = \frac{1}{2}(AB - BA). \tag{2.319}$$

It is found that the commutator product satisfies the *Jacobi identity*, that is,

$$A \times (B \times C) + C \times (A \times B) + B \times (C \times A) = 0, \tag{2.320}$$

for any multivectors A, B and C, which can be verified by applying directly the definition of the commutator product.

Note that, given a bivector B and a vector $\mathbf{u}$, one has

$$B \times \mathbf{u} = \frac{1}{2}(B\mathbf{u} - \mathbf{u}B) = B \cdot \mathbf{u}, \tag{2.321}$$

which results in a vector. In this way, for a bivector B and a k-vector $A_k = \mathbf{u}_1 \cdots \mathbf{u}_k$, being the k vectors $(2 \leq k \leq 4)$ $\mathbf{u}_1, \ldots, \mathbf{u}_k$ mutually orthogonal, one has

$$\begin{aligned} B(\mathbf{u}_1 \cdots \mathbf{u}_k) &= (2B \cdot \mathbf{u}_1 + \mathbf{u}_1 B)\mathbf{u}_2 \cdots \mathbf{u}_k \\ &= 2(B \cdot \mathbf{u}_1)\mathbf{u}_2 \cdots \mathbf{u}_k + \mathbf{u}_1 B \mathbf{u}_2 \cdots \mathbf{u}_k \\ &= 2(B \cdot \mathbf{u}_1)\mathbf{u}_2 \cdots \mathbf{u}_k + \mathbf{u}_1(2B \cdot \mathbf{u}_2 + \mathbf{u}_2 B)\mathbf{u}_3 \cdots \mathbf{u}_k \\ &= 2(B \cdot \mathbf{u}_1)\mathbf{u}_2 \cdots \mathbf{u}_k + \cdots + 2\mathbf{u}_1 \cdots \mathbf{u}_{k-1}(B \cdot \mathbf{u}_k) + (\mathbf{u}_1 \cdots \mathbf{u}_k)B, \end{aligned} \tag{2.322}$$

that is,

$$B \times (\mathbf{u}_1 \cdots \mathbf{u}_k) = \sum_{i=1}^{k} \mathbf{u}_1 \cdots (B \cdot \mathbf{u}_i) \cdots \mathbf{u}_k. \tag{2.323}$$

The right-hand side of the resulting expression is at principle a multivector with grades k and $k-2$. But, it follows that

$$\begin{aligned} (\widetilde{B \times A_k}) &= \frac{1}{2}\left(\widetilde{A}_k \widetilde{B} - \widetilde{B}\widetilde{A}_k\right) \\ &= \frac{1}{2}\left(-\widetilde{A}_k B + B\widetilde{A}_k\right) \\ &= B \times \widetilde{A}_k \\ &= (-1)^{\frac{1}{2}k(k-1)} B \times A_k \end{aligned} \tag{2.324}$$

that is, $B \times A_k$ transforms in the same way as A_k under reversion. Since multivectors of grade k and $k-2$ transform in different ways under reversion, which follows from the fact that $(-1)^{\frac{1}{2}k(k-1)}/(-1)^{\frac{1}{2}(k-2)(k-3)} = -1$, the multivector $B \times A_k$ must have grade k. Therefore, the commutator product of a bivector with any multivector preserves the grade of the multivector:

$$B \times A_k = \langle B \times A_k \rangle_k. \tag{2.325}$$

This result is general and applies to multivectors of an arbitrary geometric algebra.

Inequalities, Parallelism and Orthogonality

The hyperbolic angle α between two time-like vectors $\mathbf{u}$ and $\mathbf{v}$, both either directed to the future or to the past, is given by

$$\cosh(\alpha) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|_h \|\mathbf{v}\|_h}. \tag{2.326}$$

From this expression, one can write also

$$\sinh(\alpha) = \frac{\sqrt{-\|\mathbf{u} \wedge \mathbf{v}\|_h{}^2}}{\|\mathbf{u}\|_h \|\mathbf{v}\|_h}. \tag{2.327}$$

Furthermore, expression (2.326) implies also in the reversed Cauchy-Schwarz inequality:

$$(\mathbf{u}\cdot\mathbf{v})^2 \geq \|\mathbf{u}\|_h{}^2\|\mathbf{v}\|_h{}^2. \tag{2.328}$$

From this inequality it follows, in turn, that

$$\|\mathbf{u}+\mathbf{v}\|_h{}^2 = \|\mathbf{u}\|_h{}^2 + \|\mathbf{v}\|_h{}^2 + 2(\mathbf{u}\cdot\mathbf{v}) \geq \|\mathbf{u}\|_h{}^2 + \|\mathbf{v}\|_h{}^2 + 2\|\mathbf{u}\|_h\|\mathbf{v}\|_h = (\|\mathbf{u}\|_h + \|\mathbf{v}\|_h)^2, \tag{2.329}$$

which implies the reversed triangular inequality:

$$\|\mathbf{u}+\mathbf{v}\|_h \geq \|\mathbf{u}\|_h + \|\mathbf{v}\|_h. \tag{2.330}$$

The angle θ between two space-like vectors $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|_h\|\mathbf{v}\|_h}. \tag{2.331}$$

From this expression, one can write

$$\sin(\theta) = \frac{\|\mathbf{u}\wedge\mathbf{v}\|_h}{\|\mathbf{u}\|_h\|\mathbf{v}\|_h}. \tag{2.332}$$

For space-like vectors, the Cauchy-Schwarz inequality in its usual form holds:

$$(\mathbf{u}\cdot\mathbf{v})^2 \leq \|\mathbf{u}\|_h{}^2\|\mathbf{v}\|_h{}^2. \tag{2.333}$$

For two arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$, one can define the condition for orthogonality as $\mathbf{u}\cdot\mathbf{v}$, or, in terms of the geometric product, $\mathbf{uv} = -\mathbf{vu}$. The condition for parallelism can be taken as $\mathbf{u}\wedge\mathbf{v} = 0$, that is, $\mathbf{uv} = \mathbf{vu}$. As in the three-dimensional Euclidean case, given a vector $\mathbf{u}$ and a bivector B, the condition for orthogonality can be taken as $\mathbf{u}\cdot B = 0$, and the condition for parallelism can be taken as $\mathbf{u}\wedge B = 0$. In the same way, given two bivectors A and B, the condition for orthogonality is taken to be $\langle AB\rangle = 0$, and the condition for parallelism is taken to be $A\times B = 0$.

Now, observe that a trivector, like $\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2$, determine a *hyperplane* (a hyperplane in an n-dimensional space is an $(n-1)$-dimensional subspace), which can also be determined by its orthogonal line. Observe then that, for example, the vector $\mathbf{e}_3$ is orthogonal to the vectors $\mathbf{e}_0$, $\mathbf{e}_1$ and $\mathbf{e}_2$, and one has

$$\mathbf{e}_3\cdot(\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2) = 0. \tag{2.334}$$

In general, it follows that

$$\mathbf{e}_\mu\cdot(\mathbf{e}_\nu\mathbf{e}_\rho\mathbf{e}_\sigma) = 0, \tag{2.335}$$

for distinct μ, ν, ρ and σ, and one can say that the vector $\mathbf{e}_\mu$ is orthogonal to the trivector $\mathbf{e}_\nu\mathbf{e}_\rho\mathbf{e}_\sigma$, and also to the hyperplane determined by this trivector. Thus the condition for orthogonality of a vector $\mathbf{u}$ and a trivector T can be taken as $\mathbf{u}\cdot T = 0$. On the other hand, since the exterior product of four linearly dependent vectors from $\mathbb{R}^{1,3}$ is zero, the exterior product of a vector $\mathbf{u}$ with a trivector $T = \mathbf{v}\wedge\mathbf{w}\wedge\mathbf{x}$ is zero if and only if the set of vectors $\{\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{x}\}$ is linearly dependent, which means that $\mathbf{u}$ belongs to the hyperplane determined by T. The condition for parallelism of a vector $\mathbf{u}$ and a trivector T is then $\mathbf{u}\wedge T = 0$.

Pseudoscalars, Orientation and Duality

Quadrivectors are also called *pseudoscalars*, since any quadrivector is a scalar multiple of the unit pseudoscalar $I = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. Note that the unit pseudoscalar is equal to its reverse,

$$\widetilde{I} = I, \tag{2.336}$$

and that

$$I^2 = \widetilde{I}I = (\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1\mathbf{e}_0)(\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = -1. \tag{2.337}$$

Another important property of pseudoscalars is that they anticommute with vectors, from which follows that they also anticommute with trivectors and commute with any even grade multivector. Note also that

$$\widehat{(A_k I)} = \widehat{A}_k I, \tag{2.338}$$

for any k-vector A_k from $\mathcal{C}\ell_{1,3}$. Therefore, given a vector $\mathbf{u}$ and a k-vector A_k, it follows that

$$\begin{aligned}
\mathbf{u}\cdot(A_k I) &= \frac{1}{2}\left(\mathbf{u}A_k I - \widehat{(A_k I)}\mathbf{u}\right) \\
&= \frac{1}{2}\left(\mathbf{u}A_k I - \widehat{A}_k I\mathbf{u}\right) \\
&= \frac{1}{2}\left(\mathbf{u}A_k + \widehat{A}_k\mathbf{u}\right) I \\
&= (\mathbf{u}\wedge A_k)I.
\end{aligned} \tag{2.339}$$

The unit pseudoscalar I defines an *orientation* for Minkowski spacetime, which is conventionally considered as a positive orientation. Another unit pseudoscalar defines either the same or the opposite orientation, depending on whether its sign is the same or different from that of I. In the case of the three-dimensional Euclidean space, the orientation conventionally considered to be positive is that determined by the pseudoscalar $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.

The *Hodge isomorphism* or *Hodge duality* is defined for $\mathcal{C}\ell_{1,3}$ in a way analogous to that for $\mathcal{C}\ell_{3,0}$. The *Hodge dual* of a k-vector A_k is a $(4-k)$-vector $\star A_k$ given by

$$\star A_k = \widetilde{A}_k I. \tag{2.340}$$

Observe that, the Hodge dual of a scalar is a pseudoscalar, and vice versa, the Hodge dual of a vector is a trivector (also called in this context a *pseudovector*), and vice versa, and the Hodge dual of a bivector is another bivector. In particular, one has the relations in the following table.

$\star 1 = I = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$
$\star\mathbf{e}_0 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$
$\star\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_3\mathbf{e}_0$
$\star\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_1\mathbf{e}_0$
$\star\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_0$
$\star(\mathbf{e}_1\mathbf{e}_0) = \mathbf{e}_2\mathbf{e}_3$
$\star(\mathbf{e}_2\mathbf{e}_0) = \mathbf{e}_3\mathbf{e}_1$
$\star(\mathbf{e}_3\mathbf{e}_0) = \mathbf{e}_1\mathbf{e}_2$
$\star(\mathbf{e}_1\mathbf{e}_2) = -\mathbf{e}_3\mathbf{e}_0$
$\star(\mathbf{e}_3\mathbf{e}_1) = -\mathbf{e}_2\mathbf{e}_0$
$\star(\mathbf{e}_2\mathbf{e}_3) = -\mathbf{e}_1\mathbf{e}_0$
$\star(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = \mathbf{e}_0$
$\star(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_0) = \mathbf{e}_3$
$\star(\mathbf{e}_3\mathbf{e}_1\mathbf{e}_0) = \mathbf{e}_2$
$\star(\mathbf{e}_2\mathbf{e}_3\mathbf{e}_0) = \mathbf{e}_1$
$\star I = \star(\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = -1$

TABLE 2.2 – Hodge duals of the basic multivectors from $\mathcal{C}\ell_{1,3}$.

In general, $\star A_k$ furnishes a $(4-k)$-vector determining the orthogonal complement of the subspace determined by A_k.

The Even Subalgebra and the Algebra of Biquaternions

Let ${\mathcal{C}\ell_{1,3}}^+$ be the set formed by even grade multivectors from $\mathcal{C}\ell_{1,3}$, that is, the set of multivectors A satisfying $\widehat{A} = A$:

$${\mathcal{C}\ell_{1,3}}^+ = \left\{ A \;\middle|\; A \in \mathcal{C}\ell_{1,3} \text{ and } \widehat{A} = A \right\}. \tag{2.341}$$

If $\mathcal{C}\ell_{1,3}{}^{+}$, then A is the sum of a scalar, a bivector and a pseudoscalar, and it can be written

$$A = a + a_{10}\mathbf{e}_1\mathbf{e}_0 + a_{20}\mathbf{e}_2\mathbf{e}_0 + a_{30}\mathbf{e}_3\mathbf{e}_0 + a_{12}\mathbf{e}_1\mathbf{e}_2 + a_{31}\mathbf{e}_3\mathbf{e}_1 + a_{23}\mathbf{e}_2\mathbf{e}_3 + a_{0123}I. \quad (2.342)$$

An even grade multivector can be expressed without reference to any basis as

$$M = \alpha + B + \beta I, \quad (2.343)$$

where α and β are scalars and B is a bivector. So, given the even grade multivectors $M_1 = \alpha_1 + B_1 + \beta_1 I$ and $M_2 = \alpha_2 + B_2 + \beta_2 I$, it follows that

$$\begin{aligned} M_1M_2 &= (\alpha_1 + B_1 + \beta_1 I)(\alpha_2 + B_2 + \beta_2 I) \\ &= \alpha_1\alpha_2 + \alpha_1 B_2 + \alpha_1\beta_2 I + \alpha_2 B_1 + B_1B_2 + \beta_2 B_1 I + \beta_1\alpha_2 I + \beta_1 B_2 I - \beta_1\beta_2 \\ &= \Big(\alpha_1\alpha_2 + \langle B_1B_2\rangle - \beta_1\beta_2\Big) + \\ &\qquad + \Big(\alpha_1 B_2 + \alpha_2 B_1 + B_1 \times B_2 + \beta_2 B_1 I + \beta_1 B_2 I\Big) + \\ &\qquad\qquad + \Big(\alpha_1\beta_2 I + \beta_1\alpha_2 I + B_1 \wedge B_2\Big). \end{aligned} \quad (2.344)$$

Therefore, the geometric product of two even grade multivectos is also an even grade multivector, in such way that the vector subspace determined by elements of $\mathcal{C}\ell_{1,3}{}^{+}$ endowed with the geometric product is a subalgebra of $\mathcal{C}\ell_{1,3}$, the *even subalgebra* of $\mathcal{C}\ell_{1,3}$, which is also denoted by $\mathcal{C}\ell_{1,3}{}^{+}$.

Consider the following notation: $\mathbf{I} = \mathbf{e}_2\mathbf{e}_3$, $\mathbf{J} = \mathbf{e}_3\mathbf{e}_1$, $\mathbf{K} = \mathbf{e}_1\mathbf{e}_2$, and $\mathbf{i} = I$. Observe then from (2.342), and from the relations on the table 2.2, that an element of $\mathcal{C}\ell_{1,3}{}^{+}$ can be written in the form

$$A = (x_0 + y_0\mathbf{i}) + (x_1 + y_1\mathbf{i})\mathbf{I} + (x_2 + y_2\mathbf{i})\mathbf{J} + (x_3 + y_3\mathbf{i})\mathbf{K}, \quad (2.345)$$

which resembles a quaternion, although with complex components. Quaternions with complex components are called *biquaternions* and form an 8-dimensional real algebra (or a 4-dimensional complex algebra), denoted by $\mathbb{C} \otimes \mathbb{H}$. Note then that the bivectors $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$ satisfy

$$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = \mathbf{IJK} = -1, \quad (2.346)$$

which are identical to the basic relations defining the product of quaternions (cf. relations (2.164)), which also hold for biquaternions. Therefore, one can conclude that the even subalgebra $\mathcal{C}\ell_{1,3}{}^{+}$ is isomorphic to the real algebra of biquaternions, $\mathbb{C} \otimes \mathbb{H}$, through the identification of the bivectors $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$ with the unit quaternions i, j and k, and through the identification of the unit pseudoscalar $\mathbf{i} = I$ with the imaginary unit $\sqrt{-1}$, in addition to the identification of the geometric product with the product of biquaternions.

Reflections and Rotations

As seen before, the form of the conditions for parallelism and orthogonality for vectors are preserved in the pseudo-Euclidean case, although the scalar product (hence the geometric product) is different. This implies that the expression for a reflection transformation has the same form: the reflection of a vector $\mathbf{v}$ through a hyperplane with orthogonal vector $\mathbf{u}$ is given by

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = -\mathbf{u}\mathbf{v}\mathbf{u}^{-1}. \tag{2.347}$$

As in the cases already considered, it is found that two reflections describe a rotation in Minkowski spacetime. But, as in the case of the pseudo-Euclidean plane, there are two types of reflections to consider, that performed through a hyperplane with time-like orthogonal vector and that performed through a hyperplane with space-like orthogonal vector. A generic reflection of a vector $\mathbf{v}$ through a hyperplane with orthogonal vector $\mathbf{u}$ reverses the component of $\mathbf{v}$ parallel to $\mathbf{u}$ and preserves the component of $\mathbf{v}$ orthogonal to $\mathbf{u}$. In this way, a time-like reflection reverses the corresponding time-like component of a vector, but preserves its space-like component, whereas a space-like reflection reverses the corresponding space-like component of a vector, reversing also the orientation of a three-dimensional volume element, but preserves time-like components. In this respect, the "proper rotations", understood as orthogonal transformations with determinant +1 which preserve both the orientation of time-like components (future or past) and the orientation of a three-dimensional volume element, consist in either (I) a pair of time-like reflections, or (II) a pair space-like reflections, or, more generally, the composition of a type I and a type II rotations. These types of rotations are considered in the following.

A rotation of type I can be expressed by

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = R\mathbf{v}R^{-1} = R\mathbf{v}\widetilde{R}, \tag{2.348}$$

where $R = \mathbf{u}_2\mathbf{u}_1$ and ${\mathbf{u}_1}^2 = {\mathbf{u}_2}^2 = 1$. If the unit time-like vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are both either future-oriented or past-oriented, the rotor R can be written

$$R = \mathbf{u}_2 \cdot \mathbf{u}_1 + \mathbf{u}_2 \wedge \mathbf{u}_1 = \cosh\left(\frac{\alpha}{2}\right) + \sinh\left(\frac{\alpha}{2}\right) B, \tag{2.349}$$

where $\alpha/2$ is the hyperbolic angle between $\mathbf{u}_1$ and $\mathbf{u}_2$, and

$$B = \frac{\mathbf{u}_2 \wedge \mathbf{u}_1}{\sqrt{-\|\mathbf{u}_2 \wedge \mathbf{u}_1\|_h{}^2}}. \tag{2.350}$$

The bivector B satisfies $B^2 = -\|B\|_h{}^2 = 1$ and is called a *time-like* bivector. Expressing the cosine and sine hyperbolic functions in the expression for R as power series, one can write

$$R = \exp\left(\frac{1}{2}\alpha B\right). \tag{2.351}$$

Analogously to the case of the pseudo-Euclidean plane, the rotation given by (2.348), where the rotor R is given by (2.351), is a hyperbolic rotation by a hyperbolic angle α through the plane given by the unit time-like bivector B, in this case, in the sense from $\mathbf{u}_1$ to $\mathbf{u}_2$. For the case where the unit time-like vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are such that one is future-oriented and the other is past-oriented, one can write the rotor R in (2.348) as

$$R = \mathbf{u}_2 \cdot \mathbf{u}_1 + \mathbf{u}_2 \wedge \mathbf{u}_1 = -\Big(\mathbf{u}_2 \cdot (-\mathbf{u}_1) + \mathbf{u}_2 \wedge (-\mathbf{u}_1)\Big), \tag{2.352}$$

where the resulting scalar and exterior products involve a pair of either future-oriented or past-oriented vectors. In this case, the rotor R can be written as

$$R = -\left(\cosh\left(\frac{\alpha}{2}\right) + \sinh\left(\frac{\alpha}{2}\right) B\right), \tag{2.353}$$

where $\alpha/2$ is the hyperbolic angle between $-\mathbf{u}_1$ and $\mathbf{u}_2$ (or, equivalently, the hyperbolic angle between $\mathbf{u}_1$ and $-\mathbf{u}_2$), and

$$B = \frac{\mathbf{u}_2 \wedge (-\mathbf{u}_1)}{\sqrt{-\|\mathbf{u}_2 \wedge (-\mathbf{u}_1)\|_h{}^2}} \tag{2.354}$$

is a unit time-like bivector. The rotor R in this case can be expressed by

$$R = -\exp\left(\frac{1}{2}\alpha B\right). \tag{2.355}$$

The rotation in question is a hyperbolic rotation by a hyperbolic angle α through the plane given by the unit time-like bivector B in the sense from $-\mathbf{u}_1$ to $\mathbf{u}_2$ (or, equivalently, in the sense from $\mathbf{u}_1$ to $-\mathbf{u}_2$). Note that the negative sign in the expression for the rotor R has no influence in the result of its application in a vector.

A rotation of type II can be expressed by

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = R\mathbf{v}R^{-1} = R\mathbf{v}\widetilde{R}, \tag{2.356}$$

where $R = \mathbf{u}_2\mathbf{u}_1$ and $\mathbf{u}_1{}^2 = \mathbf{u}_2{}^2 = -1$. In this case, the rotor R can be written

$$R = \mathbf{u}_2 \cdot \mathbf{u}_1 + \mathbf{u}_2 \wedge \mathbf{u}_1 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) B, \tag{2.357}$$

where $\theta/2$ is the angle between $\mathbf{u}_1$ and $\mathbf{u}_2$, and

$$B = \frac{\mathbf{u}_2 \wedge \mathbf{u}_1}{\|\mathbf{u}_2 \wedge \mathbf{u}_1\|_h}. \tag{2.358}$$

The bivector B satisfies $B^2 = -\|B\|_h{}^2 = -1$ and is called a *space-like* bivector. Expressing the cosine and sine functions in the expression for R as power series, one can write

$$R = \exp\left(\frac{1}{2}\theta B\right). \tag{2.359}$$

Analogously to the case of rotations in the three-dimensional Euclidean space, the rotation given by (2.356), where the rotor R is given by (2.359), it is found to be a "circular" or a "spatial" rotation (to differ from hyperbolic rotations) by an angle θ through the plane given by the unit space-like bivector B, in this case, in the sense from $\mathbf{u}_1$ to $\mathbf{u}_2$.

A general rotation in spacetime is given by

$$\mathbf{v} \;\mapsto\; \mathbf{v}' = R\mathbf{v}R^{-1} = R\mathbf{v}\widetilde{R}, \tag{2.360}$$

where

$$R = LU, \tag{2.361}$$

with L being a rotor describing a hyperbolic rotation and U being a rotor describing a spatial rotation. Such a general rotation in spacetime can be extended to be applied to a generic multivector A through the expression

$$A \;\mapsto\; A' = RAR^{-1} = RA\widetilde{R}. \tag{2.362}$$

Note that, as in the three-dimensional Euclidean case, the set of the rotors of $\mathcal{C}\ell_{1,3}{}^{+}$ can be characterized as

$$\left\{R \;\middle|\; R \in \mathcal{C}\ell_{1,3}{}^{+} \text{ and } \widetilde{R}R = R\widetilde{R} = 1\right\}. \tag{2.363}$$

As in the three-dimensional Euclidean case, it is easy to verify that the set of rotors endowed with the geometric product has the structure of a group. This group is denoted by $\mathrm{Spin}_+(1,3)$, and a rotor of $\mathcal{C}\ell_{1,3}{}^{+}$ can be characterized as an element of this group.

The fact that the rotors R and $-R$ produce the same rotation is an expression of the fact that $\mathrm{Spin}_+(1,3)$ is a double covering of $\mathrm{SO}_+(1,3)$ (i.e. there is a two-to-one correspondence between rotors from $\mathrm{Spin}_+(1,3)$ and special orthogonal transformations in spacetime). Since it is known that the group $\mathrm{SL}(2,\mathbb{C})$ is also a double covering of $\mathrm{SO}_+(1,3)$, one can conclude that $\mathrm{Spin}_+(1,3)$ is isomorphic to $\mathrm{SL}(2,\mathbb{C})$.

In the same way as an even grade multivector ψ from ${\mathcal{C}\ell_{3,0}}^{+}$ can be written in the form $\psi = \sqrt{\rho}R$, where $\rho \in \mathbb{R}$ and $R \in \mathrm{Spin}(3)$, an even grade multivector from ${\mathcal{C}\ell_{1,3}}^{+}$ admits also a factored form, as can be seen below. Given an even grade multivector ψ from ${\mathcal{C}\ell_{1,3}}^{+}$, it follows that $\psi\tilde{\psi}$ is also an even grade multivector, and it is equal to its reverse. Then, $\psi\tilde{\psi}$ is a scalar plus a pseudoscalar, which can be written as

$$\psi\tilde{\psi} = \rho e^{I\beta}, \tag{2.364}$$

where $\rho, \beta \in \mathbb{R}$ and $\rho > 0$. In this way, from ψ one can define an even grade multivector of unit pseudo-norm, more precisely a rotor, given by

$$R = \psi\big(\psi\tilde{\psi}\big)^{-\frac{1}{2}} = \psi\rho^{-\frac{1}{2}}e^{-\frac{1}{2}I\beta}. \tag{2.365}$$

From this expression it follows that ψ can be written as

$$\psi = \rho^{\frac{1}{2}}e^{\frac{1}{2}I\beta}R, \tag{2.366}$$

which is the desired factored form for ψ. Thus, any even grade multivector of the geometric algebra of spacetime can be factored as a geometric product of a scalar, an exponential of a pseudoscalar and a rotor.

3 Relativistic Physics in terms of Clifford Algebras

Based essentially on chapters 5 and 7 of the textbook by Doran and Lasenby (2003), this chapter is intended to serve as a concise introduction to the basics of the relativistic formalism in terms of the geometric algebra of spacetime. The focus is on the Lorentz transformations and the covariant formulation of Maxwell's equations. Any standard introduction to relativity can be considered as a background reference, e.g. the texts of Rindler (2006) and d'Inverno (1992). For the part on Maxwell's equations, the texts by Jackson (1999) and Schwinger *et al.* (1998) are taken as background references.

3.1 Preliminaries

It is natural that the three-dimensional Euclidean space is included in the Minkowski spacetime. In this way, it is very sensible to propose an inclusion of $\mathcal{C}\ell_{3,0}$ into $\mathcal{C}\ell_{1,3}$. In order to introduce and apply this inclusion, appropriate notations and conventions must be used.

3.1.1 Notation and Conventions

Regarding the notation to be used, vectors from $\mathbb{R}^3$ are represented by letters in boldface (usually lowercase, not necessarily Latin), e.g. $\mathbf{a}$, $\mathbf{b}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, etc. In particular, the canonical basis from $\mathbb{R}^3$ is represented by $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$. Vectors from $\mathbb{R}^{1,3}$ are represented by letters in normal font (usually lowercase letters, not necessarily Latin), e.g. a, b, α, β, etc. In particular, the canonical basis from $\mathbb{R}^{1,3}$ is represented by $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$. As a convention, unless specified, lowercase Greek letters (e.g. μ, ν) are used to represent indices assuming integer values from 0 to 3, and lowercase Latin letters (e.g. i, j, k) are used to represent indices assuming integer values from 1 to 3. The Einstein summation convention is also used: if an index appear twice in a term, once as subscript and once as superscript, then a summation is implied with relation to this index (e.g. the vector $a = \sum_{\mu=0}^{3} a^\mu \gamma_\mu$ can be expressed simply by $a = a^\mu \gamma_\mu$).

3.1.2 The Isomorphism $\mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,3}{}^+$

Note that the basic time-like bivectors from $\mathcal{C}\ell_{1,3}$, $\gamma_1\gamma_0$, $\gamma_2\gamma_0$ and $\gamma_3\gamma_0$, are all square roots of the unit and satisfy

$$\frac{1}{2}\Big((\gamma_i\gamma_0)(\gamma_j\gamma_0) + (\gamma_j\gamma_0)(\gamma_i\gamma_0)\Big) = \frac{1}{2}(-\gamma_i\gamma_j - \gamma_j\gamma_i) = -\gamma_i \cdot \gamma_j = \delta_{ij}, \tag{3.1}$$

Note also that

$$\begin{aligned}\frac{1}{2}\Big((\gamma_i\gamma_0)(\gamma_j\gamma_0) - (\gamma_j\gamma_0)(\gamma_i\gamma_0)\Big) &= \frac{1}{2}(-\gamma_i\gamma_j + \gamma_j\gamma_i)\\ &= \frac{1}{2}\Big(- \epsilon_{ijk} \star (\gamma_k\gamma_0) + \epsilon_{ji\ell} \star (\gamma_\ell\gamma_0)\Big)\\ &= -\epsilon_{ijk} \star (\gamma_k\gamma_0)\\ &= \epsilon_{ijk} I(\gamma_k\gamma_0) = \gamma_j \wedge \gamma_i, \end{aligned} \tag{3.2}$$

where $I = \gamma_0\gamma_1\gamma_2\gamma_3$ is the unit pseudoscalar of $\mathcal{C}\ell_{1,3}$. The sum of (3.1) and (3.2) then furnishes

$$(\gamma_i\gamma_0)(\gamma_j\gamma_0) = \delta_{ij} + \epsilon_{ijk} I(\gamma_k\gamma_0), \tag{3.3}$$

which is either a scalar or a space-like bivector. The basic space-like bivectors can be expressed as $I(\gamma_i\gamma_0)$, and from relations (3.1) and (3.2) one can write also

$$\frac{1}{2}\Big((I\gamma_i\gamma_0)(I\gamma_j\gamma_0) + (I\gamma_j\gamma_0)(I\gamma_i\gamma_0)\Big) = -\frac{1}{2}\Big((\gamma_i\gamma_0)(\gamma_j\gamma_0) + (\gamma_j\gamma_0)(\gamma_i\gamma_0)\Big) = -\delta_{ij} \tag{3.4}$$

and

$$\frac{1}{2}\Big((I\gamma_i\gamma_0)(I\gamma_j\gamma_0) - (I\gamma_j\gamma_0)(I\gamma_i\gamma_0)\Big) = -\frac{1}{2}\Big((\gamma_i\gamma_0)(\gamma_j\gamma_0) - (\gamma_j\gamma_0)(\gamma_i\gamma_0)\Big) = -\epsilon_{ijk} I(\gamma_k\gamma_0), \tag{3.5}$$

the sum of which gives

$$(I\gamma_i\gamma_0)(I\gamma_j\gamma_0) = -\delta_{ij} - \epsilon_{ijk} I(\gamma_k\gamma_0), \tag{3.6}$$

which in turn is either a scalar or a space-like bivector. Also from relations (3.1) and (3.2) one can write

$$\frac{1}{2}\Big((\gamma_i\gamma_0)(I\gamma_j\gamma_0) + (I\gamma_j\gamma_0)(\gamma_i\gamma_0)\Big) = I\frac{1}{2}\Big((\gamma_i\gamma_0)(\gamma_j\gamma_0) + (\gamma_j\gamma_0)(\gamma_i\gamma_0)\Big) = \delta_{ij} I \tag{3.7}$$

and

$$\frac{1}{2}\Big((\gamma_i\gamma_0)(I\gamma_j\gamma_0)-(I\gamma_j\gamma_0)(\gamma_i\gamma_0)\Big) = I\frac{1}{2}\Big((\gamma_i\gamma_0)(\gamma_j\gamma_0)-(\gamma_j\gamma_0)(\gamma_i\gamma_0)\Big) = -\epsilon_{ijk}(\gamma_k\gamma_0). \quad (3.8)$$

whose sum gives

$$(\gamma_i\gamma_0)(I\gamma_j\gamma_0) = \delta_{ij}I - \epsilon_{ijk}(\gamma_k\gamma_0), \quad (3.9)$$

which in turn is either a pseudoscalar or a time-like bivector. The relations (3.3), (3.6) and (3.9), in conjunction with the relations for the products involving 1 and I, determine the product of the even subalgebra $\mathcal{C}\ell_{1,3}{}^{+}$ in terms of the basic even grade multivectors.

On the other hand, the basic vectors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$ from $\mathcal{C}\ell_{3,0}$ are also found to be square roots of the unit and satisfy relations analogous to (3.1) and (3.2), that is,

$$\frac{1}{2}(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j + \boldsymbol{\sigma}_j\boldsymbol{\sigma}_i) = \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \delta_{ij} \quad (3.10)$$

and

$$\frac{1}{2}(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j\boldsymbol{\sigma}_i) = \boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j = \epsilon_{ijk}I\boldsymbol{\sigma}_k, \quad (3.11)$$

where, in this case, $I = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$ is the unit pseudoscalar from $\mathcal{C}\ell_{3,0}$. The sum of the relations (3.10) and (3.11) provides

$$\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j = \delta_{ij} + \epsilon_{ijk}I\boldsymbol{\sigma}_k, \quad (3.12)$$

which is either a scalar or a bivector. From relations (3.10) and (3.11) one can write

$$\frac{1}{2}(I\boldsymbol{\sigma}_iI\boldsymbol{\sigma}_j + I\boldsymbol{\sigma}_jI\boldsymbol{\sigma}_i) = -\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = -\delta_{ij} \quad (3.13)$$

and

$$\frac{1}{2}(I\boldsymbol{\sigma}_iI\boldsymbol{\sigma}_j - I\boldsymbol{\sigma}_jI\boldsymbol{\sigma}_i) = -\boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j = -\epsilon_{ijk}I\boldsymbol{\sigma}_k, \quad (3.14)$$

whose sum gives

$$(I\boldsymbol{\sigma}_i)(I\boldsymbol{\sigma}_j) = -\delta_{ij} - \epsilon_{ijk}I\boldsymbol{\sigma}_k, \quad (3.15)$$

which in turn is either a scalar or a bivector. Also from relations (3.10) and (3.11) one can write

$$\frac{1}{2}(\boldsymbol{\sigma}_iI\boldsymbol{\sigma}_j + I\boldsymbol{\sigma}_j\boldsymbol{\sigma}_i) = I(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) = \delta_{ij}I \quad (3.16)$$

and

$$\frac{1}{2}(\boldsymbol{\sigma}_iI\boldsymbol{\sigma}_j - I\boldsymbol{\sigma}_j\boldsymbol{\sigma}_i) = I(\boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j) = -\epsilon_{ijk}\boldsymbol{\sigma}_k, \quad (3.17)$$

whose sum gives

$$(\boldsymbol{\sigma}_i)(I\boldsymbol{\sigma}_j) = \delta_{ij}I - \epsilon_{ijk}\boldsymbol{\sigma}_k, \quad (3.18)$$

which in turn is either a pseudoscalar or a vector. The relations (3.12), (3.15) and (3.18), in conjunction with the relations for the products involving 1 and I, determine the product of the algebra $\mathcal{C}\ell_{3,0}$. It is remarkable that these relations are identical to (3.3), (3.6) and (3.9), respectively, if one replaces $\boldsymbol{\sigma}_i$ by $\gamma_i\gamma_0$. In addition, note that this replacement implies

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 = (\gamma_1\gamma_0)(\gamma_2\gamma_0)(\gamma_3\gamma_0) = \gamma_0\gamma_1\gamma_2\gamma_3, \tag{3.19}$$

which is compatible with the correspondence between the pseudoscalars from both sets of relations. The conclusion is that the geometric algebra of the three-dimensional Euclidean space is isomorphic to the even subalgebra of the algebra of Minkowski spacetime through the identification of $\boldsymbol{\sigma}_i$ with $\gamma_i\gamma_0$, and through the identification of the units and the geometric products of both. This fact is denoted by $\mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,3}{}^+$. This isomorphism determines the inclusion of $\mathcal{C}\ell_{3,0}$ into $\mathcal{C}\ell_{1,3}$.

In applications involving the geometric algebra of spacetime, it is common practice to use the isomorphism $\mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,3}{}^+$ and set $\boldsymbol{\sigma}_i = \gamma_i\gamma_0$, in addition to the use of the same notation for the pseudoscalars from the equivalent algebras. In this context, the time-like bivectors, which correspond to three-dimensional vectors, are denoted as such, in boldface, and scalar and exterior products involving only vectors denoted in boldface correspond to the three-dimensional Euclidean scalar and exterior products. Otherwise the products are interpreted as those from the geometric algebra of spacetime.

According to the isomorphism $\mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,3}{}^+$, scalars from $\mathcal{C}\ell_{3,0}$ are mapped to scalars from $\mathcal{C}\ell_{1,3}{}^+$, vectors from $\mathcal{C}\ell_{3,0}$ are mapped to time-like bivectors from $\mathcal{C}\ell_{1,3}{}^+$, bivectors from $\mathcal{C}\ell_{3,0}$ are mapped to space-like bivectors from $\mathcal{C}\ell_{1,3}{}^+$, and pseudoscalars from $\mathcal{C}\ell_{3,0}$ are mapped to pseudoscalars from $\mathcal{C}\ell_{1,3}{}^+$. In this respect, one notes that the reversion operation for $\mathcal{C}\ell_{3,0}$ is not coincident to that for $\mathcal{C}\ell_{1,3}{}^+$. The three-dimensional Euclidean reversion operation is then denoted by a superscript dagger symbol, e.g. $A^\dagger$, while the spacetime reversion continues to be denoted by an overwritten tilde symbol, e.g. $\tilde{A}$.

3.2 Relativistic Observables

As outlined before (cf. subsection 2.3.3), adopted a canonical reference frame, determined by the canonical basis of $\mathbb{R}^{1,3}$, the trajectory of a particle with non-null mass in spacetime is a time-like curve, that is, a curve with a time-like tangent vector. The trajectory of light (or a massless particle, in general) is a light-like curve. Since two events with a space-like separation do not have a causal connection, space-like curves cannot represent trajectories of known particles. These facts are consequences of the two postulates of the special theory of relativity, the *principle of relativity* and the *invariance of the speed of light c*. Thus, all of the above applies only if the canonical reference frame adopted complies with the principle of inertia of classical mechanics.

A natural parameterization for a curve is made through its length ℓ, since a curve parameterized by its length has unit tangent vector. For a time-like curve, one writes $\ell = c\tau$, where the new parameter τ corresponds to the elapsed time for an observer following this curve, the proper time associated to the curve. For a massive particle following a curve $x = x(\tau)$, the *spacetime velocity* $v = v(\tau)$ is defined by

$$v = \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}\tau}, \tag{3.20}$$

where the dot denotes the derivative relative to the proper time. It follows that the spacetime velocity for such a particle is proportional to the unit tangent vector, with proportionality constant c, so that

$$v^2 = c^2. \tag{3.21}$$

Since different time-like curves can have different lengths, particles in relative motion experience time elapsing differently.

An observer is defined by a time-like curve parameterized by its proper time and oriented to the future. An inertial observer, in particular, is given by a time-like straight line. Such an observer has constant spacetime velocity. In this way, an inertial observer can construct a reference frame given by a basis $\{e_0, e_1, e_2, e_3\}$, where $e_0 = \hat{v} = v/c$, with v being the spacetime velocity for the observer, and $\{e_1, e_2, e_3\}$ a set of orthogonal unit space-like vectors mutually orthogonal to e_0 and whose implied orientation follows the right-hand convention. The *reciprocal basis* $\{e^0, e^1, e^2, e^3\}$ is defined by

$$e^\mu \cdot e_\nu = \delta^\mu{}_\nu, \tag{3.22}$$

in such way that

$$e^0 = e_0 \quad \text{and} \quad e^i = -e_i. \tag{3.23}$$

Therefore, if an event is given by a vector x, its coordinates relative to the reference frame constructed by the above inertial observer are

$$x^\mu = x \cdot e^\mu \quad \text{and} \quad x_\mu = x \cdot e_\mu, \tag{3.24}$$

and the event can be expressed in terms of the basis constructed by the inertial observer and its reciprocal basis as

$$x = x^\mu e_\mu = (x \cdot e^\mu) e_\mu \quad \text{and} \quad x = x_\mu e^\mu = (x \cdot e_\mu) e^\mu. \tag{3.25}$$

Such an event can then be written

$$x = cte_0 + x^i e_i, \tag{3.26}$$

where $ct = x^0 = x_0$. Its space-like component can be written as

$$x^i e_i = x - (x \cdot e^0)e_0 = x\hat{v}\hat{v} - (x \cdot \hat{v})\hat{v} = (x\hat{v} - x \cdot \hat{v})\hat{v} = (x \wedge \hat{v})\hat{v}. \tag{3.27}$$

The time-like bivector $x \wedge \hat{v}$ corresponds to a vector in the three-dimensional Euclidean *rest space* of the inertial observer of spacetime velocity v, defining the *relative position* of the event x, which is denoted by

$$\mathbf{x} = x \wedge \hat{v}. \tag{3.28}$$

This correspondence is explained by the fact that an inertial observer with spacetime velocity v "splits" the Minkowski spacetime in two subspaces, "time" and "space", the first generated by its spacetime velocity, while the second is the orthogonal complement of the first and corresponds to the hyperplane orthogonal to the vector v. A vector $x^i e_i = (x \wedge \hat{v})\hat{v}$ from the hyperplane orthogonal to v is identified with the time-like bivector $x \wedge \hat{v}$, which according to the ismorphism $\mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,3}{}^{+}$, corresponds to the three-dimensional vector $\mathbf{x}$ representing the three-dimensional position of the event x according to the above mentioned inertial observer. For this observer, the time of the event x is $ct = x \cdot \hat{v}$. Note then that

$$x\hat{v} = x \cdot \hat{v} + x \wedge \hat{v} = ct + \mathbf{x}, \tag{3.29}$$

which allows one write the magnitude of x as

$$x^2 = x\hat{v}\hat{v}x = (x \cdot \hat{v} + x \wedge \hat{v})(x \cdot \hat{v} + \hat{v} \wedge x) = (ct + \mathbf{x})(ct - \mathbf{x}) = c^2t^2 - \mathbf{x}^2. \tag{3.30}$$

This corresponds to the invariant interval between x and the origin in terms of the time and distance as measured by the inertial observer with spacetime velocity v. Another inertial observer, with a different spacetime velocity, performs a different "split" of spacetime, so that it expresses the event x in a different, but equivalent way, in such a manner that the measured interval is the same.

Given an inertial observer with spacetime velocity v and a massive particle following a trajectory given by $x = x(\tau)$, with a spacetime velocity $u = u(\tau) = \dot{x}$, where τ is the proper time of the particle, one has

$$u\hat{v} = \frac{\mathrm{d}}{\mathrm{d}\tau}(x\hat{v}) = \frac{\mathrm{d}}{\mathrm{d}\tau}(ct + \mathbf{x}), \tag{3.31}$$

where $x \cdot \hat{v} = ct$ and $\mathbf{x} = x \wedge \hat{v}$ correspond to the relative time and position for the particle as measured by the observer. In this way, it follows that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \hat{u} \cdot \hat{v} \quad \text{and} \quad \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = u \wedge \hat{v}. \tag{3.32}$$

The *relative velocity* $\mathbf{u}$ of the particle, as measured by the inertial observer of spacetime velocity v, is therefore given by

$$\mathbf{u} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{u \wedge \hat{v}}{\hat{u} \cdot \hat{v}} = c\frac{\hat{u} \wedge \hat{v}}{\hat{u} \cdot \hat{v}}. \tag{3.33}$$

From this expression, it follows that

$$\begin{aligned}\left(\frac{\mathbf{u}}{c}\right)^2 = \frac{(\hat{u} \wedge \hat{v})^2}{(\hat{u} \cdot \hat{v})^2} &= -\frac{(\hat{u} \wedge \hat{v})(\hat{v} \wedge \hat{u})}{(\hat{u} \cdot \hat{v})^2} \\ &= -\frac{(\hat{u}\hat{v} - \hat{u} \cdot \hat{v})(\hat{v}\hat{u} - \hat{u} \cdot \hat{v})}{(\hat{u} \cdot \hat{v})^2} \\ &= -\frac{\hat{u}\hat{v}\hat{v}\hat{u} - (\hat{u}\hat{v} + \hat{v}\hat{u})(\hat{u} \cdot \hat{v}) + (\hat{u} \cdot \hat{v})^2}{(\hat{u} \cdot \hat{v})^2} \\ &= -\frac{1 - 2(\hat{u} \cdot \hat{v})(\hat{u} \cdot \hat{v}) + (\hat{u} \cdot \hat{v})^2}{(\hat{u} \cdot \hat{v})^2} \\ &= 1 - \frac{1}{(\hat{u} \cdot \hat{v})^2} \\ &< 1, \end{aligned} \tag{3.34}$$

that is, the relative velocity of a massive particle as measured by an inertial observer has magnitude less than the speed of light. The Lorentz factor γ is given by

$$\gamma^2 = \frac{1}{1 - \left(\frac{\mathbf{u}}{c}\right)^2} = \frac{1}{1 - \left(1 - \frac{1}{(\hat{u}\cdot\hat{v})^2}\right)} = (\hat{u} \cdot \hat{v})^2. \tag{3.35}$$

The spacetime velocity of the particle can then be written

$$u = u\hat{v}\hat{v} = (u \cdot \hat{v} + u \wedge \hat{v})\,\hat{v} = c\gamma\hat{v} + \gamma\mathbf{u}\hat{v}, \tag{3.36}$$

which is the sum of a component along the spacetime velocity v of the observer and a component belonging to the hyperplane orthogonal to v.

If the above considered massive particle has mass m, its *spacetime momentum* or *energy-momentum* is defined by

$$p = mu. \tag{3.37}$$

The inertial observer with spacetime velocity v measures the energy and momentum for the particle as

$$\frac{E}{c} = p \cdot \hat{v} \quad \text{and} \quad \mathbf{p} = p \wedge \hat{v}. \tag{3.38}$$

The energy-momentum of the particle can then be written

$$p = p\hat{v}\hat{v} = (p \cdot \hat{v} + p \wedge \hat{v})\,\hat{v} = \frac{E}{c}\hat{v} + \mathbf{p}\hat{v}, \tag{3.39}$$

which is the sum of a component along the spacetime velocity v of the observer and a component belonging to the hyperplane orthogonal to v. Note that the magnitude of the energy-momentum of the particle is

$$p^2 = (mu)^2 = m^2c^2, \tag{3.40}$$

which in terms of the energy and momentum as measured by the observer is given by

$$p^2 = p\hat{v}\hat{v}p = (p \cdot \hat{v} + p \wedge \hat{v})(p \cdot \hat{v} + \hat{v} \wedge p) = \left(\frac{E}{c} + \mathbf{p}\right)\left(\frac{E}{c} - \mathbf{p}\right) = \frac{E^2}{c^2} - \mathbf{p}^2. \tag{3.41}$$

Thus, it follows that

$$\frac{E^2}{c^2} - \mathbf{p}^2 = m^2c^2. \tag{3.42}$$

The *spacetime acceleration* of the considered particle is defined by

$$\dot{u} = \frac{\mathrm{d}u}{\mathrm{d}\tau}. \tag{3.43}$$

It follows that spacetime acceleration and velocity are orthogonal:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(u^2\right) = 2\dot{u} \cdot u. \tag{3.44}$$

A useful concept is that of the *acceleration bivector*, given by

$$B_u = \dot{u} \wedge u = \dot{u}u, \tag{3.45}$$

which corresponds to the three-dimensional acceleration of the particle relative to its own instantaneous reference frame.

3.3 Lorentz Transformations

3.3.1 Lorentz Boosts

Consider two inertial observers, each with its reference frame (constructed as described in the previous section), given by the bases $\{e_0, e_1, e_2, e_3\}$ and $\{e'_0, e'_1, e'_2, e'_3\}$. A generic event x has the respective coordinates

$$x^\mu = x \cdot e^\mu \quad \text{and} \quad x'^\mu = x \cdot e'^\mu \tag{3.46}$$

relative to these reference frames. If the inertial observers construct their reference frames in such way that $x^1 = x'^1$ and $x^2 = x'^2$ for any event, and so that for $(x^1, x^2, x^3) =$

$(x'^1, x'^2, x'^3) = (0,0,0)$, one has $ct = x^0 = ct' = x'^0 = 0$, then the coordinates of a generic event according to these inertial observers are related by the coordinate transformation

$$ct' = \gamma(ct - \beta x^3), \quad x'^1 = x^1, \quad x'^2 = x^2, \quad x'^3 = \gamma(x^3 - \beta ct), \tag{3.47}$$

where β is the speed of the second observer relative to the first, in units of the speed of light, and $\gamma = e_0 \cdot e'_0 = (1-\beta^2)^{-1/2}$ is the Lorentz factor. Such a transformation is called a *Lorentz boost*, or simply *boost*. The inverse transformation is given by

$$ct = \gamma(ct' + \beta x'^3), \quad x^1 = x'^1, \quad x^2 = x'^2, \quad x^3 = \gamma(x'^3 + \beta ct'). \tag{3.48}$$

Since the generic event x can be expressed by

$$x = x^\mu e_\mu = x'^\mu e'_\mu, \tag{3.49}$$

the inverse coordinate transformation, taking into account that $e'_1 = e_1$ and $e'_2 = e_2$, imply:

$$\begin{aligned} ct'e'_0 + x'^3 e'_3 &= cte_0 + x^3 e_3 \\ &= \gamma(ct' + \beta x'^3)e_0 + \gamma(x'^3 + \beta ct')e_3 \\ &= ct'\Big(\gamma(e_0 + \beta e_3)\Big) + x'^3\Big(\gamma(\beta e_0 + e_3)\Big). \end{aligned} \tag{3.50}$$

This relation in turn implies

$$e'_0 = \gamma(e_0 + \beta e_3) \quad \text{and} \quad e'_3 = \gamma(\beta e_0 + e_3), \tag{3.51}$$

which express the considered Lorentz boost in terms of a reference frame transformation. The relation

$$\gamma^2(1-\beta^2) = 1 \tag{3.52}$$

suggests a parameterization of the Lorentz boost in terms of a parameter α such that

$$\gamma = \cosh(\alpha) \quad \text{and} \quad \gamma\beta = \sinh(\alpha). \tag{3.53}$$

In this way, one can write:

$$\begin{aligned} e'_0 &= \cosh(\alpha)e_0 + \sinh(\alpha)e_3 \\ &= \Big(\cosh(\alpha) + \sinh(\alpha)e_3e_0\Big)e_0 \\ &= \exp(\alpha e_3 e_0)e_0 \end{aligned} \tag{3.54}$$

and

$$\begin{aligned} e'_3 &= \sinh(\alpha)e_0 + \cosh(\alpha)e_3 \\ &= \Big(\cosh(\alpha) + \sinh(\alpha)e_3e_0\Big)e_3 \\ &= \exp(\alpha e_3 e_0)e_3. \end{aligned} \tag{3.55}$$

But, both e_0 and e_3 anticommutes with the bivector e_3e_0, which allows one write also:

$$e'_0 = \exp\left(\frac{1}{2}\alpha e_3 e_0\right) e_0 \exp\left(-\frac{1}{2}\alpha e_3 e_0\right) \tag{3.56}$$

and

$$e'_3 = \exp\left(\frac{1}{2}\alpha e_3 e_0\right) e_3 \exp\left(-\frac{1}{2}\alpha e_3 e_0\right). \tag{3.57}$$

Since both e_1 and e_2 commutes with the bivector e_3e_0, one can express the considered Lorentz boost in a general way by

$$e'_\mu = Re_\mu\tilde{R}, \tag{3.58}$$

where

$$R = \exp\left(\frac{1}{2}\alpha e_3 e_0\right). \tag{3.59}$$

It is immediate that the considered Lorentz boost is precisely the hyperbolic rotation of the basis $\{e_\mu\}$ through the time-like plane e_3e_0 by a hyperbolic angle α in the sense of increasing hyperbolic angle. Since the relative speed $\beta = \tanh(\alpha)$ increases with the increase of α and $\beta \to 1$ as $\alpha \to \infty$, the hyperbolic angle in this context is usually called *rapidity*.

3.3.2 The Lorentz Group

A *restricted Lorentz transformation* is the composition of a Lorentz boost and a spatial rotation. Since a Lorentz boost corresponds to a hyperbolic rotation, a restricted Lorentz transformation corresponds to a general rotation in spacetime. In this way, the group formed by restricted Lorentz transformations, called the *restricted Lorentz group*, it is found correspond to the group $\mathrm{SO}_+(1,3)$. Since $\mathrm{Spin}_+(1,3)$ is a double covering of $\mathrm{SO}_+(1,3)$, the restricted Lorentz group can be represented by $\mathrm{Spin}_+(1,3)$, and a restricted Lorentz transformation can be expressed by

$$v \;\mapsto\; v' = Rv\tilde{R}, \quad \text{where} \quad R \in \mathrm{Spin}_+(1,3). \tag{3.60}$$

A restricted Lorentz transformation is also known as a *proper orthochronous Lorentz transformation*, because it is both a *time-order-preserving* transformation and a *parity-preserving* transformation, that is, a transformation preserving both the orientation of time-like components of vectors (future or past) and the orientation of a three-dimensional volume element. A reflection through a hyperplane with time-like orthogonal vector, also called a *time reversal*, is a *time-reversing* transformation and is also a parity-preserving transformation, e.g. $I \mapsto -\gamma_0 I \gamma_0 = I$. Conversely, a reflection through a hyperplane with space-like orthogonal vector is a time-order-preserving transformation, but a non-parity-preserving transformation — the composition of three such reflections, in non-coplanar directions, is called a *parity inversion*, and is also a time-order-preserving but non-parity-preserving transformation, e.g. $I \mapsto \gamma_0 I \gamma_0 = -I$. The composition of a proper orthochronous Lorentz transformation with a time reversal is a time-reversing and parity-preserving transformation. On the other hand, the composition of a proper orthochronous Lorentz transformation with a parity inversion is a time-order-preserving and non-parity-preserving transformation. In turn, the composition of a proper orthochronous Lorentz transformation with a time reversal and a parity inversion is a time-reversing and non-parity-preserving transformation. All transformations considered preserve distances and angles in Minkowski spacetime and compose a group of transformations called the *Lorentz group*. As described above, the Lorentz group has four sectors, which are summarized in the table 3.1. The sector of proper orthocrhonous transformations, corresponding to the subgroup of restricted Lorentz transformations containing the identity transformation, is the most relevant in physics and is often called itself the Lorentz group.

	parity-preserving	non-parity-preserving
time-order-preserving	proper orthochronous (PO)	PO with a parity inversion
time-reversing	PO with a time reversal	PO with $a \mapsto -a$

TABLE 3.1 – The four sectors of the Lorentz group.

3.3.3 Invariant Decomposition of a Rotor

Any rotor from $\mathcal{G}(\mathbb{R}^{1,3})$ can be written in terms of a bivector B in the form

$$R = \pm e^{\frac{1}{2}B}. \tag{3.61}$$

If the bivector B is non-null, that is $B^2 \neq 0$, and since $B^2 = \widetilde{B^2}$, one can write

$$B^2 = \langle B^2 \rangle_0 + \langle B^2 \rangle_4 = \rho e^{I\phi}, \tag{3.62}$$

where ρ and ϕ are scalars and $\rho \neq 0$. Consider then the bivector

$$\hat{B} = \rho^{-\frac{1}{2}} e^{-\frac{1}{2}I\phi} B, \tag{3.63}$$

so that

$$\hat{B}^2 = \left(\rho^{-\frac{1}{2}} e^{-\frac{1}{2}I\phi} B\right)\left(\rho^{-\frac{1}{2}} e^{-\frac{1}{2}I\phi} B\right) = \rho^{-1} e^{-I\phi} B^2 = 1. \tag{3.64}$$

Therefore, the bivector B can be written in terms of the unit time-like bivector $\hat{B}$ as

$$B = \rho^{\frac{1}{2}} e^{\frac{1}{2}I\phi} \hat{B}, \tag{3.65}$$

that is,

$$B = \alpha \hat{B} + \beta I \hat{B}, \tag{3.66}$$

where $\alpha = \rho^{\frac{1}{2}} \cos\left(\frac{1}{2}\phi\right)$ and $\beta = \rho^{\frac{1}{2}} \sin\left(\frac{1}{2}\phi\right)$. Since

$$\hat{B}\left(I\hat{B}\right) = \left(I\hat{B}\right)\hat{B} = I, \tag{3.67}$$

the rotor $R = \pm e^{\frac{1}{2}B}$ can be decomposed as

$$R = e^{\frac{1}{2}\alpha\hat{B}} e^{\frac{1}{2}\beta I\hat{B}} = e^{\frac{1}{2}\beta I\hat{B}} e^{\frac{1}{2}\alpha\hat{B}}. \tag{3.68}$$

This is an invariant decomposition of the rotor R into a boost, generated by $\hat{B}$, and a spatial rotation, generated by $I\hat{B}$.

3.3.4 Observer-Dependent Decomposition of a Rotor

Given spacetime velocities u and v, the rotor L transforming u into v through a pure boost,

$$v = Lu\tilde{L}, \tag{3.69}$$

is necessarily generated by the unit time-like bivector determined by u and v,

$$\frac{v \wedge u}{|v \wedge u|}, \tag{3.70}$$

where $|v \wedge u|$ is an abbreviated manner to write $\sqrt{-\|v \wedge u\|_h{}^2}$. The rotor L can then be written as

$$L = \exp\left(\frac{1}{2}\alpha \frac{v \wedge u}{|v \wedge u|}\right), \tag{3.71}$$

where α is the hyperbolic angle between u and v. It is also possible to determine the rotor L transforming u into v through a pure boost, taking into account that such a boost can be decomposed into two reflections, first the reflection through the hyperplane orthogonal to the vector $w = (u+v)/|u+v|$, then the reflection through the hyperplane orthogonal to the vector v. The resulting transformation is given by

$$u \mapsto (-v(-wuw)v) = vwuwv, \tag{3.72}$$

so that the rotor L can be written

$$L = vw = v\left(\frac{u+v}{|u+v|}\right) = \frac{vu+1}{\sqrt{(u+v)\cdot(u+v)}}, \tag{3.73}$$

that is,

$$L = \frac{1+vu}{\sqrt{2(1+u\cdot v)}}. \tag{3.74}$$

This is not the unique rotor performing the required boost, the rotor

$$-vw = -\frac{1+vu}{\sqrt{2(1+u\cdot v)}} \tag{3.75}$$

also performs the same boost.

It is natural to question the form for a general rotor transforming u into v. For simplicity, set $u = \gamma_0$. The pure boost for this transformation can be taken as given by the rotor

$$L = \frac{1+v\gamma_0}{\sqrt{2(1+v\cdot\gamma_0)}} = \exp\left(\frac{1}{2}\alpha\frac{v\wedge\gamma_0}{|v\wedge\gamma_0|}\right). \tag{3.76}$$

Then, one can define the rotor U given by

$$U = \tilde{L}R, \tag{3.77}$$

where R is the general rotor desired. Note that U satisfies

$$U\tilde{U} = \tilde{L}R\tilde{R}L = 1, \tag{3.78}$$

as required for a rotor. The rotor U also satisfies

$$U\gamma_0\tilde{U} = \tilde{L}R\gamma_0\tilde{R}L = \tilde{L}vL = \gamma_0, \tag{3.79}$$

that is, U commutes with γ_0 and its action has no effect under γ_0. Thus, one must have

$$U = \exp\left(\frac{1}{2}I\mathbf{b}\right), \tag{3.80}$$

where $\mathbf{b}$ is relative vector according to an inertial observer with spacetime velocity γ_0, so that $I\mathbf{b}$ is a space-like bivector generating pure spatial rotations in the reference frame constructed by its observer. The rotor U is then a pure spatial rotation and the general rotor R can be written

$$R = LU. \tag{3.81}$$

Differently from the invariant decomposition given by (3.68), the rotors L and U do not commute in general. Since the rotor U is constructed in such way that its action does not affect the velocity γ_0 of the observer, this decomposition is observer-dependent.

3.4 Maxwell's Equations

Classical electrodynamics is founded in *Maxwell's equations*, which in terms of SI units are usually written as:

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho, \qquad & \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, \qquad & \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= \mathbf{J}. \end{aligned} \tag{3.82}$$

In these equations, ρ and $\mathbf{J}$ are the *free electric charge density* and *free electric current density*, $\mathbf{E}$ and $\mathbf{B}$ are the *electric field* and *magnetic induction field*, $\mathbf{D}$ and $\mathbf{H}$ the *electric displacement field* and the *magnetic field*. The latter two quantities are defined by

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad \text{and} \qquad \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}, \tag{3.83}$$

where the constants ϵ_0 and μ_0 are the *electric permittivity* and *magnetic permeability* of vacuum, $\mathbf{P}$ is the *electric polarization field* (the electric dipole moment density) and $\mathbf{M}$ is the *magnetization field* (the magnetic dipole moment density). In general, Maxwell's equations must be complemented with constitutive relations which relate $\mathbf{P}$ and $\mathbf{M}$ to $\mathbf{E}$ and $\mathbf{B}$, or equivalently, relate $\mathbf{D}$ and $\mathbf{H}$ to $\mathbf{E}$ and $\mathbf{B}$. In some applications, additional constitutive relations may be needed. Also essential, especially for description of the motion of electric charged particles, is the *Lorentz force law*,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{3.84}$$

which gives the force acting on a particle of electric charge q and velocity $\mathbf{v}$ in the presence of electromagnetic fields.

In the vacuum, the polarization fields are null, the free charge and current densities correspond to the total charge and current densities, and the macroscopic fields are given by $\mathbf{D} = \epsilon_0 \mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu_0$. In this case, defining $c^2 = 1/(\mu_0 \epsilon_0)$, Maxwell's equations can be written as:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{B} &= \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}. \end{aligned} \tag{3.85}$$

Using the relation $\nabla \times \mathbf{a} = -I(\nabla \wedge \mathbf{a})$, the equations can be rewritten as:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, & \nabla \wedge \mathbf{E} &= -\frac{\partial}{\partial t}(I\mathbf{B}), \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \wedge \mathbf{B} &= I\left(\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}\right). \end{aligned} \tag{3.86}$$

Summing the first and the second pair of the above equations gives:

$$\begin{aligned} \nabla \mathbf{E} &= \frac{1}{\epsilon_0}\rho - \frac{\partial}{\partial t}(I\mathbf{B}), \\ \nabla \mathbf{B} &= I\left(\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}\right). \end{aligned} \tag{3.87}$$

This pair of equations can be rewritten as:

$$\begin{aligned} \nabla\left(\frac{1}{c}\mathbf{E}\right) &= \frac{1}{c\epsilon_0}\rho - \frac{1}{c}\frac{\partial}{\partial t}(I\mathbf{B}), \\ \nabla(I\mathbf{B}) &= -\mu_0 \mathbf{J} - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\mathbf{E}\right). \end{aligned} \tag{3.88}$$

This new pair of equations can be combined to form

$$\nabla\left(\frac{1}{c}\mathbf{E} + I\mathbf{B}\right) + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\mathbf{E} + I\mathbf{B}\right) = \mu_0(c\rho - \mathbf{J}). \tag{3.89}$$

The equality between even grade terms and the equality between odd grade terms (vectors, in this case) in this equation imply that the equations (3.88) are equivalent to equation (3.89). This resulting equation is entirely written in terms of the geometric algebra of three-dimensional Euclidean space, $\mathcal{C}\ell_{3,0}$. But there is the isomorphism between this algebra and the even subalgebra of spacetime algebra: $\mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,3}{}^{+}$. This isomorphism consists in the correspondence between vectors from $\mathcal{C}\ell_{3,0}$ and time-like bivectors from $\mathcal{C}\ell_{1,3}{}^{+}$, and between bivectors from $\mathcal{C}\ell_{3,0}$ and space-like bivectors from $\mathcal{C}\ell_{1,3}{}^{+}$, in addition to the correspondence between scalars from both algebras and the correspondence between pseudoscalars from both algebras. In terms of the spacetime algebra, the multivector

$\mathbf{E}/c + I\mathbf{B}$ in the equation (3.89) is a bivector, which is denoted by F and called the *Faraday bivector*,

$$F = \frac{1}{c}\mathbf{E} + I\mathbf{B}. \tag{3.90}$$

The Faraday bivector represents the covariant form of the *electromagnetic field strength*. The equation (3.89) can now be written

$$\boldsymbol{\nabla} F + \frac{1}{c}\frac{\partial}{\partial t}F = \mu_0(c\rho - \mathbf{J}) \tag{3.91}$$

and be understood in terms of the spacetime algebra. In order to write this equation in manifestly Lorentz covariant form, consider that the quantities present in it reflect measurements performed by an observer which, having normalized spacetime velocity γ_0, employs a spacetime reference frame represented by a basis $\{\gamma_\mu\}$ of Minkowski spacetime, satisfying

$$\gamma_0 \cdot \gamma_0 = 1, \qquad \gamma_0 \cdot \gamma_i = 0, \qquad \text{and} \qquad \gamma_i \cdot \gamma_j = -\delta_{ij}, \qquad \text{for} \qquad i, j \in \{1, 2, 3\}. \tag{3.92}$$

Consider also that this basis has reciprocal basis $\{\gamma^\mu\}$ (that is, the basic vectors γ^μ are related to those of the first considered basis by $\gamma^\mu \cdot \gamma_\nu = \delta^\mu_\nu$). In terms of these basis, the spacetime vector derivative can be written

$$\nabla = \gamma^\mu \partial_\mu = \gamma^\mu \frac{\partial}{\partial x^\mu}, \tag{3.93}$$

where $x^\mu = x \cdot \gamma^\mu$ are the coordinates of the spacetime position vector x, as measured by the above considered observer. The three-dimensional vector derivative $\boldsymbol{\nabla}$ in the equation (3.91) can then be written

$$\boldsymbol{\nabla} = \boldsymbol{\sigma}_i \partial_i = \boldsymbol{\sigma}_i \frac{\partial}{\partial x^i}, \tag{3.94}$$

where the spacetime time-like bivectors $\boldsymbol{\sigma}_i = \gamma_i\gamma_0$ correspond algebraically to the three-dimensional basic vectors defining the three-dimensional orthonormal basis employed by the observer to measure the relative quantities represented in his/her three-dimensional rest space. Now, observe that the geometric product of the spacetime derivative with γ_0, to the left, gives

$$\gamma_0\nabla = \gamma_0\gamma^0\partial_0 + \gamma_0\gamma^i\partial_i = \gamma_0\gamma_0\partial_0 - \gamma_0\gamma_i\partial_i = \partial_0 + \boldsymbol{\sigma}_i\partial_i = \frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\nabla}. \tag{3.95}$$

Therefore, the equation (3.91) can be written:

$$\gamma_0 \nabla F = \mu_0(c\rho - \mathbf{J}). \tag{3.96}$$

Finally, introduction of the *spacetime electric current density* J, which is a spacetime vector related to the charge density ρ and three-dimensional current density $\mathbf{J}$ by

$$J \cdot \gamma_0 = c\rho \qquad \text{and} \qquad J \wedge \gamma_0 = \mathbf{J}, \tag{3.97}$$

furnishes

$$c\rho - \mathbf{J} = J \cdot \gamma_0 - J \wedge \gamma_0 = \gamma_0 \cdot J + \gamma_0 \wedge J = \gamma_0 J. \tag{3.98}$$

Thus, from this equation, using the fact that the geometric product with γ_0 is invertible, the equation (3.96) can be written

$$\nabla F = \mu_0 J. \tag{3.99}$$

This is the representation of Maxwell's equations for fields in vacuum in the geometric algebra approach, which is Lorentz covariant as desired. From (3.99) it follows that

$$\nabla^2 F = \mu_0 \nabla J = \mu_0 \nabla \cdot J + \mu_0 \nabla \wedge J. \tag{3.100}$$

The scalar part of this equation is

$$\nabla \cdot J = 0, \tag{3.101}$$

or, equivalently,

$$0 = \nabla \cdot J = \langle \nabla J \rangle = \langle \gamma_0 \nabla J \gamma_0 \rangle = \left\langle \left(\frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\nabla} \right) (c\rho + \mathbf{J}) \right\rangle = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J}, \tag{3.102}$$

which is the continuity equation expressing charge conservation, implicit in Maxwell's equations.

3.4.1 Relationship with the Component-Based Version

Equation $\nabla F = \mu_0 J$ can be split into

$$\nabla \cdot F = \mu_0 J \qquad \text{and} \qquad \nabla \wedge F = 0. \tag{3.103}$$

These correspond to the tensor equations

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \qquad \text{and} \qquad \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0. \tag{3.104}$$

In these equations, $\partial_\mu = \gamma_\mu \cdot \nabla$ and $J^\mu = \gamma^\mu \cdot J$ are respectively the components of the spacetime vector derivative and spacetime current density relative to the reference frame given by the basis $\{\gamma_\mu\}$ and its reciprocal basis $\{\gamma^\mu\}$, $\epsilon^{\mu\nu\rho\sigma}$ represents the totally antisymmetric symbol of rank 4, and $F_{\mu\nu}$ and $F^{\mu\nu}$ are the components of the electromagnetic field strength in terms of the basis $\{\gamma_\mu\}$ and its reciprocal. These field components can be obtained from the Faraday bivector through

$$F_{\mu\nu} = \gamma_\nu \cdot (\gamma_\mu \cdot F) = (\gamma_\nu \wedge \gamma_\mu) \cdot F \qquad \text{and} \qquad F^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot F) = (\gamma^\nu \wedge \gamma^\mu) \cdot F. \tag{3.105}$$

In the usual matrix form, these components are represented by

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{3.106}$$

and

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}, \tag{3.107}$$

where is used $F_{0i} = -F^{0i} = E^i/c$ and $F_{ij} = F^{ij} = -\epsilon_{ijk}B^k$, with $i, j, k \in \{1, 2, 3\}$, and with $(E^1, E^2, E^3) = (E_x, E_y, E_z)$ and $(B^1, B^2, B^3) = (B_x, B_y, B_z)$.

3.4.2 Vector Potential

As seen above, equation $\nabla F = \mu_0 J$ can be split into

$$\nabla \cdot F = \mu_0 J \qquad \text{and} \qquad \nabla \wedge F = 0. \tag{3.108}$$

From the fact that $\nabla \wedge \nabla \wedge M = 0$ for any multivector field M, the second equation above is automatically satisfied if F is written as

$$F = \nabla \wedge A, \tag{3.109}$$

where A is a vector field. This vector field is known as the *vector potential.* An observer with normalized spacetime velocity γ_0 measures the vector potential A split into a *scalar potential* ϕ given by $\phi/c = A \cdot \gamma_0$ and a *three-dimensional vector potential* $\mathbf{A} = A \wedge \gamma_0$.

Note that, A is defined modulo the gradient of a scalar field λ:

$$\nabla \wedge (A + \nabla\lambda) = \nabla \wedge (A + \nabla \wedge \lambda) = \nabla \wedge A + \nabla \wedge \nabla \wedge \lambda = F. \tag{3.110}$$

For historical reasons, this freedom in defining the vector potential is known as a *gauge* *freedom*. This residual freedom can be eliminated in order to obtain a wave equation for A. This process is usually called a *gauge fixing*, and it is also necessary for the quantization of the field. From the first equation in (3.108),

$$\nabla \cdot (\nabla \wedge A) = \nabla^2 A + \nabla(\nabla \cdot A) = \mu_0 J, \tag{3.111}$$

so, a natural way to fix the gauge of the vector potential is to impose that

$$\nabla \cdot A = 0, \tag{3.112}$$

in such way that

$$F = \nabla A \qquad \text{and} \qquad \nabla^2 A = \mu_0 J. \tag{3.113}$$

Equations (3.113) provide a way to solve Maxwell's equations: solve the associated wave equation for A and then compute F from $F = \nabla A$. Equation (3.112), which is known as the *Lorenz gauge condition*, does not totally specify A, but the remaining freedom can be eliminated by imposing appropriate boundary conditions to the problem.

3.4.3 Electromagnetic Field Transformation

As seen above, in the geometric algebra approach, the electromagnetic field strength is represented by a Lorentz covariant bivector, the Faraday bivector, which has the standard form

$$F = \frac{1}{c}\mathbf{E} + I\mathbf{B}, \tag{3.114}$$

where $\mathbf{E}$ and $\mathbf{B}$ are time-like bivectors representing respectively the electric and magnetic fields as measured by an observer. Consider this observer as having spacetime velocity $c\gamma_0$. Consider then that this observer employs a spacetime basis $\{\gamma_\mu\}$ such that ${\gamma_0}^2 = 1$, $\gamma_0 \cdot \gamma_i = 0$, and $\gamma_i \cdot \gamma_j = -\delta_{ij}$, for i and j in $\{1, 2, 3\}$. In terms of this basis, the electric and magnetic fields can be expressed

$$\mathbf{E} = E^i \boldsymbol{\sigma}_i \qquad \text{and} \qquad I\mathbf{B} = B^i I \boldsymbol{\sigma}_i, \tag{3.115}$$

where $\boldsymbol{\sigma}_i = \gamma_i \gamma_0$. Thus, one finds that these fields can be obtained separately from the Faraday bivector $F = \mathbf{E}/c + I\mathbf{B}$ by

$$\frac{1}{c}\mathbf{E} = \frac{1}{2}(F - \gamma_0 F \gamma_0) \qquad \text{and} \qquad I\mathbf{B} = \frac{1}{2}(F + \gamma_0 F \gamma_0). \tag{3.116}$$

These expressions show that the decomposition of F into $\mathbf{E}/c$ and $I\mathbf{B}$ is dependent on the observer (normalized) velocity γ_0, which implies that observers in relative motion measure different fields. This can be quantified by supposing a second observer of normalized spacetime velocity $\hat{v} = R\gamma_0\tilde{R}$, where R is a spacetime rotor. This observer is associated to the basis

$$\gamma'_\mu = R\gamma_\mu\tilde{R}, \tag{3.117}$$

and measures a electric field of components E'^i such that

$$\begin{aligned} \frac{1}{c}E'^i &= F'_{0i} \\ &= (\gamma'_i \wedge \gamma'_0) \cdot F \\ &= \left(R\boldsymbol{\sigma}_i\tilde{R}\right) \cdot F \\ &= \left\langle R\boldsymbol{\sigma}_i\tilde{R}F\right\rangle \\ &= \left\langle \boldsymbol{\sigma}_i\tilde{R}FR\right\rangle \\ &= \boldsymbol{\sigma}_i \cdot \left(\tilde{R}FR\right), \end{aligned} \tag{3.118}$$

and a magnetic field of components B'^i such that (with summation only relative to j and k in all steps and using the relation $\boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j \wedge \boldsymbol{\sigma}_k = \epsilon_{ijk}I$ in the ultimate step)

$$\begin{aligned} -B'^i &= \epsilon_{ijk}F'^{jk} \\ &= \epsilon_{ijk}\left(\gamma'^k \wedge \gamma'^j\right) \cdot F \\ &= \epsilon_{ijk}\left(\gamma^k \wedge \gamma^j\right) \cdot \left(\tilde{R}FR\right) \\ &= \epsilon_{ijk}\left(\gamma_k \wedge \gamma_j\right) \cdot \left(\tilde{R}FR\right) \\ &= \epsilon_{ijk}\ (\boldsymbol{\sigma}_j \wedge \boldsymbol{\sigma}_k) \cdot \left(\tilde{R}FR\right) \\ &= \epsilon_{ijk}\ \Big(\boldsymbol{\sigma}_i \cdot (\boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j \wedge \boldsymbol{\sigma}_k)\Big) \cdot \left(\tilde{R}FR\right) \\ &= (I\boldsymbol{\sigma}_{\boldsymbol{i}}) \cdot \left(\tilde{R}FR\right). \end{aligned} \tag{3.119}$$

These correspond to the components of the electromagnetic field strength $\tilde{R}FR$ in the $\{\gamma_\mu\}$ basis, as seen in the subsection 3.4.1. Thus, under a Lorentz transformation given by the rotor R, the electromagnetic field strength transforms as $F \mapsto \tilde{R}FR$.

Remember that the rotor R can be written in terms of the normalized velocities $\hat{v}$ and γ_0 and the hyperbolic angle α between them (the rapidity) as

$$R = \exp\left(\frac{1}{2}\frac{\hat{v} \wedge \gamma_0}{|\hat{v} \wedge \gamma_0|}\alpha\right). \tag{3.120}$$

Remember also that the three-dimensional velocity of the second observer relative to the first (conveniently taken in units of speed of light) is

$$\mathbf{v} = \frac{\hat{v} \wedge \gamma_0}{\hat{v} \cdot \gamma_0} = \frac{\hat{v} \wedge \gamma_0}{\cosh(\alpha)} = \frac{\hat{v} \wedge \gamma_0}{|\hat{v} \wedge \gamma_0|} \frac{\sinh(\alpha)}{\cosh(\alpha)} = \frac{\hat{v} \wedge \gamma_0}{|\hat{v} \wedge \gamma_0|} \tanh(\alpha), \tag{3.121}$$

where $\cosh(\alpha)$ corresponds to the Lorentz factor γ and $\tanh(\alpha)$ is the relative speed β (in units of the speed of light). The rotor R can now be written

$$R = \exp\left(\frac{1}{2}\hat{\mathbf{v}}\alpha\right), \tag{3.122}$$

where $\hat{\mathbf{v}} = \mathbf{v}/\tanh(\alpha)$. Consider then F in terms of components parallel and orthogonal to $\mathbf{v}$,

$$F = F_{\|} + F_{\perp}, \tag{3.123}$$

such that

$$\mathbf{v}F_{\|} = F_{\|}\mathbf{v} \qquad \text{and} \qquad \mathbf{v}F_{\perp} = -F_{\perp}\mathbf{v}. \tag{3.124}$$

In this way, the field components parallel and orthogonal to $\mathbf{v}$ as measured by the second observer are

$$F'_{\|} = \tilde{R}F_{\|}R = \exp\left(-\frac{1}{2}\hat{\mathbf{v}}\alpha\right) F_{\|} \exp\left(\frac{1}{2}\hat{\mathbf{v}}\alpha\right) = F_{\|} \tag{3.125}$$

and

$$\begin{aligned} F'_{\perp} = \tilde{R}F_{\perp}R &= \exp\left(-\frac{1}{2}\hat{\mathbf{v}}\alpha\right) F_{\perp} \exp\left(\frac{1}{2}\hat{\mathbf{v}}\alpha\right) \\ &= \exp\left(-\hat{\mathbf{v}}\alpha\right) F_{\perp} \\ &= \Big(\cosh(\alpha) - \hat{\mathbf{v}}\sinh(\alpha)\Big) F_{\perp} \\ &= (\gamma - \gamma\mathbf{v})F_{\perp} \\ &= \gamma(1 - \mathbf{v})F_{\perp}. \end{aligned} \tag{3.126}$$

Therefore, the observers measure the same components of electric and magnetic fields in the direction of the relative motion, however the components orthogonal to the relative motion are such that

$$\begin{aligned} F'_{\perp} = \mathbf{E}'_{\perp} + I\mathbf{B}'_{\perp} &= \gamma(1 - \mathbf{v})\left(\mathbf{E}_{\perp} + I\mathbf{B}_{\perp}\right) \\ &= \gamma\mathbf{E}_{\perp} + \gamma I\mathbf{B}_{\perp} - \gamma\mathbf{v} \wedge \mathbf{E}_{\perp} - \gamma I\left(\mathbf{v} \wedge \mathbf{B}_{\perp}\right) \\ &= \gamma\mathbf{E}_{\perp} + \gamma I\mathbf{B}_{\perp} - \gamma I\mathbf{v} \times \mathbf{E}_{\perp} + \gamma\mathbf{v} \times \mathbf{B}_{\perp}, \end{aligned} \tag{3.127}$$

where is considered the fact that the orthogonality of $\mathbf{v}$ in relation to $\mathbf{E}_{\perp}$ and $\mathbf{B}_{\perp}$ implies that the geometric product of $\mathbf{v}$ with either of these fields is an exterior product. Equation

(3.127) express:

$$\mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp}\right),$$
$$\mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \mathbf{v} \times \mathbf{E}_{\perp}\right). \tag{3.128}$$

These are the known expressions for transformation of the field components orthogonal to the relative velocity between the two observers. Observe that the transformed fields has new components, but also relative to the $\{\gamma_\mu\}$ basis, that is, the transformed fields are formed as $\mathbf{E}' = E'^i \boldsymbol{\sigma}_i$ and $\mathbf{B}' = B'^i \boldsymbol{\sigma}_i$.

Since the square of a spacetime bivector is a scalar plus a pseudoscalar, it is noted that the square of the electromagnetic field strength is Lorentz invariant. Indeed, if

$$F^2 = \langle FF \rangle + \langle FF \rangle_4 = a_0 + I a_4, \tag{3.129}$$

then

$$\left(\tilde{R}FR\right)\left(\tilde{R}FR\right) = \tilde{R}F^2R = F^2 = a_0 + I a_4. \tag{3.130}$$

Both the scalar and pseudoscalar parts are independent of the reference frame. According to the first observer, these are

$$a_0 = \left\langle \left(\frac{1}{c}\mathbf{E} + I\mathbf{B}\right)\left(\frac{1}{c}\mathbf{E} + I\mathbf{B}\right) \right\rangle = \frac{1}{c^2}\mathbf{E}^2 - \mathbf{B}^2 \tag{3.131}$$

and

$$a_4 = \left\langle I^{-1}F^2 \right\rangle = \left\langle (-I)\left(\frac{1}{c}\mathbf{E} + I\mathbf{B}\right)\left(\frac{1}{c}\mathbf{E} + I\mathbf{B}\right) \right\rangle = \frac{2}{c}\,\mathbf{E}\cdot\mathbf{B}. \tag{3.132}$$

The first Lorentz invariant above appears in the expression of the Lagrangian density for the electromagnetic field. The second encodes the relative orientation of the electric and magnetic fields.

4 Quantum Mechanics Revisited: From the Classical, through the Algebraic, to the Geometric Picture

This chapter is designed to serve as an outline of the emergence of Clifford algebras in quantum mechanics, as well as to explain concisely how such algebras provide an alternative language to express it. This is accomplished by transitioning from the "classical picture", based on the classical definition of a spinor, to the "algebraic picture", based on the algebraic definition of a spinor, which leads naturally to a "geometric picture", based on the operator definition of a spinor.

4.1 Non-Relativistic Theory

This section, on non-relativistic states, begins with a contextualization, based on chapter 6 of the textbook by Piza (2003). A better-known English text, such as chapter XIII of Messiah (2014), for example, can also be taken as reference. These textbooks can be taken as background references for the well-established quantum-mechanical concepts to be introduced in the remaining subsections.

4.1.1 Introduction

The observation of the splitting of the spectral lines of the hydrogen atom evidences the existence of a structure of four states associated with the fundamental energy level of the simple model for the hydrogen atom, which is based on classical analogies. This fact suggests that such a simple model needs improvement through the introduction of additional degrees of freedom. Since this is a two-body system, one can conjecture that both the proton and the electron possess, in addition to the degrees of freedom associated with position, intrinsic properties associated to observables acting in a two-dimensional state space. This hypothesis is supported by experiments, which suggest that the supposed additional degrees of freedom have a character similar to angular momentum. If such an *intrinsic angular momentum* is represented by a vector observable $\hat{s} = (\hat{s}_1, \hat{s}_2, \hat{s}_3)$, whose component operators satisfy commutation relations analogous to those for angular momentum,

$$[\hat{s}_j, \hat{s}_k] = i\hbar\epsilon_{jkl}\hat{s}_l, \tag{4.1}$$

then the eigenvalues of $\hat{s}^2$ are of the form $\hbar^2 s(s+1)$, where s is an integer or semi-integer number. The number of states associated to a specific s is $2s+1$, which is 2 for $s = 1/2$. Thus, by associating to both proton and electron an intrinsic angular momentum, or *spin*, as it is currently termed, associated with $s = 1/2$, the state space of the hydrogen atom becomes the tensor product of the space corresponding to the simple model with the four-dimensional space corresponding to the new degrees of freedom.

The vectors of the two-dimensional state space acted on by the spin observable $\hat{s}$ of a particle with $s = 1/2$ (such as an electron or a proton) can be expressed in terms of the basis formed by the concomitant eigenvectors of $\hat{s}^2$ and $\hat{s}_3$, which can be denoted generically by $|s\ m_s\rangle$, with the eigenvalue equations taking a form analogous to those for angular momentum,

$$\hat{s}^2 \left|\tfrac{1}{2}\ m_s\right\rangle = \hbar^2 \frac{1}{2}\left(\frac{1}{2}+1\right)\left|\tfrac{1}{2}\ m_s\right\rangle \qquad \text{and} \qquad \hat{s}_3 \left|\tfrac{1}{2}\ m_s\right\rangle = \hbar m_s \left|\tfrac{1}{2}\ m_s\right\rangle, \tag{4.2}$$

where the eigenvectors, which are necessarily orthogonal, are taken to be normalized, that is,

$$\left\langle \tfrac{1}{2}\ m_s \middle| \tfrac{1}{2}\ m'_s \right\rangle = \delta_{m_s m'_s}, \qquad \text{for} \qquad m_s, m'_s \in \left\{-\tfrac{1}{2}, +\tfrac{1}{2}\right\}. \tag{4.3}$$

A generic vector $|\psi\rangle$ of the spin state space can then be expressed as the linear combination

$$|\psi\rangle = \zeta\left|\tfrac{1}{2}\ {+\tfrac{1}{2}}\right\rangle + \eta\left|\tfrac{1}{2}\ {-\tfrac{1}{2}}\right\rangle, \tag{4.4}$$

where ζ and η are complex numbers. Its representation in terms of components relative to the adopted basis is given by the column matrix

$$\varPsi = \begin{pmatrix} \left\langle \tfrac{1}{2}\ {+\tfrac{1}{2}} \middle| \psi \right\rangle \\ \left\langle \tfrac{1}{2}\ {-\tfrac{1}{2}} \middle| \psi \right\rangle \end{pmatrix} = \begin{pmatrix} \zeta \\ \eta \end{pmatrix}. \tag{4.5}$$

This is a representative of a class of objects called *spinors* — this type of spinor, in particular, is known as a *Pauli spinor*.

In the same way as spin states, a linear operator $\hat{a}$ defined on the spin state space can also be represented in terms of components, relative to the basis of normalized eigenvectors of $\hat{s}^2$ and $\hat{s}_3$, by the matrix

$$\mathrm{A} = \begin{pmatrix} \left\langle \frac{1}{2} + \frac{1}{2} \middle| \hat{a} \middle| \frac{1}{2} + \frac{1}{2} \right\rangle & \left\langle \frac{1}{2} + \frac{1}{2} \middle| \hat{a} \middle| \frac{1}{2} - \frac{1}{2} \right\rangle \\ \left\langle \frac{1}{2} - \frac{1}{2} \middle| \hat{a} \middle| \frac{1}{2} + \frac{1}{2} \right\rangle & \left\langle \frac{1}{2} - \frac{1}{2} \middle| \hat{a} \middle| \frac{1}{2} - \frac{1}{2} \right\rangle \end{pmatrix}. \tag{4.6}$$

In particular, one has the following matrix representations:

$$\hat{s}^2 \quad \sim \quad \mathrm{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.7}$$

and

$$\hat{s}_3 \quad \sim \quad \mathrm{S}_3 = \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.8}$$

In order to determine the corresponding matrix representations for $\hat{s}_1$ and $\hat{s}_2$, consider the new operators

$$\hat{s}_\pm = \hat{s}_1 \pm i\hat{s}_2. \tag{4.9}$$

Note that the addition or subtraction of $\hat{s}_+$ by $\hat{s}_-$ reproduces $\hat{s}_1$ or $\hat{s}_2$:

$$\hat{s}_1 = \frac{1}{2}(\hat{s}_+ + \hat{s}_-), \qquad \hat{s}_2 = \frac{1}{2i}(\hat{s}_+ - \hat{s}_-). \tag{4.10}$$

Observe now that, from the commutation relations (4.1), it follows that:

$$[\hat{s}_3, \hat{s}_\pm] = [\hat{s}_3, \hat{s}_1] \pm i[\hat{s}_3, \hat{s}_2] = i\hbar\hat{s}_2 \pm \hbar\hat{s}_1 = \pm\hbar\hat{s}_\pm. \tag{4.11}$$

These commutation relations, in addition to the eigenvalue equation for $\hat{s}_3$ (cf. the second equation in (4.2)), furnish:

$$\begin{aligned} \hat{s}_3\hat{s}_\pm \left| \tfrac{1}{2}\, m_s \right\rangle &= [\hat{s}_3, \hat{s}_\pm] \left| \tfrac{1}{2}\, m_s \right\rangle + \hat{s}_\pm\hat{s}_3 \left| \tfrac{1}{2}\, m_s \right\rangle \\ &= \pm\hbar\hat{s}_\pm \left| \tfrac{1}{2}\, m_s \right\rangle + \hbar m_s \hat{s}_\pm \left| \tfrac{1}{2}\, m_s \right\rangle \\ &= \hbar(m_s \pm 1)\hat{s}_\pm \left| \tfrac{1}{2}\, m_s \right\rangle. \end{aligned} \tag{4.12}$$

This resulting equation implies in the "proportionality" relation

$$\hat{s}_\pm \left| \tfrac{1}{2}\, m_s \right\rangle = \lambda_\pm \left| \tfrac{1}{2}\, m_s \pm 1 \right\rangle, \tag{4.13}$$

where the scalars $\lambda_\pm$ can be determined, disregarding a possible phase factor (that is, considering $\lambda_\pm$ as non-negative reals), as follows:

$$\begin{aligned}
|\lambda_\pm|^2 = \left\langle \tfrac{1}{2}\, m_s \middle| \hat{s}_\pm{}^\dagger \hat{s}_\pm \middle| \tfrac{1}{2}\, m_s \right\rangle &= \left\langle \tfrac{1}{2}\, m_s \middle| \hat{s}_\mp \hat{s}_\pm \middle| \tfrac{1}{2}\, m_s \right\rangle \\
&= \left\langle \tfrac{1}{2}\, m_s \middle| \left(\hat{s}_1{}^2 \pm i\hat{s}_1\hat{s}_2 \mp i\hat{s}_2\hat{s}_1 + \hat{s}_2{}^2 \right) \middle| \tfrac{1}{2}\, m_s \right\rangle \\
&= \left\langle \tfrac{1}{2}\, m_s \middle| \left(\hat{s}_1{}^2 + \hat{s}_2{}^2 \pm i[\hat{s}_1, \hat{s}_2] \right) \middle| \tfrac{1}{2}\, m_s \right\rangle \\
&= \left\langle \tfrac{1}{2}\, m_s \middle| \left(\hat{s}_1{}^2 + \hat{s}_2{}^2 \mp \hbar\hat{s}_3 \right) \middle| \tfrac{1}{2}\, m_s \right\rangle \\
&= \left\langle \tfrac{1}{2}\, m_s \middle| \left(\hat{s}^2 - \hat{s}_3(\hat{s}_3 \pm \hbar) \right) \middle| \tfrac{1}{2}\, m_s \right\rangle \\
&= \hbar^2 \left(\frac{3}{4} - m_s(m_s \pm 1) \right). \qquad (4.14)
\end{aligned}$$

The matrix representations of $\hat{s}_\pm$ relative to the basis of normalized eigenvectors of $\hat{s}^2$ and $\hat{s}_3$ (or only of $\hat{s}_3$, since $\hat{s}^2$ is proportional to the identity operator) are then obtained just applying the relation (4.13) in the standard expression (4.6), with the use of the values of $\lambda_\pm$ given by the square root of (4.14):

$$\hat{s}_+ \quad \sim \quad \mathrm{S}_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \hat{s}_- \quad \sim \quad \mathrm{S}_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \qquad (4.15)$$

Thus, from the relations (4.10), one obtains the corresponding matrix representations for $\hat{s}_1$ and $\hat{s}_2$:

$$\hat{s}_1 \quad \sim \quad \mathrm{S}_1 = \frac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \hat{s}_2 \quad \sim \quad \mathrm{S}_2 = \frac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \qquad (4.16)$$

The spin vector observable $\hat{s} = (\hat{s}_1, \hat{s}_2, \hat{s}_3)$ is now determined in terms of the matrix representations (4.16) and (4.8). It can be conveniently written as

$$\hat{s} = \frac{1}{2}\hbar\hat{\sigma} = \frac{1}{2}\hbar(\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3), \qquad (4.17)$$

where the operators $\hat{\sigma}_1$, $\hat{\sigma}_2$ and $\hat{\sigma}_3$ have the following corresponding matrix representations:

$$\hat{\sigma}_1 \sim \Sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 \sim \Sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \hat{\sigma}_3 \sim \Sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (4.18)$$

The above matrices are known as the *Pauli matrices.* Note that the square of any Pauli matrix equals to the identity matrix, which implies

$$\hat{\sigma}_1{}^2 = \hat{\sigma}_2{}^2 = \hat{\sigma}_3{}^2 = \hat{1}, \qquad (4.19)$$

where $\hat{1}$ is the identity operator. Note also that the commutation relations (4.1) imply

$$[\hat{\sigma}_j, \hat{\sigma}_k] = 2i\epsilon_{jkl}\hat{\sigma}_l. \tag{4.20}$$

These commutation relations, in addition to the property (4.19), furnish

$$\begin{aligned} 0 = [\hat{\sigma}_j{}^2, \hat{\sigma}_k] &= \hat{\sigma}_j\hat{\sigma}_j\hat{\sigma}_k - \hat{\sigma}_k\hat{\sigma}_j\hat{\sigma}_j \\ &= \hat{\sigma}_j\hat{\sigma}_j\hat{\sigma}_k - \hat{\sigma}_j\hat{\sigma}_k\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_k\hat{\sigma}_j - \hat{\sigma}_k\hat{\sigma}_j\hat{\sigma}_j \\ &= \hat{\sigma}_j[\hat{\sigma}_j, \hat{\sigma}_k] + [\hat{\sigma}_j, \hat{\sigma}_k]\hat{\sigma}_j \\ &= 2i\epsilon_{jkl}(\hat{\sigma}_j\hat{\sigma}_l + \hat{\sigma}_l\hat{\sigma}_j). \end{aligned} \tag{4.21}$$

Since ϵ_{jkl} is not identically null for $j \neq l$, it follows from the above equation that

$$\{\hat{\sigma}_j, \hat{\sigma}_k\} = \hat{\sigma}_j\hat{\sigma}_k + \hat{\sigma}_k\hat{\sigma}_j = \hat{0}, \quad \text{for} \quad j \neq k, \tag{4.22}$$

where $\hat{0}$ is the null operator. The properties (4.19) and (4.22) can be summarized by

$$\frac{1}{2}\{\hat{\sigma}_j, \hat{\sigma}_k\} = \frac{1}{2}(\hat{\sigma}_j\hat{\sigma}_k + \hat{\sigma}_k\hat{\sigma}_j) = \delta_{jk}\hat{1}. \tag{4.23}$$

Finally, observe that any observable dependent on the spin $\hat{s}$ can be written as a (real) linear combination of the "Pauli operators" $\hat{\sigma}_1$, $\hat{\sigma}_2$ and $\hat{\sigma}_3$, and that the anticommutation relations (4.23) are equivalent to the relations

$$\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k = \frac{1}{2}(\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k + \boldsymbol{\sigma}_k\boldsymbol{\sigma}_j) = \delta_{jk} \tag{4.24}$$

for the vectors of the orthonormal basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ of the three-dimensional Euclidean space $\mathbb{R}^3$, in such a way that the linear space of spin dependent observables is isomorphic to $\mathbb{R}^3$. Observe also the equivalence of the commutation relations (4.20) and the expressions

$$\boldsymbol{\sigma}_j \wedge \boldsymbol{\sigma}_k = \frac{1}{2}(\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k - \boldsymbol{\sigma}_k\boldsymbol{\sigma}_j) = \epsilon_{jkl}I\boldsymbol{\sigma}_l. \tag{4.25}$$

These equivalences imply that the algebra generated by the Pauli operators, or the algebra of spin dependent observables (with the composition operation as product), is isomorphic to the geometric algebra of the three-dimensional Euclidean space, $\mathcal{C}\ell_{3,0}$, through the identifications $\hat{1} \sim 1$ and $\hat{\sigma}_j \sim \boldsymbol{\sigma}_j$, and through the identification of the composition operation of observables with the geometric product of multivectors.

Observe now that, in calculations involving a Pauli spinor, it can be replaced by the 2×2 matrix

$$\begin{pmatrix} \zeta & 0 \\ \eta & 0 \end{pmatrix}, \tag{4.26}$$

since the product of such a matrix from the left by an arbitrary 2×2 complex matrix produces another matrix of the form (4.26). This fact allows one to work with spin dependent observables and Pauli spinors represented in the same matrix algebra. It is natural then to imagine the possibility to work with spin dependent observables and Pauli spinors both represented in the geometric algebra $\mathcal{C}\ell_{3,0}$. A way to realize this idea is presented in subsequent subsections. The following subsection, based on Vaz and da Rocha (2019), is dedicated to the introduction of some mathematical concepts useful for the understanding of future considerations.

4.1.2 Ideals and Idempotents of an Algebra

Given an algebra $\mathcal{A}$, a subset of its elements which is closed with relation to the addition operation and is invariant under the product from the left by elements of the algebra is said to be a *left ideal* of the algebra $\mathcal{A}$. That is, a left ideal of an algebra $\mathcal{A}$ is a subset $\mathcal{I}$ such that $(x + y) \in \mathcal{I}$, for $x, y \in \mathcal{I}$, and $ax \in \mathcal{I}$, for $a \in \mathcal{A}$ and $x \in \mathcal{I}$. A *right ideal* is defined in a similar way. A subset of elements of an algebra which is both a left and a right ideal is called a *two-sided ideal*. Any algebra contains at least two trivial ideals, the set formed by the zero element only and the set of all elements of the algebra — both are two-sided ideals. A subset of an ideal $\mathcal{I}$ which is also an ideal is called a *subideal* of $\mathcal{I}$. An ideal is said to be *minimal* if it contains no non-trivial subideals.

An element f of an algebra $\mathcal{A}$ is said to be an *idempotent* if its square reproduces itself, $f^2 = f$. If the product of two idempotents is zero they are called *orthogonal*. An idempotent is said to be *primitive* if it can not be written as a sum of two other orthogonal idempotents.

Given an arbitrary element x of an algebra $\mathcal{A}$, the set of elements of the form ax, for any a in $\mathcal{A}$, defines a left ideal $\mathcal{I}$. Consider the case where the element x is an idempotent. In this case, the set of elements af, where $a \in \mathcal{A}$ and f is a primitive idempotent of the algebra $\mathcal{A}$, defines a minimal left ideal. Otherwise, if f is a non-primitive idempotent, it can be written as the sum of two orthogonal idempotents, $f = f_1 + f_2$, and it is possible to construct two non-trivial subideals whose elements are of the form $a_1 f_1 f$ and $a_2 f_2 f$, where $a_1, a_2 \in \mathcal{A}$. In summary, given a primitive idempotent f of an algebra $\mathcal{A}$, the set of elements of the form af, where $a \in \mathcal{A}$, is a minimal left ideal of $\mathcal{A}$. In a similar way, one can verify that, given a primitive idempotent f of an algebra $\mathcal{A}$, the set of elements of the form fa, where $a \in \mathcal{A}$, is a minimal right ideal of $\mathcal{A}$.

4.1.3 From Classical, through Algebraic, to Operator Pauli Spinors

As seen above, a column matrix with two complex entries defines a Pauli spinor. This corresponds essentially to the *classical definition* of a Pauli spinor (FIGUEIREDO *et al.*, 1990; VAZ; DA ROCHA, 2019). In this way, a *classical Pauli spinor* is given by

$$\mathit{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{4.27}$$

where $\psi_1, \psi_2 \in \mathbb{C}$. Note then that, in any calculation involving such a classical Pauli spinor, it can be replaced by the 2×2 matrix

$$\Psi = \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix}, \tag{4.28}$$

since the product of such a matrix from the left by an arbitrary 2×2 complex matrix produces another matrix with null entries in the second column. In this way, the following equivalence relation is valid:

$$\mathit{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \sim \quad \Psi = \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix}. \tag{4.29}$$

It is easy to note that the matrix Ψ can be put in the form

$$\Psi = \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} = \begin{pmatrix} \psi_1 & \psi_{12} \\ \psi_2 & \psi_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.30}$$

where the entries ψ_{12} and ψ_{22} are arbitrary complex numbers. Note now that the matrix

$$\mathrm{F} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.31}$$

is idempotent, that is, $\mathrm{F}^2 = \mathrm{F}$, and it can be expressed in terms of the identity matrix and the Pauli matrix Σ_3 by

$$\mathrm{F} = \frac{1}{2}(1 + \Sigma_3) \tag{4.32}$$

(where was used the convention that in any equation involving 2×2 matrices, 1 denotes the identity matrix). Consider then the matrix

$$\Sigma_1 \mathrm{F} = \Sigma_1 \frac{1}{2}(1 + \Sigma_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4.33}$$

and note that Ψ can be written as the linear combination

$$\Psi = \psi_1 \mathrm{F} + \psi_2 \Sigma_1 \mathrm{F} = (\psi_1 + \psi_2 \Sigma_1)\mathrm{F}. \tag{4.34}$$

By writing $\psi_A = r_A + is_A$, where $r_A, s_A \in \mathbb{R}$ and $A \in \{1, 2\}$, and using the property $\Sigma_3 \mathrm{F} = \mathrm{F}$, in addition to the basic property $\frac{1}{2}\{\Sigma_i, \Sigma_j\} = \delta_{ij} 1$ and the fact that $\Sigma_1 \Sigma_2 \Sigma_3 = i1$, one can rewrite Ψ as follows:

$$\begin{aligned} \Psi &= (r_1 + is_1 + r_2 \Sigma_1 + s_2 i \Sigma_1)\mathrm{F} \\ &= (r_1 + s_1 i \Sigma_3 + r_2 \Sigma_1 \Sigma_3 + s_2 i \Sigma_1)\mathrm{F} \\ &= (r_1 + s_1 i \Sigma_3 - r_2 i \Sigma_2 + s_2 i \Sigma_1)\mathrm{F}. \end{aligned} \tag{4.35}$$

Now, recall that the algebra of Pauli operators is isomorphic to the geometric algebra of the three-dimensional Euclidean space, $\mathcal{C}\ell_{3,0}$, via the identifications $\hat{1} \sim 1$ and $\hat{\sigma}_j \sim \boldsymbol{\sigma}_j$, and via the identification of the composition of observables with the geometric product. This isomorphism establishes also the correspondences $1 \sim 1$ (i.e. the identity matrix is equivalent to the number one) and $\Sigma_j \sim \boldsymbol{\sigma}_j$, in addition to the correspondence between the matrix product and the geometric product. These correspondences imply, in particular, the correspondences $\Sigma_1 \Sigma_2 \Sigma_3 = i1 \sim \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 = I$ and $i\Sigma_j \sim I\boldsymbol{\sigma}_j$. In this way, the square matrix Ψ given by equation (4.35) is in correspondence with the multivector

$$\underline{\psi} = (r_1 + s_1 I\boldsymbol{\sigma}_3 - r_2 I\boldsymbol{\sigma}_2 + s_2 I\boldsymbol{\sigma}_1) f, \tag{4.36}$$

where $f = \frac{1}{2}(1 + \boldsymbol{\sigma}_3)$. This representation defines a Pauli spinor as an element of a minimal left ideal of the algebra $\mathcal{C}\ell_{3,0}$. This is demonstrated below.

Assume that the idempotent $f = \frac{1}{2}(1 + \boldsymbol{\sigma}_3)$ is non-primitive, that is, that there exist idempotents f_1 and f_2 such that $f_1 f_2 = f_2 f_1 = 0$ and $f_1 + f_2 = f$. These imply that $f_A = f_A f = f f_A$, for $A \in \{1, 2\}$. The commutativity of the product between f_A and $f = \frac{1}{2}(1 + \boldsymbol{\sigma}_3)$ show that f_A must be of the form

$$f_A = a_A + b_A \boldsymbol{\sigma}_3 + c_A I\boldsymbol{\sigma}_3 + d_A I. \tag{4.37}$$

Since, by hypothesis, f_1 and f_2 are idempotents, i.e. $f_1 = {f_1}^2$ and $f_2 = {f_2}^2$, it follows that

$$\begin{aligned} a_A + b_A \boldsymbol{\sigma}_3 + c_A I\boldsymbol{\sigma}_3 + d_A I = {a_A}^2 + {b_A}^2 - {c_A}^2 - {d_A}^2 + 2(a_A b_A - c_A d_A)\boldsymbol{\sigma}_3 \\ + 2(a_A c_A + b_A d_A) I\boldsymbol{\sigma}_3 + 2(a_A d_A + b_A c_A) I. \end{aligned} \tag{4.38}$$

This equation imply a system of equations which clearly has no non-trivial solutions, so there are no idempotents f_1 and f_2 satisfying $f_1 f_2 = f_2 f_1 = 0$ and $f_1 + f_2 = f$. By contradiction, the idempotent f is primitive. Now, note that, given a multivector

$$A = a + b^i\boldsymbol{\sigma}_i + c^i I\boldsymbol{\sigma}_i + dI, \tag{4.39}$$

the multivector $\underline{\psi} = Af$, where $f = \frac{1}{2}(1+\boldsymbol{\sigma}_3)$, is an element of the left ideal $\mathcal{I} = \{Af \mid A \in \mathcal{C}\ell_{3,0} \text{ and } f = \frac{1}{2}(1+\boldsymbol{\sigma}_3)\}$, which is minimal since the idempotent f is primitive. By considering the property $\boldsymbol{\sigma}_3 f = f$ of the idempotent f, in addition to the bilinearity of the geometric product, one can rewrite the generic element $\underline{\psi} = Af$ of the ideal $\mathcal{I}$ as follows:

$$\begin{aligned}\underline{\psi} &= \Big(a + b^1\boldsymbol{\sigma}_1 + b^2\boldsymbol{\sigma}_2 + b^3\boldsymbol{\sigma}_3 + c^1 I\boldsymbol{\sigma}_1 + c^2 I\boldsymbol{\sigma}_2 + c^3 I\boldsymbol{\sigma}_3 + dI\Big)f \\ &= \Big(a - b^1 I\boldsymbol{\sigma}_2 + b^2 I\boldsymbol{\sigma}_1 + b^3 + c^1 I\boldsymbol{\sigma}_1 + c^2 I\boldsymbol{\sigma}_2 + c^3 I\boldsymbol{\sigma}_3 + dI\boldsymbol{\sigma}_3\Big)f \\ &= \Big((a+b^3) + (b^2+c^1)I\boldsymbol{\sigma}_1 + (-b^1+c^2)I\boldsymbol{\sigma}_2 + (c^3+d)I\boldsymbol{\sigma}_3\Big)f. \end{aligned} \tag{4.40}$$

This multivector represents a Pauli spinor, as seen in the last paragraph (cf. equation (4.36)). This fact allows one to define a Pauli spinor as an element of the minimal left ideal $\mathcal{I} = \{Af \mid A \in \mathcal{C}\ell_{3,0} \text{ and } f = \frac{1}{2}(1+\boldsymbol{\sigma}_3)\}$ (HESTENES, 2015; FIGUEIREDO *et al.*, 1990; LOUNESTO, 2001; VAZ; DA ROCHA, 2019). This way of characterizing a spinor, introduced by Riesz in the 1950s (RIESZ, 1993), is called the *algebraic definition* of a spinor, and a Pauli spinor defined in this way is usually called an *algebraic Pauli spinor* (FIGUEIREDO *et al.*, 1990; LOUNESTO, 2001; VAZ; DA ROCHA, 2019). In fact, an algebraic Pauli spinor can be defined as an element of a minimal left ideal generated by any idempotent obtained from f through a rotation. As explained by Hiley and Callaghan (2010), the choice of the idempotent reflects merely the choice of a quantization direction, and consequently the adoption of a matrix representation. The conventional choice for $f = \frac{1}{2}(1+\boldsymbol{\sigma}_3)$ corresponds to the choice of the z-axis as the quantization direction, and corresponds to the usual matrix representation adopted in this context.

The above developments allow one to conclude that an even grade multivector from $\mathcal{C}\ell_{3,0}$ is sufficient to describe a Pauli spinor, since an algebraic Pauli spinor as expressed by equation (4.36), or equation (4.40), is the geometric product of an even grade multivector with the idempotent $f = \frac{1}{2}(1+\boldsymbol{\sigma}_3)$, which is a fixed factor. This fact allows one to describe a Pauli spinor by an element of the even subalgebra ${\mathcal{C}\ell_{3,0}}^+$. This way of describing a spinor was implemented by Hestenes in the 1960s (HESTENES, 1967; HESTENES, 1971; HESTENES; GÜRTLER, 1971; HESTENES, 1975), and today is known as the *operator definition* of a spinor (FIGUEIREDO *et al.*, 1990; LOUNESTO, 2001; VAZ; DA ROCHA, 2019). A Pauli spinor defined in this way is usually known as an *operator Pauli spinor* (FIGUEIREDO *et al.*, 1990; LOUNESTO, 2001; VAZ; DA ROCHA, 2019). In these terms, note that the operator Pauli spinor ψ corresponding to the algebraic Pauli spinor $\underline{\psi}$ given by equation (4.36) can be expressed in terms of this latter as

$$\psi = 2\langle \underline{\psi} \rangle_+, \tag{4.41}$$

where $\langle A \rangle_+$ denotes the even grade part of the multivector A from $\mathcal{C}\ell_{3,0}$ (LOUNESTO, 2001). Another way to express ψ in terms of $\underline{\psi}$ is given by

$$\psi = 2\langle \underline{\psi} \rangle_- \boldsymbol{\sigma}_3, \tag{4.42}$$

where $\langle A \rangle_-$ denotes the odd grade part of the multivector A from $\mathcal{C}\ell_{3,0}$. Thus, the information of an operator Pauli spinor is encoded two times in the corresponding algebraic Pauli spinor, in its even grade part and in its odd grade part. This fact becomes clear when one observes that, from equation (4.36), by using the property $\boldsymbol{\sigma}_3 f = f$, one can rewrite $\underline{\psi}$ as

$$\underline{\psi} = (r_1\boldsymbol{\sigma}_3 + s_1 I + r_2\boldsymbol{\sigma}_1 + s_2\boldsymbol{\sigma}_2)f, \tag{4.43}$$

which shows the possibility of describe a Pauli spinor through an odd grade multivector. However, it seems more attractive to describe spinors through even grade multivectors, mainly because such elements compose a subalgebra.

By comparing the classical Pauli spinor given by equation (4.27) and the corresponding algebraic Pauli spinor, given by equation (4.36), where $\psi_A = r_A + is_A$ and $A \in \{1, 2\}$, and by considering the relation between an algebraic Pauli spinor and its corresponding operator Pauli spinor introduced above, one is able to express the correspondence between classical, algebraic and operator Pauli spinors through the following maps,

$$\begin{array}{ccc} \mathit{\Psi} = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} & \xmapsto{\ \alpha\ } & \underline{\psi} = (a^0 + a^i I\boldsymbol{\sigma}_i)f = \psi f \\ & \searrow^{\beta\circ\alpha} & \downarrow \beta \\ & & \psi = a^0 + a^i I\boldsymbol{\sigma}_i = 2\langle \underline{\psi} \rangle_+ = 2\langle \underline{\psi} \rangle_- \boldsymbol{\sigma}_3. \end{array} \tag{4.44}$$

These explicit transformations are presented for the first time here. The composite map $\beta\circ\alpha$ reproduces the known relation between a classical Pauli spinor and an operator Pauli spinor, as presented by Doran and Lasenby (2003).

The maps α, β and $\beta\circ\alpha$ can be used now to translate the action of observables on Pauli spinors. This translation is given by

$$\begin{array}{ccc} \Sigma_j\,\mathit{\Psi} & \xmapsto{\ \alpha\ } & \boldsymbol{\sigma}_j\underline{\psi} = \boldsymbol{\sigma}_j\psi f = \boldsymbol{\sigma}_j\psi\boldsymbol{\sigma}_3 f \\ & \searrow^{\beta\circ\alpha} & \downarrow \beta \\ & & 2\langle \boldsymbol{\sigma}_j\underline{\psi} \rangle_+ = 2\langle \boldsymbol{\sigma}_j\underline{\psi} \rangle_- \boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_j\psi\boldsymbol{\sigma}_3 \end{array} \tag{4.45}$$

and

$$\begin{array}{ccc} i\mathit{\Psi} & \overset{\alpha}{\longmapsto} & I\underline{\psi} = I\psi f = \psi I\boldsymbol{\sigma}_3 f \\ & \overset{\beta\circ\alpha}{\searrow} & \downarrow \beta \\ & & 2\langle I\underline{\psi}\rangle_+ = 2\langle I\underline{\psi}\rangle_-\boldsymbol{\sigma}_3 = \psi I\boldsymbol{\sigma}_3, \end{array} \tag{4.46}$$

where the property $\boldsymbol{\sigma}_3 f = f$ was used. It should be noted that the composite map $\beta \circ \alpha$ reproduces the transformation for action of operators from the classical to the operator representation (cf. Doran and Lasenby (2003)). The explicit expression of the transformations above is presented for the first time here.

4.1.4 Hermitian Adjoint and Hermitian Inner Products

The *Hermitian adjoint* of the classical Pauli spinor in equation (4.27) is given by

$$\mathit{\Psi}^\dagger = \left(\mathit{\Psi}^{\mathrm{T}}\right)^* = \left(\mathit{\Psi}^*\right)^{\mathrm{T}} = \begin{pmatrix} \psi_1{}^* & \psi_2{}^* \end{pmatrix}, \tag{4.47}$$

so that its representation as a square matrix is given by

$$\Psi^\dagger = \left(\Psi^{\mathrm{T}}\right)^* = \left(\Psi^*\right)^{\mathrm{T}} = \begin{pmatrix} \psi_1{}^* & \psi_2{}^* \\ 0 & 0 \end{pmatrix}. \tag{4.48}$$

This can be written as

$$\Psi^\dagger = \psi_1{}^*\mathrm{F} + \psi_2{}^*(\Sigma_1\mathrm{F})^\dagger, \tag{4.49}$$

that is,

$$\Psi^\dagger = \psi_1{}^*\mathrm{F} + \psi_2{}^*\mathrm{F}\Sigma_1 = \mathrm{F}(\psi_1{}^* + \psi_2{}^*\Sigma_1). \tag{4.50}$$

By considering that $\psi_A = r_A + is_A$, where $r_A, s_A \in \mathbb{R}$ and $A \in \{1,2\}$, and by using the property $\Sigma_3\mathrm{F} = \mathrm{F}\Sigma_3 = \mathrm{F}$, in addition to the basic property $\frac{1}{2}\{\Sigma_i, \Sigma_j\} = \delta_{ij}1$ and the fact that $\Sigma_1\Sigma_2\Sigma_3 = i1$, one can rewrite $\Psi^\dagger$ as follows:

$$\begin{aligned} \Psi^\dagger &= \mathrm{F}(r_1 - is_1 + r_2\Sigma_1 - is_2\Sigma_1) \\ &= \mathrm{F}(r_1 - s_1 i\Sigma_3 + r_2\Sigma_3\Sigma_1 - s_2 i\Sigma_1) \\ &= \mathrm{F}(r_1 - s_1 i\Sigma_3 + r_2 i\Sigma_2 - s_2 i\Sigma_1). \end{aligned} \tag{4.51}$$

Now, by using the correspondences $\Sigma_j \sim \boldsymbol{\sigma}_j$, in addition to the correspondence between the identity matrix and the number one, which imply, in particular, the correspondences $\Sigma_1\Sigma_2\Sigma_3 = i1 \sim \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = I$ and $i\Sigma_j \sim I\boldsymbol{\sigma}_j$, one can write the multivector corresponding to the adjoint Pauli spinor in question as

$$\underline{\psi}^\dagger = f(r_1 - s_1 I\boldsymbol{\sigma}_3 + r_2 I\boldsymbol{\sigma}_2 - s_2 I\boldsymbol{\sigma}_1). \tag{4.52}$$

This is just the reverse of the algebraic Pauli spinor given by equation (4.36), which justifies the use of a superscript dagger to denote the reverse of an element of the algebra $\mathcal{C}\ell_{3,0}$. (Note that this adjoint algebraic Pauli spinor is an element of the minimal right ideal $\mathcal{I}^\dagger = \{fA \mid A \in \mathcal{C}\ell_{3,0} \text{ and } f = \frac{1}{2}(1+\boldsymbol{\sigma}_3)\}$.) In this way, the map which maps the adjoint classical Pauli spinor $\varPsi^\dagger$ to the corresponding adjoint algebraic Pauli spinor $\underline{\psi}^\dagger$ is found to be $\bar{\alpha} = \mathrm{rev} \circ \alpha \circ \mathrm{adj}$, where adj denotes the Hermitian adjoint operation, rev denotes the reversion operation and the map α is that in the relations (4.44). The operator adjoint Pauli spinor $\psi^\dagger$ corresponding to the algebraic adjoint Pauli spinor $\underline{\psi}^\dagger$ is obtained in the same way as ψ is obtained from $\underline{\psi}$, through the map $\beta : \underline{\psi} \mapsto \psi = \langle\underline{\psi}\rangle_+ = \langle\underline{\psi}\rangle_- \boldsymbol{\sigma}_3$. In summary, the maps between adjoint Pauli spinors are the following:

$$\begin{array}{ccc} \varPsi^\dagger = \begin{pmatrix} a^0 - ia^3 & -a^2 - ia^1 \end{pmatrix} & \xmapsto{\ \bar{\alpha}\ } & \underline{\psi}^\dagger = f(a^0 - a^i I\boldsymbol{\sigma}_i) = f\psi^\dagger \\ & \searrow^{\beta\circ\bar{\alpha}} & \big\downarrow \beta \\ & & \psi^\dagger = a^0 - a^i I\boldsymbol{\sigma}_i = 2\langle\underline{\psi}^\dagger\rangle_+ = 2\langle\underline{\psi}^\dagger\rangle_- \boldsymbol{\sigma}_3. \end{array} \tag{4.53}$$

The *Hermitian inner product* of the classical Pauli spinors $\varPsi = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}^{\mathrm{T}}$ and $\varPhi = \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix}^{\mathrm{T}}$ is given by

$$\varPsi^\dagger \varPhi = \begin{pmatrix} {\psi_1}^* & {\psi_2}^* \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = {\psi_1}^*\phi_1 + {\psi_2}^*\phi_2. \tag{4.54}$$

In this way, if Ψ and Φ are the square matrices corresponding to the classical Pauli spinors $\varPsi$ and $\varPhi$, it follows that

$$\Psi^\dagger\Phi = \begin{pmatrix} {\psi_1}^* & {\psi_2}^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 & 0 \\ \phi_2 & 0 \end{pmatrix} = \begin{pmatrix} {\psi_1}^*\phi_1 + {\psi_2}^*\phi_2 & 0 \\ 0 & 0 \end{pmatrix} = \big({\psi_1}^*\phi_1 + {\psi_2}^*\phi_2\big)\mathrm{F}, \tag{4.55}$$

so that the Hermitian inner product $\varPsi^\dagger\varPhi$ can be expressed as the trace of the product $\Psi^\dagger\Phi$:

$$\varPsi^\dagger\varPhi = \mathrm{tr}\big(\Psi^\dagger\Phi\big). \tag{4.56}$$

Now, taking

$$\varPsi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \quad \text{and} \quad \varPhi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} b^0 + ib^3 \\ -b^2 + ib^1 \end{pmatrix}, \tag{4.57}$$

it follows that

$$\varPsi^\dagger\varPhi = \big(a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3\big) + i\big(a^0b^3 - a^3b^0 - a^2b^1 + a^1b^2\big), \tag{4.58}$$

so that

$$\mathrm{Re}\big(\mathit{\Psi}^\dagger \mathit{\Phi}\big) = \mathrm{Re}\big(\mathrm{tr}\big(\Psi^\dagger \Phi\big)\big) = a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \tag{4.59}$$

and

$$\begin{aligned}\mathrm{Im}\big(\mathit{\Psi}^\dagger \mathit{\Phi}\big) &= -\mathrm{Re}\big(i\,\mathit{\Psi}^\dagger \mathit{\Phi}\big) \\ &= -\mathrm{Re}\big(\mathrm{tr}\big(i\Psi^\dagger \Phi\big)\big) \\ &= a^0 b^3 - a^3 b^0 - a^2 b^1 + a^1 b^2. \end{aligned}\tag{4.60}$$

Given that $\underline{\psi} = (a^0 + a^i I\boldsymbol{\sigma}_i)f = \psi f$ and $\underline{\phi} = (b^0 + b^i I\boldsymbol{\sigma}_i)f = \phi f$ are the algebraic Pauli spinors corresponding to Ψ and Φ, the geometric product corresponding to $\Psi^\dagger\Phi$ is given by

$$\begin{aligned}\underline{\psi}^\dagger \underline{\phi} &= f\psi^\dagger \phi f \\ &= f\big(a^0 - a^1 I\boldsymbol{\sigma}_1 - a^2 I\boldsymbol{\sigma}_2 - a^3 I\boldsymbol{\sigma}_3\big)\big(b^0 + b^1 I\boldsymbol{\sigma}_1 + b^2 I\boldsymbol{\sigma}_2 + b^3 I\boldsymbol{\sigma}_3\big) f \\ &= f\Big(\big(a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3\big) + \\ &\quad + \big(a^0 b^1 I\boldsymbol{\sigma}_1 + a^0 b^2 I\boldsymbol{\sigma}_2 + a^0 b^3 I\boldsymbol{\sigma}_3 + a^1 b^0 I\boldsymbol{\sigma}_1 + a^1 b^2 I\boldsymbol{\sigma}_3 - a^1 b^3 I\boldsymbol{\sigma}_2 \\ &\qquad - a^2 b^0 I\boldsymbol{\sigma}_2 - a^2 b^1 I\boldsymbol{\sigma}_3 + a^2 b^3 I\boldsymbol{\sigma}_1 - a^3 b^0 I\boldsymbol{\sigma}_3 + a^3 b^1 I\boldsymbol{\sigma}_2 - a^3 b^2 I\boldsymbol{\sigma}_1\big)\Big) f. \end{aligned}\tag{4.61}$$

Then, by noting that $\frac{1}{2}(1 + \boldsymbol{\sigma}_3)I\boldsymbol{\sigma}_k = I\boldsymbol{\sigma}_k\frac{1}{2}(1 - \boldsymbol{\sigma}_3)$ for $k = 1$ or $k = 2$, and that $\frac{1}{2}(1 + \boldsymbol{\sigma}_3)\frac{1}{2}(1 - \boldsymbol{\sigma}_3) = 0$, which imply $fI\boldsymbol{\sigma}_k f = 0$ for $k = 1$ or $k = 2$, one can rewrite the above expression as

$$\begin{aligned}\underline{\psi}^\dagger \underline{\phi} &= \Big(\big(a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3\big) + I\boldsymbol{\sigma}_3\big(a^0 b^3 - a^3 b^0 - a^2 b^1 + a^1 b^2\big)\Big) f \\ &= \Big(\big(a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3\big) + I\big(a^0 b^3 - a^3 b^0 - a^2 b^1 + a^1 b^2\big)\Big) f. \end{aligned}\tag{4.62}$$

This is the corresponding to the equation (4.55) in terms of algebraic Pauli spinors. By comparing it with equations (4.59) and (4.60) one notes that the real and imaginary parts of the Hermitian inner product $\mathit{\Psi}^\dagger \mathit{\Phi}$ can be written respectively as $2\langle\underline{\psi}^\dagger\underline{\phi}\rangle$ and $-2\langle I\underline{\psi}^\dagger\underline{\phi}\rangle$. These in turn can be expressed in terms of operator Pauli spinors as

$$2\langle\underline{\psi}^\dagger\underline{\phi}\rangle = 2\langle f\psi^\dagger\phi f\rangle = 2\langle\psi^\dagger\phi f\rangle = \langle\psi^\dagger\phi\rangle \tag{4.63}$$

and

$$-2\langle I\underline{\psi}^\dagger\underline{\phi}\rangle = -2\langle If\psi^\dagger\phi f\rangle = -2\langle\psi^\dagger\phi If\rangle = -\langle\psi^\dagger\phi I\boldsymbol{\sigma}_3\rangle, \tag{4.64}$$

where was considered the invarance of the scalar part of a geometric product under cyclic permutations of the factors, the fact that odd grade multivectors has null scalar part, and the fact that the geometric product of even grade multivectors is also an even grade multivector. Then, one can write:

$$\mathrm{Re}\big(\mathit{\Psi}^\dagger \mathit{\Phi}\big) = \mathrm{Re}\big(\mathrm{tr}\big(\Psi^\dagger \Phi\big)\big) = 2\langle \underline{\psi}^\dagger \underline{\phi}\rangle = \langle \psi^\dagger \phi\rangle \tag{4.65}$$

and

$$\mathrm{Im}\big(\mathit{\Psi}^\dagger \mathit{\Phi}\big) = -\mathrm{Re}\big(i\mathit{\Psi}^\dagger \mathit{\Phi}\big) = -\mathrm{Re}\big(\mathrm{tr}\big(i\Psi^\dagger \Phi\big)\big) = -2\langle I\underline{\psi}^\dagger \underline{\phi}\rangle = -\langle \psi^\dagger \phi I\boldsymbol{\sigma}_3\rangle. \tag{4.66}$$

Finally, note that the expression (4.62) can be understood as an element of the minimal left ideal $\mathcal{I} = \{Af \mid A \in \mathcal{C}\ell_{3,0} \text{ and } f = \frac{1}{2}(1+\boldsymbol{\sigma}_3)\}$. In particular, it is the image of the classical Pauli spinor $(\mathit{\Psi}^\dagger \mathit{\Phi})\begin{pmatrix}1 & 0\end{pmatrix}^{\mathrm{T}}$ by the map α given in relations (4.44). In this way, the map β in relations (4.44) transforms the expression (4.62) into

$$\langle \psi^\dagger \phi\rangle - \langle \psi^\dagger \phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3, \tag{4.67}$$

which represents the Hermitian inner product $\mathit{\Psi}^\dagger \mathit{\Phi}$ in terms of operator Pauli spinors, in agreement with Doran and Lasenby (2003), who denote this expression by $\langle \psi^\dagger \phi\rangle_q$. In summary, Hermitian inner products of classical, algebraic and operator Pauli spinors can be translated through the mappings

$$\begin{array}{ccc} (\mathit{\Psi}^\dagger \mathit{\Phi})F & \overset{\alpha}{\longmapsto} & \underline{\psi}^\dagger \underline{\phi} = \Big(\langle \psi^\dagger \phi\rangle - \langle \psi^\dagger \phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3\Big) f \\ & \overset{\beta\circ\alpha}{\searrow} & \big\downarrow \beta \\ & & \langle \psi^\dagger \phi\rangle_q = \langle \psi^\dagger \phi\rangle - \langle \psi^\dagger \phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3, \end{array} \tag{4.68}$$

where

$$F = \begin{pmatrix}1\\0\end{pmatrix}. \tag{4.69}$$

In general, given a third Pauli spinor $\mathit{\Xi} \overset{\alpha}{\mapsto} \underline{\xi} \overset{\beta}{\mapsto} \xi$, it follows the maps

$$\begin{array}{ccc} \big(\mathit{\Psi}^\dagger \mathit{\Phi}\big)\mathit{\Xi} = \mathit{\Xi}\big(\mathit{\Psi}^\dagger \mathit{\Phi}\big) & \overset{\alpha}{\longmapsto} & \underline{\xi}\big(\underline{\psi}^\dagger \underline{\phi}\big) = \xi\Big(\langle \psi^\dagger \phi\rangle - \langle \psi^\dagger \phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3\Big) f \\ & \overset{\beta\circ\alpha}{\searrow} & \big\downarrow \beta \\ & & \xi\langle \psi^\dagger \phi\rangle_q = \xi\Big(\langle \psi^\dagger \phi\rangle - \langle \psi^\dagger \phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3\Big), \end{array} \tag{4.70}$$

where it is worth noting the ordering of the product in each case: although the Hermitian inner product commutes with the third Pauli spinor in the classical case, it necessarily appears as a factor on the right in the algebraic case, and this order for the product is preserved in terms of operator spinors, which is in agreement with Doran and Lasenby

(2003).

The *probability density* for a particle with spin $\frac{1}{2}$ described by the classical Pauli spinor $\mathit{\Psi}$ is given by the real and positive-definite quantity

$$\rho = \mathit{\Psi}^\dagger \mathit{\Psi} = |\psi_1|^2 + |\psi_2|^2 = (a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2. \tag{4.71}$$

It can be written in terms of square matrices as $\mathrm{tr}\big(\Psi^\dagger\Psi\big)$. In terms of algebraic Pauli spinors, it is given by $2\langle \underline{\psi}^\dagger \underline{\psi}\rangle$. In terms of operator Pauli spniors, it reduces to $\langle \psi^\dagger\psi\rangle_q = \langle \psi^\dagger\psi\rangle = \psi^\dagger\psi$. In summary,

$$\rho = \mathit{\Psi}^\dagger \mathit{\Psi} = \mathrm{tr}\big(\Psi^\dagger\Psi\big) = 2\langle \underline{\psi}^\dagger \underline{\psi}\rangle = \psi^\dagger\psi. \tag{4.72}$$

For the considered particle, the Hermitian inner product

$$\rho s_j = \frac{\hbar}{2}\mathit{\Psi}^\dagger \Sigma_j \mathit{\Psi} \tag{4.73}$$

defines the components of a vector, which can be understood as a *spin density*. In terms of square matrices these components are given by $\frac{\hbar}{2}\mathrm{tr}\big(\Psi^\dagger\Sigma_j\Psi\big)$. The expression in terms of algebraic Pauli spinors is given by $\hbar\langle \underline{\psi}^\dagger \boldsymbol{\sigma}_j \underline{\psi}\rangle$. These furnish the corresponding expressions in terms of operator Pauli spinors as

$$\hbar\langle f\psi^\dagger\boldsymbol{\sigma}_j\psi f\rangle = \hbar\langle \boldsymbol{\sigma}_j\psi f\psi^\dagger\rangle = \frac{\hbar}{2}\langle \boldsymbol{\sigma}_j\psi\boldsymbol{\sigma}_3\psi^\dagger\rangle. \tag{4.74}$$

Note then that $\psi\boldsymbol{\sigma}_3\psi^\dagger$ is an odd grade multivector (since the product of an even grade multivector with an odd grade multivector is an odd grade multivector) and it is equal to its reverse. This shows that $\psi\boldsymbol{\sigma}_3\psi^\dagger$ is a vector, so that the above expression is a scalar product,

$$\frac{\hbar}{2}\langle \boldsymbol{\sigma}_j\psi\boldsymbol{\sigma}_3\psi^\dagger\rangle = \frac{\hbar}{2}\boldsymbol{\sigma}_j\cdot\big(\psi\boldsymbol{\sigma}_3\psi^\dagger\big). \tag{4.75}$$

In this way, the components of the spin density can be expressed by

$$\frac{\hbar}{2}\mathit{\Psi}^\dagger \Sigma_j \mathit{\Psi} = \frac{\hbar}{2}\mathrm{tr}\big(\Psi^\dagger\Sigma_j\Psi\big) = \hbar\langle \underline{\psi}^\dagger \underline{\psi}\rangle = \frac{\hbar}{2}\boldsymbol{\sigma}_j\cdot\big(\psi\boldsymbol{\sigma}_3\psi^\dagger\big). \tag{4.76}$$

The spin density vector is then identified as

$$\rho\mathbf{s} = \frac{\hbar}{2}\psi\boldsymbol{\sigma}_3\psi^\dagger \tag{4.77}$$

(cf. Doran and Lasenby (2003)). Since ψ is an even grade multivector, it can be expressed by

$$\psi = \rho^{\frac{1}{2}}R, \tag{4.78}$$

where ρ is the probability density and R is a rotor, that is, an even grade multivector satisfying $R^\dagger R = RR^\dagger = 1$. Application of this expression in the expression for the spin density vector furnishes the *spin vector* as

$$\mathbf{s} = \frac{\hbar}{2} R \boldsymbol{\sigma}_3 R^\dagger \tag{4.79}$$

(cf. Doran and Lasenby (2003)). This expression shows as the spin vector can be obtained through the rotation transformation $\mathbf{u} \mapsto R\mathbf{u}R^\dagger$ applied on the reference vector $\frac{\hbar}{2}\boldsymbol{\sigma}_3$, which makes clear also that the spin density vector can be obtained through such a rotation transformation followed by a dilation transformation, given by multiplication by ρ, applied on the same reference vector $\frac{\hbar}{2}\boldsymbol{\sigma}_3$ (cf. Doran and Lasenby (2003)).

4.1.5 Pauli Equation

The Lagrangian function for a non-relativistic particle of mass m and charge q moving under the action of an electromagnetic field described by a scalar potential ϕ and a vector potential $\mathbf{A}$ is given by

$$L = \frac{1}{2} m \dot{\mathbf{x}}^2 - q\phi + q\dot{\mathbf{x}} \cdot \mathbf{A}. \tag{4.80}$$

Since the canonical momentum of the particle is given by

$$\mathbf{p} = \boldsymbol{\nabla}_{\dot{\mathbf{x}}} L = m\dot{\mathbf{x}} + q\mathbf{A}, \tag{4.81}$$

which is clearly different from the "kinematic" momentum $m\dot{\mathbf{x}}$, the Hamiltonian function for the particle is

$$H = \dot{\mathbf{x}} \cdot \mathbf{p} - L = \frac{1}{2} m \dot{\mathbf{x}}^2 + q\phi = \frac{1}{2m}\big(\mathbf{p} - q\mathbf{A}\big)^2 + q\phi. \tag{4.82}$$

As explained by Fock (1978), the transition to the quantum treatment of a non-relativistic particle with spin $\frac{1}{2}$ is made by expressing the vector observables for the particle in terms of the Pauli matrices, in the same way as the spin observable. In this way, the wave equation for such a particle can be written by converting the functional value of the Hamiltonian function H, i.e. the energy, into the operator $i\hbar\frac{\partial}{\partial t}$ and the canonical momentum $\mathbf{p}$ into the momentum operator $-i\hbar\Sigma_j\partial^j$, as usual in the Schrödinger representation, but now expressing the vector observables as linear combinations of the Pauli matrices (the scalar potential is converted simply through a multiplication by the identity matrix), and then applying the resulting expression for the Hamiltonian operator on a "two-component wave function" $\mathit{\Psi}$, i.e. a classical Pauli spinor. The resulting wave equation is then

$$\left(\frac{1}{2m}\Big(-i\hbar\Sigma_j\partial^j - q\Sigma_j A^j\Big)^2 + q\phi\right)\mathit{\Psi} = i\hbar\frac{\partial}{\partial t}\mathit{\Psi}. \tag{4.83}$$

This equation, known as the *Pauli equation*, encodes the interaction of the non-relativistic spin-$\frac{1}{2}$ particle with the external electromagnetic field.

In terms of the algebraic Pauli spinor $\underline{\psi}$ corresponding to the classical Pauli spinor Ψ, and by converting the matrix operators into the corresponding elements of the geometric algebra of the three-dimensional Euclidean space, $\mathcal{C}\ell_{3,0}$, one can write the algebraic version of the Pauli equation as

$$\left(\frac{1}{2m}\Big(-I\hbar\boldsymbol{\sigma}_j\partial^j - q\boldsymbol{\sigma}_jA^j\Big)^2 + q\phi\right)\underline{\psi} = I\hbar\frac{\partial}{\partial t}\underline{\psi}, \tag{4.84}$$

or better,

$$\left(\frac{1}{2m}\Big(-I\hbar\boldsymbol{\nabla} - q\mathbf{A}\Big)^2 + q\phi\right)\underline{\psi} = I\hbar\frac{\partial}{\partial t}\underline{\psi}, \tag{4.85}$$

where $\boldsymbol{\nabla} = \boldsymbol{\sigma}_j\partial^j$ is the vector derivative for $\mathcal{C}\ell_{3,0}$, and $\mathbf{A} = \boldsymbol{\sigma}_jA^j$ is the vector potential. Now, this equation can be expressed in terms of the operator Pauli spinor ψ corresponding to the algebraic Pauli spinor $\underline{\psi} = \psi f$ as follows,

$$\left(\frac{1}{2m}\Big(-I\hbar\boldsymbol{\nabla} - q\mathbf{A}\Big)^2 + q\phi\right)\psi\frac{1}{2}(1+\boldsymbol{\sigma}_3) = I\hbar\frac{\partial}{\partial t}\psi\frac{1}{2}(1+\boldsymbol{\sigma}_3). \tag{4.86}$$

From the linear independence of the even grade and odd grade parts of a multivector, the above equation must be equivalent to its even grade and odd grade parts, which can be written respectively as

$$\left(\frac{1}{2m}\Big(\hat{\mathbf{p}} - q\mathbf{A}\Big)^2 + q\phi\right)(\psi) = I\hbar\frac{\partial}{\partial t}\psi\boldsymbol{\sigma}_3 \tag{4.87}$$

and

$$\left(\frac{1}{2m}\Big(\hat{\mathbf{p}} - q\mathbf{A}\Big)^2 + q\phi\right)(\psi)\boldsymbol{\sigma}_3 = I\hbar\frac{\partial}{\partial t}\psi, \tag{4.88}$$

where it was necessary to introduce the multivector operator $\hat{\mathbf{p}}$, given by

$$\hat{\mathbf{p}}(\psi) = -I\hbar\boldsymbol{\nabla}\psi\boldsymbol{\sigma}_3 = -\hbar\boldsymbol{\nabla}\psi I\boldsymbol{\sigma}_3, \tag{4.89}$$

which is the corresponding to the momentum operator. The equations (4.87) and (4.88) are exactly the same equation, since the product from the right by $\boldsymbol{\sigma}_3$ is invertible, and it can be written as

$$\left(\frac{1}{2m}\Big(\hat{\mathbf{p}} - q\mathbf{A}\Big)^2 + q\phi\right)(\psi) = \hbar\frac{\partial\psi}{\partial t}I\boldsymbol{\sigma}_3. \tag{4.90}$$

This is the Pauli equation for an operator Pauli spinor, in agreement with Hestenes (1971). It follows that this operator version of the Pauli equation is encoded two times in the algebraic version, equation (4.85), as its even grade and odd grade parts. It is worth to note that these translations of the Pauli equation can be understood in terms of applications of the maps α and β in relations (4.44).

4.2 Relativistic Theory

4.2.1 Introduction

As can be seen in the vast literature on quantum mechanics (see, for example, the already quoted texts by Piza (2003) and Messiah (2014), and the text on quantum field theory by Ryder (1996)), the relativistic generalization of the Schödinger equation can be obtained directly from the energy-momentum-mass relation,

$$p^\mu p_\mu = \frac{E^2}{c^2} - \mathbf{p}^2 = (mc)^2, \tag{4.91}$$

by performing the replacement of p_μ by the differential operator $i\hbar\partial_\mu$ and applying the resulting differential operator in a wave function ϕ (a complex scalar field) to obtain the equation

$$\left(\partial^\mu\partial_\mu + \kappa^2\right)\phi = 0, \tag{4.92}$$

where $\kappa = mc/\hbar$. This equation of motion, known as the *Klein-Gordon equation*, describes the wave function for a free spin-0 particle of mass m.

The relativistic quantum mechanics of a particle with spin $\frac{1}{2}$ is founded on the *Dirac equation*,

$$i\hbar\Gamma^\mu\partial_\mu\mathit{\Psi} = mc\mathit{\Psi}, \tag{4.93}$$

obtained by Dirac through a heuristic procedure which can be intuitively understood as an "extraction of the square root" of the differential operator in the Klein-Gordon equation to obtain the differential operator $i\hbar\Gamma^\mu\partial_\mu - mc$, where Γ^μ, with $\mu \in \{0, 1, 2, 3\}$, form a set of operators which can be represented by matrices, m is the mass and $\mathit{\Psi}$ represents the wave function for the particle, which can be represented by a column matrix with complex entries (DIRAC, 1982). Following this idea, the equation can be justified by requiring that the energy-momentum-mass relation must be satisfied also in this case, so that $\mathit{\Psi}$ must satisfy also the Klein-Gordon equation. Along this line, by applying again the matrix operator $i\hbar\Gamma^\mu\partial_\mu = mc$ in the Dirac equation, one obtains

$$-\hbar^2\Gamma^\mu\Gamma^\nu\partial_\mu\partial_\nu\mathit{\Psi} = (mc)^2\,\mathit{\Psi}. \tag{4.94}$$

The requirement of symmetry of the second derivatives allows one to write

$$-\hbar^2 \frac{1}{2}(\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu)\partial_\mu \partial_\nu \Psi = (mc)^2 \Psi, \tag{4.95}$$

which reduces to the Klein-Gordon equation for Ψ provided that

$$\frac{1}{2}(\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu)\partial_\mu \partial_\nu = \partial^\mu \partial_\nu, \tag{4.96}$$

that is, in the context of the Dirac theory, the symmetric part of the product of two of the Dirac matrices Γ^μ must act "raising spacetime indices" in the same way as the metric tensor for Minkowski spacetime: the product of operators associated to the observables, which are generated by the Dirac matrices Γ^μ, must satisfy the basic property

$$\frac{1}{2}\left(\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu\right) = \eta^{\mu\nu} 1, \tag{4.97}$$

where 1 is the identity matrix and $\eta^{\mu\nu}$ is the metric tensor for Minkowski spacetime, given by $\eta^{00} = 1$, $\eta^{ij} = -\delta^{ij}$ for $i, j \in \{1, 2, 3\}$ and $\eta^{\mu\nu} = 0$ for $\mu, \nu \in \{0, 1, 2, 3\}$ and $\mu \neq \nu$. In the same way, the symmetric part of the product of a pair of the covariant version of the Dirac matrices, Γ_μ, must act "lowering spacetime indices" in the same way as the covariant version of the metric tensor for Minkowski spacetime $\eta_{\mu\nu}$, which is such that $\eta^{\mu\lambda}\eta_{\lambda\nu} = \delta^\mu{}_\nu$. The index of a Dirac matrix itself acts as a spacetime index, which can be "raised" or "lowered" by the metric tensor. In this way, it follows the relations $\Gamma^0 = \Gamma_0$ and $\Gamma^i = -\Gamma_i$, for $i \in \{1, 2, 3\}$.

Dirac concluded that the matrices representing his operators must be 4×4 complex matrices, and so he obtained such a representation (PIZA, 2003; MESSIAH, 2014; RYDER, 1996). However, there is not a unique matrix representation for the Dirac operators, although it is always possible to represent them as 2×2 block matrices in terms of the Pauli matrices and the 2×2 identity and null matrices. The most usual representation is given by

$$\Gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \Gamma_i = \begin{pmatrix} 0 & -\Sigma_i \\ \Sigma_i & 0 \end{pmatrix}, \tag{4.98}$$

where the entries 1 and 0 are respectively the 2×2 identity and null matrices, and Σ_i are the Pauli matrices. The matrix

$$\Gamma_5 = -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 \tag{4.99}$$

is important and, according to the above representation is given by

$$\Gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.100}$$

In agreement with this representation the wave function is given by a column matrix with four complex entries,

$$\mathit{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{4.101}$$

Such a wave function corresponds to a *Dirac spinor*, more precisely, it defines a *classical Dirac spinor* (FIGUEIREDO *et al.*, 1990; VAZ; DA ROCHA, 2019).

4.2.2 From Dirac Equation to Dirac-Hestenes Equation

It is a known fact, whose possibility had already been outlined in the early 1930s by Sauter (1930) and Juvet (1930, 1932), that the traditional Dirac equation can be rewritten in an equivalent way by replacing the wave function by a suitable element from the Dirac algebra, which is currently recognized to be $\mathcal{C}\ell_{1,3}(\mathbb{C}) \simeq \mathbb{C} \otimes \mathcal{C}\ell_{1,3}$, that is, the geometric algebra of spacetime with the field of real scalars replaced by the field of complex scalars, usually called the *complexified* geometric algebra of spacetime (FIGUEIREDO *et al.*, 1990; LOUNESTO, 2001; DA ROCHA; VAZ, 2007). More precisely, thinking in terms of matrices, the following equivalence relation holds,

$$\mathit{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad \sim \quad \Psi = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix}, \tag{4.102}$$

so that the Dirac equation can be written in the form

$$i\hbar\Gamma^\mu \partial_\mu \Psi - q\Gamma^\mu A_\mu \Psi = mc\Psi, \tag{4.103}$$

where the term encoding the interaction of the spin-$\frac{1}{2}$ particle with an electromagnetic field, expressible through the potential A_μ, is included. Note then that the wave function Ψ can be put in the form

$$\Psi = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \psi_1 & \psi_{12} & \psi_{13} & \psi_{14} \\ \psi_2 & \psi_{22} & \psi_{23} & \psi_{24} \\ \psi_3 & \psi_{32} & \psi_{33} & \psi_{34} \\ \psi_4 & \psi_{42} & \psi_{43} & \psi_{44} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.104}$$

where the entries of the matrix to the left at the right-hand side, with exception of the first column, are arbitrary. Note also that the matrix

$$\mathrm{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.105}$$

is idempotent, that is $\mathrm{F}^2 = \mathrm{F}$, and it can be expressed in terms of Dirac matrices by

$$\mathrm{F} = \frac{1}{2}(1 + \Gamma_0)\frac{1}{2}(1 + i\Gamma_1\Gamma_2). \tag{4.106}$$

Consider then the matrices

$$i\Gamma_2\Gamma_3\mathrm{F} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \Gamma_3\Gamma_0\mathrm{F} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \Gamma_1\Gamma_0\mathrm{F} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{4.107}$$

and note that the wave function Ψ can be expressed as

$$\Psi = \psi_1\mathrm{F} + \psi_2 i\Gamma_2\Gamma_3\mathrm{F} + \psi_3\Gamma_3\Gamma_0\mathrm{F} + \psi_4\Gamma_1\Gamma_0\mathrm{F}. \tag{4.108}$$

Now the matrix representation can be abandoned and the Dirac equation can be written

$$i\hbar\gamma^\mu\partial_\mu\underline{\psi} - q\gamma^\mu A_\mu\underline{\psi} = mc\underline{\psi}, \tag{4.109}$$

where the Dirac matrices were replaced by the corresponding elements of the canonical basis $\{\gamma^\mu\}$ of the Minkowski vector space, $\mathbb{R}^{1,3}$, the matrix product was replaced by the geometric product of the geometric algebra of spacetime, $\mathcal{C}\ell_{1,3}$, and the wave function is now given by

$$\underline{\psi} = (\psi_1 + \psi_2 i\gamma_{23} + \psi_3\gamma_{30} + \psi_4\gamma_{10})f, \tag{4.110}$$

with f being an idempotent from $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ given by

$$f = \frac{1}{2}(1 + \gamma_0)\frac{1}{2}(1 + i\gamma_{12}) \tag{4.111}$$

(note that the notation $\gamma_{\mu\nu} = \gamma_\mu\gamma_\nu$ has been introduced). At this point, a connection has been established with the algebraic definition of a spinor, as an element of a minimal left ideal of a Clifford algebra (FIGUEIREDO *et al.*, 1990; LOUNESTO, 2001; VAZ; DA ROCHA, 2019), since f is a primitive idempotent from the algebra $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ and the multivector in parentheses in the expression (4.110) is an element from $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$. Such an element need not to be a complex multivector. Indeed, by writing $\psi_A = r_A + is_A$, where $r_A, s_A \in \mathbb{R}$ and $A \in \{1,2,3,4\}$, and by considering the property $i\gamma_{12}f = f$ of the idempotent f, one can rewrite $\underline{\psi}$ in (4.110) as follows,

$$\begin{aligned}\underline{\psi} &= \Big((r_1 + is_1) + (r_2 + is_2)i\gamma_{23} + (r_3 + is_3)\gamma_{30} + (r_4 + is_4)\gamma_{10}\Big)f \\ &= \Big((r_1 + s_1\gamma_{21}) + (r_2\gamma_{31} + s_2\gamma_{32}) + (r_3\gamma_{30} + s_3 I) + (r_4\gamma_{10} + s_4\gamma_{20})\Big)f, \end{aligned} \tag{4.112}$$

where $I = \gamma_0\gamma_1\gamma_2\gamma_3$. In this way, it is noted that an *algebraic Dirac spinor* can be written in the form

$$\underline{\psi} = \psi f, \tag{4.113}$$

where $\psi \in \mathcal{C}\ell_{1,3}{}^{+}$, which allows one to define it as an element of the minimal left ideal $\mathcal{I} = \{Af \mid A \in \mathbb{C}\otimes\mathcal{C}\ell_{1,3} \text{ and } f = \frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_{12})\}$. In fact, similarly to the case of the algebraic Pauli spinors, an algebraic Dirac spinor can be defined as an element of a minimal left ideal generated by any idempotent obtained from f through a Lorentz transformation (HILEY; CALLAGHAN, 2010), and the choice of an idempotent reflects the choice of a reference frame and a quantization direction, which consequently define the particular choice for the matrix representation. The conventional choice for $f = \frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+\boldsymbol{\sigma}_3)$ corresponds to the choice of the reference frame determined by γ_0 and the choice of the z-axis as the quantization direction. This choice determines the standard matrix representation adopted in this context.

As observed from the expression (4.113), all the content of the wave function reduces to an even grade element of the real algebra $\mathcal{C}\ell_{1,3}$. Such an element defines an *operator Dirac spinor* (FIGUEIREDO *et al.*, 1990; LOUNESTO, 2001; VAZ; DA ROCHA, 2019). It is natural to ask for an equation, or a set of equations, equivalent to the Dirac equation but expressed entirely in terms of the real algebra $\mathcal{C}\ell_{1,3}$. This can be accomplished as follows.

The algebraic version of Dirac equation, that is, equation (4.109), can be expressed as

$$i\hbar\gamma^\mu\partial_\mu\psi\frac{1}{4}(1+\gamma_0)(1+i\gamma_{12}) - q\gamma^\mu A_\mu\psi\frac{1}{4}(1+\gamma_0)(1+i\gamma_{12}) = mc\psi\frac{1}{4}(1+\gamma_0)(1+i\gamma_{12}), \tag{4.114}$$

where, as seen above, ψ is an even grade multivector from the real geometric algebra of spacetime, $\mathcal{C}\ell_{1,3}$. This equation is equivalent to its real and imaginary parts, respectively given by

$$\hbar\gamma^\mu\partial_\mu\psi(1+\gamma_0)\gamma_{21} - q\gamma^\mu A_\mu\psi(1+\gamma_0) = mc\psi(1+\gamma_0) \tag{4.115}$$

and

$$\hbar\gamma^\mu\partial_\mu\psi(1+\gamma_0) - q\gamma^\mu A_\mu\psi(1+\gamma_0)\gamma_{12} = mc\psi(1+\gamma_0)\gamma_{12}. \tag{4.116}$$

These equations, which are equivalent (since the product from the right by the bivector

γ_{12} is invertible), are expressed in terms of the real algebra $\mathcal{C}\ell_{1,3}$. In additon, each of these equations should be equivalent to its even grade and odd grade parts. In particular, equation (4.115) is equivalent to its even grade and odd grade parts, respectively given by

$$\hbar\gamma^\mu\partial_\mu\psi\gamma_0\gamma_{21} - q\gamma^\mu A_\mu\psi\gamma_0 = mc\psi \tag{4.117}$$

and

$$\hbar\gamma^\mu\partial_\mu\psi\gamma_{21} - q\gamma^\mu A_\mu\psi = mc\psi\gamma_0, \tag{4.118}$$

and equation (4.116) it is also equivalent to its even grade and odd grade parts, respectively given by

$$\hbar\gamma^\mu\partial_\mu\psi\gamma_0 - q\gamma^\mu A_\mu\psi\gamma_0\gamma_{12} = mc\psi\gamma_{12} \tag{4.119}$$

and

$$\hbar\gamma^\mu\partial_\mu\psi - q\gamma^\mu A_\mu\psi\gamma_{12} = mc\psi\gamma_0\gamma_{12}. \tag{4.120}$$

Equations (4.117), (4.118), (4.119) and (4.120) are all equivalent and correspond to the usual Dirac equation for an operator Dirac spinor (LOUNESTO, 2001; DORAN; LASENBY, 2003), which is usually known as the *Dirac-Hestenes equation*, since it was obtained by Hestenes in the 1960s (HESTENES, 1967; HESTENES, 1975). The quadruplicate derivation above is not known to the author prior to this work.

This result means that not only is the traditional Dirac equation equivalent to the Dirac-Hestenes equation as the latter is encoded four times in the former, under the form of (I) the even grade part of the real part, (II) the odd grade part of the real part, (III) the even grade part of the imaginary part, and (IV) the odd grade part of the imaginary part of the algebraic version of Dirac equation, that is, equation (4.109), or equation (4.114). This fact suggests that both the classical version and the algebraic version of the Dirac equation contain redundant information and that the Dirac-Hestenes equation eventually contains the minimum information required to describe a spin-$\frac{1}{2}$ particle. This idea seems sensible, given that the traditional Dirac theory is described through a complex algebra of operators, corresponding to $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$, acting on a Hilbert space, and the algebraic version, although described through a single structure, employs also the complexified geometric algebra of spacetime, $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$. The formulation in terms of the Dirac-Hestenes equation, in turn, is based on the real geometric algebra of spacetime, $\mathcal{C}\ell_{1,3}$, which has half of the real dimension of the complexified algebra $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$.

4.2.3 The Relation between Classical, Algebraic and Operator Dirac Spinors

Once a version of the Dirac equation expressed entirely of a real algebra has been obtained, it is desirable to obtain a scheme of translation from the classical wave function to its operator version. This has been done implicitly in the previous subsection. When relation (4.112) was obtained, at the same time, the following correspondence between a classical and an algebraic Dirac spinor was established (cf. equations (4.102), (4.108) and (4.112)),

$$\Psi = \begin{pmatrix} a^0 + ia^{21} \\ a^{31} + ia^{32} \\ a^{30} + ia^5 \\ a^{10} + ia^{20} \end{pmatrix} \quad \sim \quad \underline{\psi} = \left(a^0 + \sum_{\mu>\nu} a^{\mu\nu}\gamma_{\mu\nu} + a^5 I \right) f, \tag{4.121}$$

where $a^0, a^{\mu\nu}, a^5 \in \mathbb{R}$. Note that, by considering the isomorphism $\mathcal{C}\ell_{1,3}{}^+ \simeq \mathcal{C}\ell_{3,0}$ via the correspondences $\boldsymbol{\sigma}_i = \gamma_i\gamma_0$ (cf. section 3.1), where $\{\boldsymbol{\sigma}_i\}$ is an orthonormal basis of $\mathbb{R}^3$, the above equivalence relation can be put in the form

$$\Psi = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \\ b^0 + ib^3 \\ -b^2 + ib^1 \end{pmatrix} \quad \sim \quad \underline{\psi} = \Big((a^0 + a^k I\boldsymbol{\sigma}_k) + (b^0 + b^k I\boldsymbol{\sigma}_k)\boldsymbol{\sigma}_3 \Big) f, \tag{4.122}$$

where $a^\mu, b^\mu \in \mathbb{R}$, which is similar to the usual correspondence between a classical and an operator Dirac spinor (cf. e.g. Doran and Lasenby (2003)) except for the idempotent f as factor on the right. This fact shows that, given an algebraic Dirac spinor $\underline{\psi}$ and its corresponding operator spinor ψ, it follows that the latter is four times the even grade part of the real part of the former (cf. e.g. Lounesto (2001)). This fact can be denoted by $\psi = 4\langle \mathrm{Re}(\underline{\psi})\rangle_+$, where $\langle A\rangle_+$ denotes the even grade part of the multivector A from $\mathcal{C}\ell_{1,3}$. It can be noted that ψ can also be obtained from the odd grade part of the real part of $\underline{\psi}$, more precisely, $\psi = 4\langle \mathrm{Re}(\underline{\psi})\rangle_-\gamma_0$, where $\langle A\rangle_-$ denotes the odd grade part of the multivector A from $\mathcal{C}\ell_{1,3}$. But there are also two other ways of expressing ψ in terms of $\underline{\psi}$, namely, $4\langle \mathrm{Im}(\underline{\psi})\rangle_+\gamma_{21} = 4\langle \mathrm{Im}(\underline{\psi})\rangle_-\gamma_0\gamma_{21}$. Thus, the correspondence relations between classical, algebraic and operator Dirac spinors, respectively, can be expressed by the following maps:

$$
\begin{array}{ccc}
\Psi = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \\ b^0 + ib^3 \\ -b^2 + ib^1 \end{pmatrix} & \xmapsto{\alpha} & \underline{\psi} = \Big((a^0 + a^k \boldsymbol{I}\boldsymbol{\sigma}_k) + (b^0 + b^k \boldsymbol{I}\boldsymbol{\sigma}_k)\boldsymbol{\sigma}_3\Big) f = \psi f \\
& \searrow^{\beta\circ\alpha} & \downarrow \beta \\
& & \begin{aligned} \psi = (a^0 + a^k \boldsymbol{I}\boldsymbol{\sigma}_k) + (b^0 + b^k \boldsymbol{I}\boldsymbol{\sigma}_k)\boldsymbol{\sigma}_3 &= 4\langle \mathrm{Re}(\underline{\psi})\rangle_+ \\ &= 4\langle \mathrm{Re}(\underline{\psi})\rangle_- \gamma_0 \\ &= 4\langle \mathrm{Im}(\underline{\psi})\rangle_+ \gamma_{21} \\ &= 4\langle \mathrm{Im}(\underline{\psi})\rangle_- \gamma_0\gamma_{21}. \end{aligned}
\end{array}
\tag{4.123}
$$

In this way, an operator Dirac spinor can be obtained in four equivalent ways from its algebraic counterpart, and consequently from its classical counterpart, a fact already outlined by Hiley and Callaghan (2010). The transformations above are presented explicitly for the first time here.

The action of operators is now translated in a straightforward manner, given by the correspondences

$$
\begin{array}{ccc}
\Gamma_\mu \Psi & \xmapsto{\alpha} & \gamma_\mu \underline{\psi} = \gamma_\mu \psi f = \gamma_\mu \psi \gamma_0 f \\
& \searrow^{\beta\circ\alpha} & \downarrow \beta \\
& & 4\langle \mathrm{Re}(\gamma_\mu \underline{\psi})\rangle_+ = \gamma_\mu \psi \gamma_0,
\end{array}
\tag{4.124}
$$

$$
\begin{array}{ccc}
\Gamma_5 \Psi & \xmapsto{\alpha} & \gamma_5 \underline{\psi} = \gamma_5 \psi f = \psi \gamma_5 i \gamma_{12} f = \psi \boldsymbol{\sigma}_3 f \\
& \searrow^{\beta\circ\alpha} & \downarrow \beta \\
& & 4\langle \mathrm{Re}(\gamma_5 \underline{\psi})\rangle_+ = \psi \boldsymbol{\sigma}_3,
\end{array}
\tag{4.125}
$$

and

$$
\begin{array}{ccc}
i \Psi & \xmapsto{\alpha} & i \underline{\psi} = i \psi f = i \psi i \gamma_{12} f = \psi \boldsymbol{I}\boldsymbol{\sigma}_3 f \\
& \searrow^{\beta\circ\alpha} & \downarrow \beta \\
& & 4\langle \mathrm{Re}(i \underline{\psi})\rangle_+ = \psi \boldsymbol{I}\boldsymbol{\sigma}_3,
\end{array}
\tag{4.126}
$$

where $\gamma_5 = -iI = -i\gamma_0\gamma_1\gamma_2\gamma_3$ and the properties $\gamma_0 f = f$ and $i\gamma_{12} f = f$ have been used. As before, the map $\beta \circ \alpha$ reproduces the transformation for action of operators from the classical to the operator representation (cf. Doran and Lasenby (2003)). The explicit derivation of the above relations are presented for the first time here.

4.2.4 Dirac Adjoint and Bilinear Covariant Expressions

The *Dirac adjoint* of the classical Dirac spinor $\mathit{\Psi}$ in relation (4.102) is given by

$$\bar{\mathit{\Psi}} = \mathit{\Psi}^\dagger \Gamma_0 = \begin{pmatrix} \psi_1{}^* & \psi_2{}^* & -\psi_3{}^* & -\psi_4{}^* \end{pmatrix}, \tag{4.127}$$

so that its representation as a square matrix is given by

$$\bar{\Psi} = \Psi^\dagger \Gamma_0 = \begin{pmatrix} \psi_1{}^* & \psi_2{}^* & -\psi_3{}^* & -\psi_4^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.128}$$

This can be written as

$$\bar{\Psi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1{}^* & \psi_2{}^* & -\psi_3{}^* & -\psi_4{}^* \\ \psi_{12}{}^* & \psi_{22}{}^* & -\psi_{32}{}^* & -\psi_{42}{}^* \\ \psi_{13}{}^* & \psi_{23}{}^* & -\psi_{33}{}^* & -\psi_{43}{}^* \\ \psi_{14}{}^* & \psi_{24}{}^* & -\psi_{34}{}^* & -\psi_{44}{}^* \end{pmatrix}, \tag{4.129}$$

where the entries of the matrix to the right at the right-hand side, with exception of the first row, are arbitrary. In terms of Dirac matrices, this expression reads

$$\bar{\Psi} = \psi_1{}^*\mathrm{F} + \psi_2{}^*(i\Gamma_2\Gamma_3\mathrm{F})^\dagger - \psi_3{}^*(\Gamma_3\Gamma_0\mathrm{F})^\dagger - \psi_4{}^*(\Gamma_1\Gamma_0\mathrm{F})^\dagger, \tag{4.130}$$

that is,

$$\bar{\Psi} = \psi_1{}^*\mathrm{F} - \psi_2{}^* i\mathrm{F}\Gamma_3\Gamma_2 + \psi_3{}^*\mathrm{F}\Gamma_0\Gamma_3 + \psi_4{}^*\mathrm{F}\Gamma_0\Gamma_1. \tag{4.131}$$

As before, the matrix representation can be abandoned, and the algebraic version of the above adjoint classical Dirac spinor can be written as

$$\bar{\psi} = f(\psi_1{}^* + \psi_2{}^* i\gamma_{23} - \psi_3{}^*\gamma_{30} - \psi_4{}^*\gamma_{10}). \tag{4.132}$$

Again as before, by writing $\psi_A = r_A + is_A$, where $r_A, s_A \in \mathbb{R}$ and $A \in \{1,2,3,4\}$, and by considering the property $i\gamma_{12}f = f$ of the idempotent f, one can rewrite $\bar{\psi}$ as

$$\begin{aligned} \bar{\psi} &= f\Big((r_1 - is_1) + (r_2 - is_2)i\gamma_{23} - (r_3 - is_3)\gamma_{30} - (r_4 - is_4)\gamma_{10}\Big) \\ &= f\Big((r_1 - s_1\gamma_{21}) + (-r_2\gamma_{31} - s_2\gamma_{32}) + (-r_3\gamma_{30} + s_3 I) + (-r_4\gamma_{10} - s_4\gamma_{20})\Big), \end{aligned} \tag{4.133}$$

which corresponds to the complex conjugate of the reverse of $\underset{\sim}{\psi}$. In this way, in view of the relations (4.123), one obtains the following correspondence relations between the classical adjoint Dirac spinor and its algebraic and operator counterparts:

$$\begin{array}{ccc} \bar{\mathit{\Psi}} = \mathit{\Psi}^\dagger\Gamma_0 & \xmapsto{\alpha} & \bar{\underline{\psi}} = \tilde{\underline{\psi}}^* = f\tilde{\psi} \\ & \searrow^{\beta\circ\alpha} & \downarrow \beta \\ & & 4\langle \mathrm{Re}(\bar{\underline{\psi}})\rangle_+ = \tilde{\psi}. \end{array} \tag{4.134}$$

Now, products of the form $\bar{\mathit{\Psi}}\mathit{\Phi}$, where $\mathit{\Psi}$ and $\mathit{\Phi}$ are classical Dirac spinors, can be expressed in terms of algebraic and operator Dirac spinors. For this, note that, since such a product is a scalar, it corresponds to the trace of the corresponding product in terms of square matrices, that is,

$$\bar{\mathit{\Psi}}\mathit{\Phi} = \mathrm{tr}(\bar{\Psi}\Phi), \tag{4.135}$$

where Ψ and Φ are the square matrices corresponding to $\mathit{\Psi}$ and $\mathit{\Phi}$. This is easily visualized by noting that

$$\bar{\Psi}\Phi = ({\psi_1}^*\phi_1 + {\psi_2}^*\phi_2 - {\psi_3}^*\phi_3 - {\psi_4}^*\phi_4)\mathrm{F}, \tag{4.136}$$

whose trace corresponds to $\bar{\mathit{\Psi}}\mathit{\Phi} = {\psi_1}^*\phi_1 + {\psi_2}^*\phi_2 - {\psi_3}^*\phi_3 - {\psi_4}^*\phi_4$. Since the algebraic version of the above expression is

$$\bar{\underline{\psi}}\underline{\phi} = ({\psi_1}^*\phi_1 + {\psi_2}^*\phi_2 - {\psi_3}^*\phi_3 - {\psi_4}^*\phi_4)f, \tag{4.137}$$

where $\underline{\psi}$ and $\underline{\phi}$ are the algebraic spinors corresponding to $\mathit{\Psi}$ and $\mathit{\Phi}$, the trace operation furnishing $\bar{\mathit{\Psi}}\mathit{\Phi}$ corresponds to four times the scalar part of $\bar{\underline{\psi}}\underline{\phi}$:

$$\bar{\mathit{\Psi}}\mathit{\Phi} = \mathrm{tr}(\bar{\Psi}\Phi) = 4\langle\bar{\underline{\psi}}\underline{\phi}\rangle. \tag{4.138}$$

In this way, if $\psi = 4\langle\mathrm{Re}(\underline{\psi})\rangle_+$ and $\phi = 4\langle\mathrm{Re}(\underline{\phi})\rangle_+$, that is, ψ and ϕ are the operator spinors corresponding to $\underline{\psi}$ and $\underline{\phi}$, then $\underline{\psi} = \psi f$ and $\underline{\phi} = \phi f$, and the above product can be written also as

$$4\langle\tilde{\underline{\psi}}^*\underline{\phi}\rangle = 4\langle f\tilde{\psi}\phi f\rangle = 4\langle\tilde{\psi}\phi f\rangle, \tag{4.139}$$

where the property of invariance of the scalar part of a geometric product with relation to cyclic permutations of the factors and the fact that f is idempotent were used. Then, by expressing the idempotent f in expanded form, one can write the last expression as

$$4\langle\tilde{\psi}\phi f\rangle = \langle\tilde{\psi}\phi(1 + \gamma_0 + i\gamma_{12} + i\gamma_0\gamma_{12})\rangle. \tag{4.140}$$

From the fact that the geometric product of even grade multivectors is also an even grade multivector and the geometric product of an even grade multivector and an odd grade multivector is an odd grade multivector, it follows that the multivectors $\tilde{\psi}\phi\gamma_0$ and $\tilde{\psi}\phi\gamma_0\gamma_{12}$ are odd grade and, consequently, $\langle\tilde{\psi}\phi\gamma_0\rangle = \langle\tilde{\psi}\phi\gamma_0\gamma_{12}\rangle = 0$. Thus, the above expression reduces to

$$\langle\tilde{\psi}\phi\rangle + i\langle\tilde{\psi}\phi\gamma_{12}\rangle. \tag{4.141}$$

In summary, one has the following equivalent expressions:

$$\bar{\mathit{\Psi}}\mathit{\Phi} = \mathrm{tr}(\bar{\Psi}\Phi) = 4\langle\bar{\psi}\underline{\phi}\rangle = \langle\tilde{\psi}\phi\rangle + i\langle\tilde{\psi}\phi\gamma_{12}\rangle. \tag{4.142}$$

Analogously to the non-relativistic case, the classical Dirac spinor $(\bar{\mathit{\Psi}}\mathit{\Phi})\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^{\mathrm{T}}$ can be transformed through the maps α and β, defined in relations (4.123), to furnish the equivalent expressions for the "Dirac inner product" $\bar{\mathit{\Psi}}\mathit{\Phi}$ in the algebraic and operator forms,

$$\begin{array}{ccc} (\bar{\mathit{\Psi}}\mathit{\Phi})F & \stackrel{\alpha}{\longmapsto} & 4\langle\bar{\psi}\underline{\phi}\rangle f = \Big(\langle\tilde{\psi}\phi\rangle + i\langle\tilde{\psi}\phi\gamma_{12}\rangle\Big) f \\ & \searrow^{\beta\circ\alpha} & \big\downarrow \beta \\ & & \langle\tilde{\psi}\phi\rangle_q = \langle\tilde{\psi}\phi\rangle - \langle\tilde{\psi}\phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3, \end{array} \tag{4.143}$$

where

$$F = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}. \tag{4.144}$$

In general, given a third Dirac spinor $\mathit{\Xi} \stackrel{\alpha}{\mapsto} \underline{\xi} \stackrel{\beta}{\mapsto} \xi$, it follows the maps

$$\begin{array}{ccc} \big(\bar{\mathit{\Psi}}\mathit{\Phi}\big)\mathit{\Xi} = \mathit{\Xi}\big(\bar{\mathit{\Psi}}\mathit{\Phi}\big) & \stackrel{\alpha}{\longmapsto} & \underline{\xi}\big(4\langle\bar{\psi}\underline{\phi}\rangle f\big) = \underline{\xi}\Big(\langle\tilde{\psi}\phi\rangle + i\langle\tilde{\psi}\phi\gamma_{12}\rangle\Big) f \\ & \searrow^{\beta\circ\alpha} & \big\downarrow \beta \\ & & \xi\langle\tilde{\psi}\phi\rangle_q = \xi\Big(\langle\tilde{\psi}\phi\rangle - \langle\tilde{\psi}\phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3\Big), \end{array} \tag{4.145}$$

where it is worth noting that, although the Dirac inner product commutes with the third Dirac spinor in the classical expression, it necessarily appears as a factor on the right in the algebraic expression, and this order for the product is preserved in terms of operator spinors. These maps are in agreement with the relation between the classical and the operator version of the Dirac inner product as presented by Doran and Lasenby (2003).

A particular case of a Dirac inner product $\bar{\mathit{\Psi}}\mathit{\Phi}$ of great importance is that for which $\mathit{\Phi} = \mathrm{A}\,\mathit{\Psi}$, where the matrix A represents a linear operator which is covariant under Lorentz transformations. A product of the form $\bar{\mathit{\Psi}}\mathrm{A}\,\mathit{\Psi}$ is usually known as a *bilinear covariant expression* and in general it corresponds to an observable quantity. The basic bilinear covariant expressions and the corresponding observable quantities are considered case by case in the following in their classical, algebraic and operator forms with assistance of the relations (4.143).

The simplest bilinear covariant expression is $\bar{\Psi}\Psi$, which is usually associated with the *probability density* ρ. For this case, relations (4.143) furnish the following mappings

$$\begin{array}{ccc} (\bar{\Psi}\Psi)F & \xmapsto{\alpha} & 4\langle\bar{\psi}\underline{\psi}\rangle f = \Big(\langle\psi\tilde{\psi}\rangle + i\langle\psi\gamma_{12}\tilde{\psi}\rangle\Big) f \\ & \overset{\beta\circ\alpha}{\searrow} & \downarrow \beta \\ & & \langle\tilde{\psi}\psi\rangle_q = \langle\psi\tilde{\psi}\rangle - \langle\psi \boldsymbol{I}\boldsymbol{\sigma}_3\tilde{\psi}\rangle \boldsymbol{I}\boldsymbol{\sigma}_3, \end{array} \tag{4.146}$$

where the invariance of the scalar part of a geometric product with relation to cyclic permutations of the factors was used. The multivector $\psi\gamma_{12}\tilde{\psi} = -\psi\boldsymbol{I}\boldsymbol{\sigma}_3\tilde{\psi}$ is even grade and is the opposite of its reverse, then it is a bivector and has no scalar part. In this way, the above relations reduce to

$$\begin{array}{ccc} (\bar{\Psi}\Psi)F & \xmapsto{\alpha} & 4\langle\bar{\psi}\underline{\psi}\rangle f = \langle\psi\tilde{\psi}\rangle f \\ & \overset{\beta\circ\alpha}{\searrow} & \downarrow \beta \\ & & \langle\tilde{\psi}\psi\rangle_q = \langle\psi\tilde{\psi}\rangle. \end{array} \tag{4.147}$$

By using the factored expression of an even grade multivector from ${\mathcal{C}\ell_{1,3}}^+$ for the operator spinor ψ, given by $\psi = \rho^{\frac{1}{2}} e^{\frac{1}{2}I\beta} R$ (see the final paragraph of the chapter 2), one obtains $\psi\tilde{\psi} = \rho e^{I\beta}$, which allows one to express

$$\langle\tilde{\psi}\psi\rangle_q = \langle\psi\tilde{\psi}\rangle = \rho\cos(\beta). \tag{4.148}$$

Another basic bilinear covariant expression is $\bar{\Psi}\Gamma_\mu\Psi$, which multiplied by c defines the components j_μ of the *probability current density vector*. In this case, taking into account the translation for the action of operators (cf. the mappings (4.124)), relations (4.143) furnish:

$$\begin{array}{ccc} (\bar{\Psi}\Gamma_\mu\Psi)F & \xmapsto{\alpha} & 4\langle\bar{\psi}\gamma_\mu\underline{\psi}\rangle f = \Big(\langle\tilde{\psi}\gamma_\mu\psi\gamma_0\rangle + i\langle\tilde{\psi}\gamma_\mu\psi\gamma_0\gamma_{12}\rangle\Big) f \\ & \overset{\beta\circ\alpha}{\searrow} & \downarrow \beta \\ & & \langle\tilde{\psi}\gamma_\mu\psi\gamma_0\rangle_q = \langle\tilde{\psi}\gamma_\mu\psi\gamma_0\rangle - \langle\tilde{\psi}\gamma_\mu\psi\gamma_0\boldsymbol{I}\boldsymbol{\sigma}_3\rangle\boldsymbol{I}\boldsymbol{\sigma}_3. \end{array} \tag{4.149}$$

These can be rewritten as

$$\begin{array}{ccc} (\bar{\Psi}\Gamma_\mu\Psi)F & \xmapsto{\alpha} & 4\langle\bar{\psi}\gamma_\mu\underline{\psi}\rangle f = \Big(\langle\gamma_\mu\psi\gamma_0\tilde{\psi}\rangle + i\langle\gamma_\mu\psi\gamma_0\gamma_{12}\tilde{\psi}\rangle\Big) f \\ & \overset{\beta\circ\alpha}{\searrow} & \downarrow \beta \\ & & \langle\tilde{\psi}\gamma_\mu\psi\gamma_0\rangle_q = \langle\gamma_\mu\psi\gamma_0\tilde{\psi}\rangle - \langle\gamma_\mu\psi\gamma_0\boldsymbol{I}\boldsymbol{\sigma}_3\tilde{\psi}\rangle\boldsymbol{I}\boldsymbol{\sigma}_3. \end{array} \tag{4.150}$$

Since $\psi\gamma_0\gamma_{12}\tilde{\psi} = -\psi\gamma_0 I\boldsymbol{\sigma}_3\tilde{\psi}$ is an odd grade multivector which is the opposite of its reverse, it is a trivector and has no scalar part. The multivector $\psi\gamma_0\tilde{\psi}$ is odd grade and is equal to its reverse, so it is a vector and, consequently, the projection $\langle\gamma_\mu\psi\gamma_0\tilde{\psi}\rangle$ corresponds to the scalar product $\gamma_\mu\cdot(\psi\gamma_0\tilde{\psi})$. In this way, the above mappings can be written simply as

$$\begin{array}{ccc} (\bar{\Psi}\Gamma_\mu\Psi)F & \xrightarrow{\ \alpha\ } & 4\langle\bar{\psi}\gamma_\mu\underline{\psi}\rangle f = \Big(\gamma_\mu\cdot(\psi\gamma_0\tilde{\psi})\Big)f \\ & \searrow^{\beta\circ\alpha} & \downarrow\beta \\ & & \langle\tilde{\psi}\gamma_\mu\psi\gamma_0\rangle_q = \gamma_\mu\cdot(\psi\gamma_0\tilde{\psi}). \end{array} \tag{4.151}$$

Given that $\bar{\Psi}\Gamma_\mu\Psi = \gamma_\mu\cdot(\psi\gamma_0\tilde{\psi})$ multiplied by c correspond to the components of the probability current density, such a vector is given by

$$j = c\psi\gamma_0\tilde{\psi}. \tag{4.152}$$

The next basic bilinear covariant expression to be considered is $\bar{\Psi}\frac{i}{2}[\Gamma_\mu,\Gamma_\nu]\Psi$, where the square brackets denote a commutator, $[\Gamma_\mu,\Gamma_\nu] = \Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu$. This bilinear covariant expression, as multiplied by $\frac{\hbar}{2}$, defines the components $S_{\mu\nu}$ of the *spin tensor*, which is clearly antisymmetric. In this case, again taking into account the mappings for the action of operators, relations (4.143) furnish:

$$\begin{array}{ccc} \big(\bar{\Psi}\tfrac{i}{2}[\Gamma_\mu,\Gamma_\nu]\Psi\big)F & \xrightarrow{\ \alpha\ } & 4\langle\bar{\psi}i(\gamma_\mu\wedge\gamma_\nu)\underline{\psi}\rangle f = \Big(i\langle\tilde{\psi}(\gamma_\mu\wedge\gamma_\nu)\psi\rangle - \langle\tilde{\psi}(\gamma_\mu\wedge\gamma_\nu)\psi\gamma_{12}\rangle\Big)f \\ & & = \Big(\langle\tilde{\psi}(\gamma_\mu\wedge\gamma_\nu)\psi\gamma_{21}\rangle + i\langle\tilde{\psi}(\gamma_\mu\wedge\gamma_\nu)\psi\rangle\Big)f \\ & \searrow^{\beta\circ\alpha} & \downarrow\beta \\ & & \langle\tilde{\psi}(\gamma_\mu\wedge\gamma_\nu)\psi I\boldsymbol{\sigma}_3\rangle_q = \langle\tilde{\psi}(\gamma_\mu\wedge\gamma_\nu)\psi I\boldsymbol{\sigma}_3\rangle + \langle\tilde{\psi}(\gamma_\mu\wedge\gamma_\nu)\psi\rangle I\boldsymbol{\sigma}_3. \end{array} \tag{4.153}$$

Since $\tilde{\psi}(\gamma_\mu\wedge\gamma_\nu)\psi$ is an even grade multivector which is the opposite of its reverse, it corresponds to a bivector and has no scalar part. In this way, by noting also that

$$\langle\tilde{\psi}(\gamma_\mu\wedge\gamma_\nu)\psi I\boldsymbol{\sigma}_3\rangle = \langle(\gamma_\mu\wedge\gamma_\nu)\psi I\boldsymbol{\sigma}_3\tilde{\psi}\rangle, \tag{4.154}$$

and noting that $\psi I\boldsymbol{\sigma}_3\tilde{\psi}$ is a bivector, one can rewrite the above mappings as

$$\begin{array}{ccc} \big(\bar{\Psi}\tfrac{i}{2}[\Gamma_\mu,\Gamma_\nu]\Psi\big)F & \xrightarrow{\ \alpha\ } & 4\langle\bar{\psi}i(\gamma_\mu\wedge\gamma_\nu)\underline{\psi}\rangle f = \Big((\gamma_\mu\wedge\gamma_\nu)\cdot(\psi\gamma_{21}\tilde{\psi})\Big)f \\ & \searrow^{\beta\circ\alpha} & \downarrow\beta \\ & & \langle\tilde{\psi}(\gamma_\mu\wedge\gamma_\nu)\psi I\boldsymbol{\sigma}_3\rangle_q = (\gamma_\mu\wedge\gamma_\nu)\cdot(\psi I\boldsymbol{\sigma}_3\tilde{\psi}). \end{array} \tag{4.155}$$

Once $\bar{\Psi}\frac{i}{2}[\Gamma_\mu, \Gamma_\nu]\Psi = (\gamma_\mu \wedge \gamma_\nu) \cdot (\psi I\boldsymbol{\sigma}_3\tilde{\psi})$ multiplied by $\frac{\hbar}{2}$ correspond to the components of the spin tensor, one can define the *spin bivector* by

$$S = \frac{\hbar}{2}\psi I\boldsymbol{\sigma}_3\tilde{\psi}. \tag{4.156}$$

Consider now the basic bilinear covariant expression given by $\bar{\Psi}\Gamma_\mu\Gamma_5\Psi$, which are usually associated to the components of an axial vector. For this case, taking into account the translation for action of operators (cf. the mappings (4.124) and (4.125)), relations (4.143) furnish:

$$\begin{array}{ccl} (\bar{\Psi}\Gamma_\mu\Gamma_5\Psi)F & \overset{\alpha}{\longmapsto} & 4\langle\bar{\psi}\gamma_\mu\gamma_5\underline{\psi}\rangle f = \Big(\langle\tilde{\psi}\gamma_\mu\psi\boldsymbol{\sigma}_3\gamma_0\rangle + i\langle\tilde{\psi}\gamma_\mu\psi\boldsymbol{\sigma}_3\gamma_0\gamma_{12}\rangle\Big)f \\ & & \qquad = \Big(\langle\tilde{\psi}\gamma_\mu\psi\gamma_3\rangle + i\langle\tilde{\psi}\gamma_\mu\psi\gamma_3\gamma_{12}\rangle\Big)f \\ & \overset{\beta\circ\alpha}{\searrow} & \qquad\qquad \downarrow \beta \\ & & \langle\tilde{\psi}\gamma_\mu\psi\gamma_3\rangle_q = \langle\tilde{\psi}\gamma_\mu\psi\gamma_3\rangle - \langle\tilde{\psi}\gamma_\mu\psi\gamma_3 I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3. \end{array} \tag{4.157}$$

These can be rewritten as

$$\begin{array}{ccl} (\bar{\Psi}\Gamma_\mu\Gamma_5\Psi)F & \overset{\alpha}{\longmapsto} & 4\langle\bar{\psi}\gamma_\mu\gamma_5\underline{\psi}\rangle f = \Big(\langle\gamma_\mu\psi\gamma_3\tilde{\psi}\rangle + i\langle\gamma_\mu\psi\gamma_3\gamma_{12}\tilde{\psi}\rangle\Big)f \\ & \overset{\beta\circ\alpha}{\searrow} & \qquad\qquad \downarrow \beta \\ & & \langle\tilde{\psi}\gamma_\mu\psi\gamma_3\rangle_q = \langle\gamma_\mu\psi\gamma_3\tilde{\psi}\rangle - \langle\gamma_\mu\psi\gamma_3 I\boldsymbol{\sigma}_3\tilde{\psi}\rangle I\boldsymbol{\sigma}_3. \end{array} \tag{4.158}$$

Since $\psi\gamma_3\gamma_{12}\tilde{\psi} = -\psi\gamma_3 I\boldsymbol{\sigma}_3\tilde{\psi}$ is an odd grade multivector which is the opposite of its reverse, it is a trivector and has no scalar part. The multivector $\psi\gamma_3\tilde{\psi}$ is odd grade and is equal to its reverse, so it is a vector and, consequently, $\langle\gamma_\mu\psi\gamma_0\tilde{\psi}\rangle$ corresponds to the scalar product $\gamma_\mu \cdot (\psi\gamma_3\tilde{\psi})$. In this way, the above relations reduce to

$$\begin{array}{ccl} (\bar{\Psi}\Gamma_\mu\Gamma_5\Psi)F & \overset{\alpha}{\longmapsto} & 4\langle\bar{\psi}\gamma_\mu\gamma_5\underline{\psi}\rangle f = \Big(\gamma_\mu \cdot (\psi\gamma_3\tilde{\psi})\Big)f \\ & \overset{\beta\circ\alpha}{\searrow} & \qquad\qquad \downarrow \beta \\ & & \langle\tilde{\psi}\gamma_\mu\psi\gamma_3\rangle_q = \gamma_\mu \cdot (\psi\gamma_3\tilde{\psi}). \end{array} \tag{4.159}$$

The components $\gamma_\mu \cdot (\psi\gamma_3\tilde{\psi})$ multiplied by $\frac{\hbar}{2}$ can be identified as the components ρs_μ of the *spin density vector*, which is then given by

$$\rho s = \frac{\hbar}{2}\psi\gamma_3\tilde{\psi}. \tag{4.160}$$

The last basic bilinear covariant expression to be considered is $\bar{\Psi}i\Gamma_5\Psi$. In this case, taking into account that $\gamma_5 = -iI$ and using the mappings (4.124) for action of operators, relations (4.143) furnish:

$$\begin{array}{ccc} (\bar{\Psi} i\Gamma_5 \Psi)F & \xmapsto{\ \alpha\ } & 4\langle\bar{\psi} i\gamma_5 \underline{\psi}\rangle f = \Big(\langle\tilde{\psi} I\psi\rangle + i\langle\tilde{\psi} I\psi\gamma_{12}\rangle\Big) f \\ & \searrow^{\beta\circ\alpha} & \Big\downarrow \beta \\ & & \langle\tilde{\psi} I\psi\rangle_q = \langle\tilde{\psi} I\psi\rangle - \langle\tilde{\psi} I\psi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3. \end{array} \tag{4.161}$$

These can be rewritten as

$$\begin{array}{ccc} (\bar{\Psi} i\Gamma_5 \Psi)F & \xmapsto{\ \alpha\ } & 4\langle\bar{\psi} i\gamma_5 \underline{\psi}\rangle f = \Big(\langle\psi\tilde{\psi} I\rangle + i\langle\psi\gamma_{30}\tilde{\psi}\rangle\Big) f \\ & \searrow^{\beta\circ\alpha} & \Big\downarrow \beta \\ & & \langle\tilde{\psi} I\psi\rangle_q = \langle\psi\tilde{\psi} I\rangle + \langle\psi\boldsymbol{\sigma}_3\tilde{\psi}\rangle I\boldsymbol{\sigma}_3. \end{array} \tag{4.162}$$

The fact that $\psi\gamma_{30}\tilde{\psi} = \psi\boldsymbol{\sigma}_3\tilde{\psi}$ is even grade and is the opposite of its reverse implies that it is a bivector and has no scalar part. This reduces the above relations to

$$\begin{array}{ccc} (\bar{\Psi} i\Gamma_5 \Psi)F & \xmapsto{\ \alpha\ } & 4\langle\bar{\psi} i\gamma_5 \underline{\psi}\rangle f = \langle\psi\tilde{\psi} I\rangle f \\ & \searrow^{\beta\circ\alpha} & \Big\downarrow \beta \\ & & \langle\tilde{\psi} I\psi\rangle_q = \langle\psi\tilde{\psi} I\rangle. \end{array} \tag{4.163}$$

By considering the factored expression of an even grade multivector from $\mathcal{C}\ell_{1,3}{}^{+}$ for ψ, given by $\psi = \rho^{\frac{1}{2}} e^{\frac{1}{2}I\beta} R$ (see the final paragraph of the chapter 2), one obtains $\psi\tilde{\psi} = \rho e^{I\beta}$, which furnishes

$$\langle\tilde{\psi} I\psi\rangle_q = \langle\psi\tilde{\psi} I\rangle = -\rho\sin(\beta). \tag{4.164}$$

It should be noted that the basic bilinear covariant expressions in terms of operator spinors obtained, as well as the corresponding expressions for the observables, reproduce the expressions presented by Doran and Lasenby (2003).

4.2.5 Plane Waves

As a simple illustration of some of the above results concerning the Dirac equation and Dirac spinors, one can consider the particular case of the plane wave equation and its associated solutions.

The positive energy plane wave solutions of equation (4.109) can be stated to be of the form

$$\underline{\psi}(x) = \underline{\psi}_0(p) e^{-\frac{i}{\hbar}(p\cdot x)}, \tag{4.165}$$

where $\underline{\psi}_0(p)$ is an algebraic Dirac spinor depending only on the spacetime momentum p

of the particle. By writing $\underline{\psi}_0(p) = \psi_0(p)f$, where $\psi_0(p) = 4\langle \mathrm{Re}(\underline{\psi}_0(p))\rangle_+$ is the operator spinor corresponding to $\underline{\psi}_0(p)$, and using the property $i\gamma_{12}f = f$ this solution can be written

$$\underline{\psi}(x) = \psi_0(p)e^{-\frac{i}{\hbar}(p\cdot x)}f = \psi_0(p)e^{-\gamma_{21}\frac{1}{\hbar}(p\cdot x)}f. \tag{4.166}$$

Substitution of this expression in the equation (4.109), with $A_\mu = 0$, furnishes

$$i\hbar\gamma^\mu\psi_0(p)\left(-\frac{1}{\hbar}\gamma_{21}p_\mu\right)e^{-\frac{1}{\hbar}\gamma_{21}(p\cdot x)}f = mc\psi_0(p)e^{-\frac{1}{\hbar}\gamma_{21}(p\cdot x)}f, \tag{4.167}$$

that is,

$$p\psi_0(p)\frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_{12}) = mc\psi_0(p)\frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_{12}) \tag{4.168}$$

(note that the form $\underline{\psi}(x) = \psi_0(p)e^{-\frac{i}{\hbar}(p\cdot x)}f$ for the assumed solution could be used in place of $\underline{\psi}(x) = \psi_0(p)e^{-\gamma_{21}\frac{1}{\hbar}(p\cdot x)}f$ in the same way to obtain the above expression). The real and imaginary parts of this equation correspond to the same equation,

$$p\psi_0(p)(1+\gamma_0) = mc\psi_0(p)(1+\gamma_0), \tag{4.169}$$

and the even grade and odd grade parts of this new equation are also the same, and correspond to

$$p\psi_0(p) = mc\psi_0(p)\gamma_0. \tag{4.170}$$

This is the expected equation in terms of the algebra $\mathcal{C}\ell_{1,3}$ (cf. Doran and Lasenby (2003)). In addition, this equation is obtained four times from the equation (4.168), and this latter is known to be equivalent to the traditional plane wave equation, $\not{p}u(p) = mcu(p)$.

The negative energy plane wave solutions of the equation (4.109) can be stated to be of the form

$$\underline{\psi}(x) = \underline{\psi}_0(p)e^{\frac{i}{\hbar}(p\cdot x)} = \psi_0(p)e^{\gamma_{21}\frac{1}{\hbar}(p\cdot x)}f. \tag{4.171}$$

Similarly to the above above, this solution furnishes the equation

$$p\psi_0(p)\frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_{12}) = -mc\psi_0(p)\frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_{12}), \tag{4.172}$$

where either the even or odd grade part of either its real or imaginary part corresponds to

$$p\psi_0(p) = -mc\psi_0(p)\gamma_0. \tag{4.173}$$

Again this is the expected plane wave equation in terms of $\mathcal{C}\ell_{1,3}$ (cf. Doran and Lasenby (2003)), which is encoded four times in equation (4.172), as well as in its classical counterpart, $\not{p}v(p) = -mcv(p)$.

The plane wave solutions in terms of the real geometric algebra of spacetime can be obtained as follows (DORAN; LASENBY, 2003). The equations (4.170) and (4.173) imply that

$$p\psi^{\pm} = \pm mc\psi^{\pm}\gamma_0, \tag{4.174}$$

where the plus and minus signs superscripts distinguish positive and negative energy solutions, given by

$$\psi^{\pm} = \psi^{\pm}(x) = \psi_0^{\pm}(p)e^{\mp \boldsymbol{I\sigma}_3 \frac{1}{\hbar}(p\cdot x)}. \tag{4.175}$$

The above equations allow one to write

$$p\psi^{\pm}\tilde{\psi}^{\pm} = \pm mc\psi^{\pm}\gamma_0\tilde{\psi}^{\pm} = \pm mj, \tag{4.176}$$

where j is the probability current density vector. Now, by using the factored expression $\psi = \rho^{\frac{1}{2}}e^{\frac{1}{2}I\beta}R$ for an even grade multivector from $\mathcal{C}\ell_{1,3}{}^{+}$ (see the final paragraph of the chapter 2) in the equation (4.176), one obtains

$$p\rho e^{I\beta} = \pm mj. \tag{4.177}$$

Since both p and j are vectors, one must have $e^{I\beta} = \pm 1$, that is, $\beta = 0$ or $\beta = \pi$. Given that $j\cdot\gamma_0 = c\rho > 0$ and $p\cdot\gamma_0 = E/c > 0$, where E is the energy, one must have $\beta = 0$ for the positive energy solutions and $\beta = \pi$ for the negative energy solutions. The plane wave solutions can then be written in the form

$$\psi^{\pm} = \rho^{\frac{1}{2}}e^{\frac{1}{2}I\beta_{\pm}}LUe^{\mp \boldsymbol{I\sigma}_3 \frac{1}{\hbar}(p\cdot x)}, \tag{4.178}$$

where $\beta_{+} = 0$, $\beta_{-} = \pi$ and $R = LU$ is a spacetime rotor, L being a rotor describing a boost and U being a rotor describing a spatial rotation. The rotor $Re^{\mp \boldsymbol{I\sigma}_3 \frac{1}{\hbar}(p\cdot x)} = LUe^{\mp \boldsymbol{I\sigma}_3 \frac{1}{\hbar}(p\cdot x)}$ must transform $mc\gamma_0$ in the momentum p of the particle, so that the rotor U must be a spatial rotor relative to an observer of normalized spacetime velocity γ_0; in particular, it must correspond to a definite spin state. In this way, one must have

$$\frac{p}{m} = LUe^{\pm \boldsymbol{I\sigma}_3 \frac{1}{\hbar}(p\cdot x)}\gamma_0 e^{\pm \boldsymbol{I\sigma}_3 \frac{1}{\hbar}(p\cdot x)}\tilde{U}\tilde{L} = L\gamma_0\tilde{L}. \tag{4.179}$$

From the known expression for a boost transforming a time-like vector into another (cf. subsection 3.3.4), the rotor L can be written as

$$L = \frac{1 + p\gamma_0/mc}{\sqrt{2(1 + p\cdot\gamma_0/mc)}} = \frac{mc + p\gamma_0}{\sqrt{2mc(mc + p\cdot\gamma_0)}} = \frac{mc + E/c + \mathbf{p}}{\sqrt{2mc(mc + E/c)}}, \tag{4.180}$$

where $\mathbf{p} = p\wedge\gamma_0$ is the relative momentum of the particle. In summary, the plane wave solutions of positive and negative energies are respectively given by

$$\psi_r^+ = \rho^{\frac{1}{2}} L(p) U_r e^{-\boldsymbol{I}\boldsymbol{\sigma}_3 \frac{1}{\hbar}(p\cdot x)} \quad \text{and} \quad \psi_r^- = \rho^{\frac{1}{2}} I L(p) U_r e^{\boldsymbol{I}\boldsymbol{\sigma}_3 \frac{1}{\hbar}(p\cdot x)}, \tag{4.181}$$

where $L(p)$ is the rotor given by equation (4.180) and the spatial rotor U_r describes a well-defined spin state (a "spin-up" or a "spin-down" state), with $U_1 = 1$ and $U_2 = -\boldsymbol{I}\boldsymbol{\sigma}_2$.

4.2.6 Energy-Momentum Tensor

The ambiguity in the definition of the *energy-momentum tensor* allows one to define it for the Dirac field to be

$$T^\mu{}_\nu = \frac{1}{2}\Big(\bar{\varPsi} i\hbar \Gamma^\mu \partial_\nu \varPsi + \big(\bar{\varPsi} i\hbar \Gamma^\mu \partial_\nu \varPsi \big)^\dagger \Big). \tag{4.182}$$

This can be rewritten as

$$T^\mu{}_\nu = \frac{1}{2}\Big(\bar{\varPsi} i\hbar \Gamma^\mu \partial_\nu \varPsi - \partial_\nu \bar{\varPsi} \Gamma_0 i\hbar \Gamma^{\mu\dagger} \Gamma_0 \varPsi \Big). \tag{4.183}$$

Since $\Gamma_0 \Gamma^{\mu\dagger} \Gamma_0 = \Gamma^\mu$, the above expression can be simplified to

$$T^\mu{}_\nu = \frac{1}{2}\Big(i\hbar \bar{\varPsi} \Gamma^\mu \partial_\nu \varPsi - i\hbar \partial_\nu \bar{\varPsi} \Gamma^\mu \varPsi \Big). \tag{4.184}$$

This corresponds to a choice similar to that made by Hiley and Callaghan (2010), for the energy-momentum tensor, except for a sign. Note now that the above expression can be written in terms of square matrices as follows:

$$T^\mu{}_\nu = \frac{1}{2}\Big(\mathrm{tr}\big(i\hbar \bar{\Psi} \Gamma^\mu \partial_\nu \Psi \big) - \mathrm{tr}\big(i\hbar \partial_\nu \bar{\Psi} \Gamma^\mu \Psi \big) \Big). \tag{4.185}$$

In terms of algebraic Dirac spinors, this expression reads

$$T^\mu{}_\nu = \frac{1}{2}\Big(4\langle i\hbar \bar{\underline{\psi}} \gamma^\mu \partial_\nu \underline{\psi} \rangle - 4\langle i\hbar \partial_\nu \bar{\underline{\psi}} \gamma^\mu \underline{\psi} \rangle \Big). \tag{4.186}$$

In terms of operator Dirac spinors, this is given by

$$T^\mu{}_\nu = \frac{1}{2}\Big(4\langle i\hbar f \tilde{\psi} \gamma^\mu \partial_\nu \psi f \rangle - 4\langle i\hbar f \partial_\nu \tilde{\psi} \gamma^\mu \psi f \rangle \Big), \tag{4.187}$$

or, considering the property of the scalar part of invariance under cyclic permutations of the factors in its argument,

$$T^\mu{}_\nu = \frac{1}{2}\Big(4\langle i\hbar \gamma^\mu \partial_\nu \psi f \tilde{\psi} \rangle - 4\langle i\hbar \gamma^\mu \psi f \partial_\nu \tilde{\psi} \rangle \Big). \tag{4.188}$$

Since γ^μ is a vector, only the terms in the expressions

$$\partial_\nu \psi f \tilde{\psi} = \partial_\nu \psi \frac{1}{4}(1 + \gamma_0 + i\gamma_{12} + i\gamma_0\gamma_{12})\tilde{\psi} \tag{4.189}$$

and

$$\psi f \partial_\nu \tilde{\psi} = \psi \frac{1}{4}(1 + \gamma_0 + i\gamma_{12} + i\gamma_0\gamma_{12})\partial_\nu \tilde{\psi} \tag{4.190}$$

which have a non-null vector part furnish in principle non-null scalar parts in the above expression for the energy-momentum tensor. It is found that the terms $\partial_\nu \psi \tilde{\psi}$, $\partial_\nu \psi i\gamma_{12}\tilde{\psi}$, $\psi \partial_\nu \tilde{\psi}$ and $\psi i\gamma_{12}\partial_\nu \tilde{\psi}$ are all even grade multivectors (since they are products of even grade multivectors), so that they furnish null contributions for the energy-momentum tensor. The remaining terms are odd grade multivectors which can have a vector part. The above expression for the energy-momentum tensor can then be written as

$$\begin{aligned} T^\mu{}_\nu = \frac{1}{2}\Big(&\langle i\hbar\gamma^\mu \partial_\nu \psi \gamma_0 \tilde{\psi}\rangle + \langle i\hbar\gamma^\mu \partial_\nu \psi i\gamma_0\gamma_{12}\tilde{\psi}\rangle \\ &- \langle i\hbar\gamma^\mu \psi \gamma_0 \partial_\nu \tilde{\psi}\rangle - \langle i\hbar\gamma^\mu \psi i\gamma_0\gamma_{12}\partial_\nu \tilde{\psi}\rangle\Big). \end{aligned} \tag{4.191}$$

This can be rewritten as

$$\begin{aligned} T^\mu{}_\nu = \frac{1}{2}\Big(&\langle i\hbar\gamma^\mu \partial_\nu \psi \gamma_0 \tilde{\psi}\rangle + \langle \hbar\gamma^\mu \partial_\nu \psi \gamma_0\gamma_{21}\tilde{\psi}\rangle \\ &- \langle i\hbar \psi \gamma_0 \partial_\nu \tilde{\psi}\gamma^\mu\rangle - \langle \hbar \psi \gamma_0\gamma_{21}\partial_\nu \tilde{\psi}\gamma^\mu\rangle\Big). \end{aligned} \tag{4.192}$$

Now, by using the property of invariance of the scalar part under the reversion operation, one can rewrite the above expression with the third and fourth terms reversed:

$$\begin{aligned} T^\mu{}_\nu = \frac{1}{2}\Big(&\langle i\hbar\gamma^\mu \partial_\nu \psi \gamma_0 \tilde{\psi}\rangle + \langle \hbar\gamma^\mu \partial_\nu \psi \gamma_0\gamma_{21}\tilde{\psi}\rangle \\ &- \langle i\hbar\gamma^\mu \partial_\nu \psi \gamma_0 \tilde{\psi}\rangle + \langle \hbar\gamma^\mu \partial_\nu \psi \gamma_0\gamma_{21}\tilde{\psi}\rangle\Big). \end{aligned} \tag{4.193}$$

This furnishes the following expression for the energy-momentum tensor:

$$T^\mu{}_\nu = \hbar\langle \gamma^\mu \partial_\nu \psi \gamma_0 I \boldsymbol{\sigma}_3 \tilde{\psi}\rangle. \tag{4.194}$$

These match to the components of the canonical energy-momentum tensor as presented by Doran and Lasenby (2003), which express it in a coordinate-free manner, in terms of its action in a vector a, as

$$T(a) = \hbar\langle (a\cdot\nabla)\psi\gamma_0 I \boldsymbol{\sigma}_3 \tilde{\psi}\rangle_1. \tag{4.195}$$

The relation to the expression in terms of components is given by

$$T^{\mu}{}_{\nu} = \gamma^{\mu} \cdot T(\gamma_{\nu}) = T(\gamma_{\nu}) \cdot \gamma^{\mu}. \tag{4.196}$$

5 Multi-Particle Spinors

The study in the previous chapter treats states of a single spin-$\frac{1}{2}$ particle. The objective here is to extend that study to states of a system of multiple particles with spin $\frac{1}{2}$. This objective and its motivation are clarified in the following section.

5.1 Introduction

In the context of quantum mechanics (PIZA, 2003; MESSIAH, 2014), the state space for a system composed of two or more particles is usually described by the tensor product of the state spaces for the individual particles. States for the composite system are then described in terms of tensor products of states for the individual particles, considered in a fixed order, compatible with the labeling of the particles (recalling that the tensor product is associative, but non-commutative). As an example, consider a system composed of two particles, labeled as "particle 1" and "particle 2", whose state spaces are the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. In this case, the space of possible states of the composite system is the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$, and a state of the form $|\phi\rangle \otimes |\chi\rangle$, where $|\phi\rangle \in \mathcal{H}_1$ and $|\chi\rangle \in \mathcal{H}_2$, is a possible state for the composite system. Although not all states from $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be written in the form $|\phi\rangle \otimes |\chi\rangle$, a general state for the composite system is a linear combination of tensor products of this form. For this case of a system of two particles, given the states $|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$ and $|\psi'\rangle = |\phi'\rangle \otimes |\chi'\rangle$ from $\mathcal{H}_1 \otimes \mathcal{H}_2$, the Hermitian inner product of the two is naturally defined as

$$\langle\psi|\psi'\rangle = \langle\phi|\phi'\rangle\langle\chi|\chi'\rangle. \tag{5.1}$$

The action of operators is also extended in a natural way. Given an operator $\hat{a}$ defined on the state space of particle 1, $\mathcal{H}_1$, its extension to the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be denoted by $\hat{a}_1$ and defined to act on a state $|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$ through the expression

$$\hat{a}_1|\psi\rangle = (\hat{a}|\phi\rangle) \otimes |\chi\rangle. \tag{5.2}$$

It follows that $\hat{a}_1$ is the tensor product of the operator $\hat{a}$ with the identity operator for the state space of particle 2:

$$\hat{a}_1 = \hat{a} \otimes \hat{1}. \tag{5.3}$$

The same applies to operators defined on the state space of particle 2. In general, an operator defined on the tensor product space is a linear combination of tensor products of operators defined on each factor space. All the constructions defined for the case of two-particle systems extend naturally to the case of systems composed of any number N of particles by considering tensor products with N factors, each corresponding to a particle. These are the basic elements for the description of N-particle systems in quantum mechanics.

One question that arises is whether it is possible to implement such a description in terms of Clifford algebras, as in the case of a single particle. Doran *et al.* (1993, 1996) have addressed this question by introducing the *multi-particle spacetime algebra*, which, for the case of a system of N non-relativistic spin-$\frac{1}{2}$ particles, is an algebra constructed from N copies of the Minkowski spacetime, each copy associated to a geometric algebra of spacetime, with the following defining property

$$\frac{1}{2}\left({\gamma_\mu}^a {\gamma_\nu}^b + {\gamma_\nu}^b {\gamma_\mu}^a\right) = \delta^{ab}\eta_{\mu\nu}, \tag{5.4}$$

where ${\gamma_\mu}^a$ is the μ-th canonical basic vector of the a-th copy of Minkowski spacetime, so that $\mu, \nu \in \{0, 1, 2, 3\}$ and $a, b \in \{1, \ldots, N\}$. This property implies, in particular, that the geometric product of vectors from the same copy of Minkowski spacetime obeys the standard property of the geometric algebra of spacetime, while the geometric product of vectors from different copies of Minkowski spacetime anticommutes. Through this algebra, Doran *et al.* (1993, 1996) were able to describe states for multi-particle systems, at least in the non-relativistic context, in a similar way to the description provided by Hestenes (HESTENES, 1967; HESTENES, 1971; HESTENES; GÜRTLER, 1971; HESTENES, 1975) for the case of a single particle.

Since their introduction, multi-particle spacetime algebras have been applied in several contexts — cf. e.g. Doran *et al.* (1996), Lasenby *et al.* (1993), Somaroo *et al.* (1998, 1999), Havel *et al.* (2001), Parker and Doran (2002), Havel and Doran (2002a, 2002b), Lasenby *et al.* (2004), and Arcaute and Lasenby (2008). Moreover, Doran *et al.* (1996) argue that the approach in terms of this algebra brings advances in clarity and insight to the subject of multi-particle quantum systems. Despite this, the topic is still little explored and there are still fundamental questions to be answered. Which Clifford algebra does the multi-particle spacetime algebra correspond to? Is this the same as the algebra of operators acting on classical multi-particle states? How can a spinor be defined in terms of these algebras? How do the different definitions of spinors relate in this context?

Adopting as a basic premise the adequacy of the multi-particle spacetime algebras in their descriptions, the aim of this work is to answer the above questions by extending the study realized in the previous chapter to the case of multi-particle states.

5.2 Non-Relativistic Multi-Particle Spinors

As shown above, a possible state for a system of two non-relativistic spin-$\frac{1}{2}$ particles can be expressed by the tensor product $|\phi\rangle \otimes |\chi\rangle$, where $|\phi\rangle$ and $|\chi\rangle$ are states for the particles 1 and 2, respectively. In this case, an operator defined on the tensor product space is represented by a linear combination of tensor products of the form $\hat{a} \otimes \hat{b}$, where $\hat{a}$ and $\hat{b}$ are operators acting on the spaces of the particles 1 and 2, respectively. Since the algebra of operators for each of the two particles is $\mathcal{C}\ell_{3,0}$, the algebra of operators acting on the tensor product space corresponds to $\mathcal{C}\ell_{3,0} \otimes \mathcal{C}\ell_{3,0}$. Analogously to the case of a single particle, the algebra $\mathcal{C}\ell_{3,0} \otimes \mathcal{C}\ell_{3,0}$ can be used to represent states as well as operators for the system of two particles.

5.2.1 The Tensor Product Algebra Acting on Two-Particle States

According to the usual definition of the tensor product of algebras (LANG, 2002), the tensor product $\mathcal{C}\ell_{3,0} \otimes \mathcal{C}\ell_{3,0}$ is defined to be another algebra, whose product is defined by

$$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB'), \tag{5.5}$$

where $A, A', B, B' \in \mathcal{C}\ell_{3,0}$, and extended by bilinearity. It is noted from this definition that the product of elements of the form $\mathbf{u} \otimes 1$, where $\mathbf{u}$ is a vector from $\mathcal{C}\ell_{3,0}$, obeys a property similar to that for the algebra $\mathcal{C}\ell_{3,0}$, with the element $1 \otimes 1$ playing the role of the unity. The same is observed about the product of elements of the form $1 \otimes \mathbf{v}$, where $\mathbf{v}$ is a vector from $\mathcal{C}\ell_{3,0}$. Note also from the definition (5.5) that

$$(\boldsymbol{\sigma}_i \otimes 1)(1 \otimes \boldsymbol{\sigma}_j) = (1 \otimes \boldsymbol{\sigma}_j)(\boldsymbol{\sigma}_i \otimes 1) = \boldsymbol{\sigma}_i \otimes \boldsymbol{\sigma}_j, \tag{5.6}$$

that is, the product of elements of the form $\mathbf{u} \otimes 1$ with elements of the form $1 \otimes \mathbf{v}$ is commutative. At this point, it is useful to denote

$$1 \otimes 1 = 1, \quad \boldsymbol{\sigma}_i \otimes 1 = {\boldsymbol{\sigma}_i}^1 \quad \text{and} \quad 1 \otimes \boldsymbol{\sigma}_i = {\boldsymbol{\sigma}_i}^2. \tag{5.7}$$

The defining property of the algebra $\mathcal{C}\ell_{3,0} \otimes \mathcal{C}\ell_{3,0}$ can then be expressed by

$$\frac{1}{2}({\boldsymbol{\sigma}_i}^a {\boldsymbol{\sigma}_j}^b + {\boldsymbol{\sigma}_j}^b {\boldsymbol{\sigma}_i}^a) = \delta_{ij}, \quad \text{if} \quad a = b, \tag{5.8}$$

and

$$\frac{1}{2}(\boldsymbol{\sigma}_i{}^a \boldsymbol{\sigma}_j{}^b - \boldsymbol{\sigma}_j{}^b \boldsymbol{\sigma}_i{}^a) = 0, \quad \text{if} \quad a \neq b. \tag{5.9}$$

These properties define the algebra $\mathcal{C}\ell_{3,0} \otimes \mathcal{C}\ell_{3,0}$ as a commuting product of two copies of $\mathcal{C}\ell_{3,0}$. An inconvenience is that this is not a Clifford algebra. As an alternative, one would think that a suitable algebra to replace $\mathcal{C}\ell_{3,0} \otimes \mathcal{C}\ell_{3,0}$ would be that satisfying the properties

$$\frac{1}{2}(\boldsymbol{\sigma}_i{}^a \boldsymbol{\sigma}_j{}^b + \boldsymbol{\sigma}_j{}^b \boldsymbol{\sigma}_i{}^a) = \delta_{ij}, \quad \text{if} \quad a = b, \tag{5.10}$$

and

$$\frac{1}{2}(\boldsymbol{\sigma}_i{}^a \boldsymbol{\sigma}_j{}^b + \boldsymbol{\sigma}_j{}^b \boldsymbol{\sigma}_i{}^a) = 0, \quad \text{if} \quad a \neq b. \tag{5.11}$$

This could be obtained by redefining the product (5.5) in such way that the relation

$$(\boldsymbol{\sigma}_i \hat{\otimes} 1)(1 \hat{\otimes} \boldsymbol{\sigma}_j) = -(1 \hat{\otimes} \boldsymbol{\sigma}_j)(\boldsymbol{\sigma}_i \hat{\otimes} 1) = \boldsymbol{\sigma}_i \hat{\otimes} \boldsymbol{\sigma}_j \tag{5.12}$$

holds instead of relation (5.6), but the product of elements of the form $\mathbf{u} \hat{\otimes} 1$, and in the same way the product of elements of the form $1 \hat{\otimes} \mathbf{v}$, still obey the same fundamental properties of $\mathcal{C}\ell_{3,0}$, with $1 \hat{\otimes} 1$ playing the role of the unity. This alternative way to define the tensor product yields a new Clifford algebra and is called an *alternating tensor product*, or a *graded tensor product* (VAZ; DA ROCHA, 2019; CRUMEYROLLE, 1990). But this definition does not allow one to construct the adequate idempotents and corresponding ideals to define spinors, since in this case

$$\frac{1}{2}(1 + \boldsymbol{\sigma}_3{}^1)\frac{1}{2}(1 + \boldsymbol{\sigma}_3{}^2) \neq \frac{1}{2}(1 + \boldsymbol{\sigma}_3{}^2)\frac{1}{2}(1 + \boldsymbol{\sigma}_3{}^1). \tag{5.13}$$

A solution requires the extension of the algebra defined by the usual tensor product $\mathcal{C}\ell_{3,0} \otimes \mathcal{C}\ell_{3,0}$.

5.2.2 The Two-Particle Spacetime Algebra

As shown in section 3.1, the algebra $\mathcal{C}\ell_{3,0}$ is isomorphic to the even subalgebra $\mathcal{C}\ell_{1,3}{}^+$ of the Clifford algebra $\mathcal{C}\ell_{1,3}$. Thus, a way to extend the algebra $\mathcal{C}\ell_{3,0} \otimes \mathcal{C}\ell_{3,0}$ would be to consider it as $\mathcal{C}\ell_{1,3}{}^+ \hat{\otimes} \mathcal{C}\ell_{1,3}{}^+$, included as the even subalgebra into the algebra $\mathcal{C}\ell_{1,3} \hat{\otimes} \mathcal{C}\ell_{1,3}$, which is taken as an alternating tensor product, ensuring it to be a Clifford algebra. The product of this larger algebra is then defined by expressions similar to the relations (5.10) and (5.11), which can be summarized in this case by

$$\frac{1}{2}(\gamma_\mu{}^a \gamma_\nu{}^b + \gamma_\nu{}^b \gamma_\mu{}^a) = \delta^{ab} \eta_{\mu\nu}, \tag{5.14}$$

where ${\gamma_\mu}^1 = \gamma_\mu \hat{\otimes} 1$ and ${\gamma_\mu}^2 = 1 \hat{\otimes} \gamma_\mu$. This is identical to the defining property of the multi-particle spacetime algebra introduced by Doran *et al.* (1993, 1996), equation (5.4), for a two-particle system. In this way, the Clifford algebra $\mathcal{C}\ell_{1,3} \hat{\otimes} \mathcal{C}\ell_{1,3}$, understood as an alternating tensor product, is identified as the two-particle spacetime algebra. The inclusion of the algebra $\mathcal{C}\ell_{3,0} \otimes \mathcal{C}\ell_{3,0} \simeq {\mathcal{C}\ell_{1,3}}^+ \hat{\otimes} {\mathcal{C}\ell_{1,3}}^+$ into the two-particle spacetime algebra, $\mathcal{C}\ell_{1,3} \hat{\otimes} \mathcal{C}\ell_{1,3}$, can then be defined by

$$ {\boldsymbol{\sigma}_i}^a = {\gamma_i}^a {\gamma_0}^a. \tag{5.15} $$

From this relations, it is simple to verify that

$$ \frac{1}{2}({\boldsymbol{\sigma}_i}^a {\boldsymbol{\sigma}_j}^b + {\boldsymbol{\sigma}_j}^b {\boldsymbol{\sigma}_i}^a) = \delta_{ij}, \quad \text{for} \quad a = b, \tag{5.16} $$

while

$$ {\boldsymbol{\sigma}_i}^a {\boldsymbol{\sigma}_j}^b = {\boldsymbol{\sigma}_j}^b {\boldsymbol{\sigma}_i}^a, \quad \text{for} \quad a \neq b. \tag{5.17} $$

This construction also conveniently allows one to associate the inclusion with the fixing of a common spacetime reference frame, defined by ${\gamma_0}^1$ and ${\gamma_0}^2$.

5.2.3 Two-Particle Pauli Spinors

According to the above, the tensor product $\mathit{\Phi} \otimes X$, where $\mathit{\Phi}$ and X are classical Pauli spinors, can be understood as a *classical two-particle Pauli spinor*. If Φ and X are the square matrices corresponding to $\mathit{\Phi}$ and X, then the tensor product $\Phi \otimes \mathrm{X}$ corresponds to $\mathit{\Phi} \otimes X$ in terms of square matrices. This can be written as

$$ \Phi \otimes \mathrm{X} = (\Phi \otimes 1)(1 \otimes \mathrm{X}) = (1 \otimes \mathrm{X})(\Phi \otimes 1), \tag{5.18} $$

and corresponds to a matrix representation for an element of $\mathcal{C}\ell_{3,0} \otimes \mathcal{C}\ell_{3,0} \simeq {\mathcal{C}\ell_{1,3}}^+ \hat{\otimes} {\mathcal{C}\ell_{1,3}}^+$. Such an element can be written as

$$ \underline{\phi} \hat{\otimes} \underline{\chi} = \underline{\phi}^1 \underline{\chi}^2 = \underline{\chi}^2 \underline{\phi}^1, \tag{5.19} $$

where $\underline{\phi}^1 = \underline{\phi} \hat{\otimes} 1$ and $\underline{\chi}^2 = 1 \hat{\otimes} \underline{\chi}$, and $\underline{\phi}$ and $\underline{\chi}$ are the algebraic Pauli spinors corresponding to Φ and X, and consequently to the classical Pauli spinors $\mathit{\Phi}$ and X. Since both $\underline{\phi}$ and $\underline{\chi}$ correspond to elements of the minimal left ideal $\mathcal{I} = \{Af \mid A \in \mathcal{C}\ell_{3,0} \text{ and } f = \frac{1}{2}(1 + \boldsymbol{\sigma}_3)\}$, they can be written as $\underline{\phi} = \phi f$ and $\underline{\chi} = \chi f$, where ϕ and χ are the even grade elements defining the corresponding operator Pauli spinors. In this way, from the above equation, one can write

$$ \underline{\phi} \hat{\otimes} \underline{\chi} = \underline{\phi}^1 \underline{\chi}^2 = \phi^1 f^1 \chi^2 f^2, \tag{5.20} $$

where $\phi^1 = \phi\hat{\otimes}1$ and $\chi^2 = 1\hat{\otimes}\chi$, and

$$f^1 = \frac{1}{2}(1 + \boldsymbol{\sigma}_3{}^1) \quad \text{and} \quad f^2 = \frac{1}{2}(1 + \boldsymbol{\sigma}_3{}^2). \tag{5.21}$$

Since one expects the product of elements of the algebra $\mathcal{C}\ell_{3,0} \otimes \mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,3}{}^+\hat{\otimes}\mathcal{C}\ell_{1,3}{}^+$ belonging to different copies of $\mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,3}{}^+$ to commute, one can write

$$\underline{\phi}\hat{\otimes}\underline{\chi} = \underline{\phi}^1\underline{\chi}^2 = \phi^1\chi^2 f^1 f^2. \tag{5.22}$$

The fact that f^1 and f^2 are commuting primitive idempotents imply that f^1f^2 is also a primitive idempotent. In this way, the above expression defines the *algebraic two-particle Pauli spinor* corresponding to $\Phi \otimes X$ to be an element of the minimal left ideal $\mathcal{I}_2 = \{Af \mid A \in \mathcal{C}\ell_{1,3}{}^+\hat{\otimes}\mathcal{C}\ell_{1,3}{}^+ \text{ and } f = \frac{1}{2}(1 + \boldsymbol{\sigma}_3{}^1)\frac{1}{2}(1 + \boldsymbol{\sigma}_3{}^2)\}$. The reduction of an algebraic two-particle Pauli spinor $A^1B^2f^1f^2$ to an element of the form (5.22), where the multivectors A^1 and B^2 are replaced by even grade multivectors from their respective copies of $\mathcal{C}\ell_{3,0}{}^+ \simeq \mathcal{C}\ell_{1,3}{}^{++}$ (note the notation introduced), is straightforward from the commutativity of elements of different copies of $\mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,3}{}^+$ and from the known way in which the reduction is performed in the case of a single copy (cf. subsection 4.1.3), which follows in this case from the property $\boldsymbol{\sigma}_3{}^a f^a = f^a$.

As in the single-particle case, this reduction and the fact that the idempotent f^1f^2 is a fixed factor in the expression for an algebraic two-particle Pauli spinor imply that the even grade multivectors belonging to the subalgebra $\mathcal{C}\ell_{3,0}{}^+ \otimes \mathcal{C}\ell_{3,0}{}^+ \simeq \mathcal{C}\ell_{1,3}{}^{++}\hat{\otimes}\mathcal{C}\ell_{1,3}{}^{++}$ can be sufficient to describe the states in question. This allows one to define an *operator two-particle Pauli spinor* as an element of the subalgebra $\mathcal{C}\ell_{1,3}{}^{++}\hat{\otimes}\mathcal{C}\ell_{1,3}{}^{++}$. Since the single-particle operator spinors ϕ and χ in equation (5.22) are given in terms of their algebraic counterparts by $\phi = 2\langle\underline{\phi}\rangle_+ = 2\langle\underline{\phi}\rangle_-\boldsymbol{\sigma}_3$ and $\chi = 2\langle\underline{\chi}\rangle_+ = 2\langle\underline{\chi}\rangle_-\boldsymbol{\sigma}_3$ (cf. subsection 4.1.3), there are four ways to express the operator two-particle Pauli spinor $\phi^1\chi^2$ in terms of its algebraic counterpart, $\underline{\phi}^1\underline{\chi}^2$, namely

$$\begin{aligned}
\phi^1\chi^2 &= 4\langle\underline{\phi}^1\underline{\chi}^2\rangle_{++} \\
&= 4\langle\underline{\phi}^1\underline{\chi}^2\rangle_{+-}\boldsymbol{\sigma}_3{}^1 \\
&= 4\langle\underline{\phi}^1\underline{\chi}^2\rangle_{-+}\boldsymbol{\sigma}_3{}^2 \\
&= 4\langle\underline{\phi}^1\underline{\chi}^2\rangle_{--}\boldsymbol{\sigma}_3{}^1\boldsymbol{\sigma}_3{}^2,
\end{aligned} \tag{5.23}$$

where $\langle A\rangle_{++}$ denotes the extraction of the even grade part of both factors of an element A from $\mathcal{C}\ell_{3,0} \otimes \mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,3}{}^+\hat{\otimes}\mathcal{C}\ell_{1,3}{}^+$, and $\langle A\rangle_{+-}$, $\langle A\rangle_{-+}$ and $\langle A\rangle_{--}$ are defined similarly. This is an expression of the fact that an operator two-particle Pauli spinor is encoded four times in its algebraic counterpart.

The relation between the above definitions for two-particle spinors can be expressed through the following extensions of the maps α and β, introduced in the previous chapter (cf. subsection 4.1.3, equation (4.44)):

$$
\begin{array}{ccc}
\mathit{\Phi}\otimes X=\begin{pmatrix} a^0+ia^3 \\ -a^2+ia^1 \end{pmatrix}\otimes\begin{pmatrix} b^0+ib^3 \\ -b^2+ib^1 \end{pmatrix} & \xmapsto{\alpha} & \underline{\phi}^1\underline{\chi}^2=(a^0+a^iI\boldsymbol{\sigma}_i{}^1)(b^0+b^iI\boldsymbol{\sigma}_i{}^2)f^1f^2=\phi^1\chi^2f^1f^2 \\
 & \searrow^{\beta\circ\alpha} & \downarrow\beta \\
 & & \phi^1\chi^2=(a^0+a^iI\boldsymbol{\sigma}_i{}^1)(b^0+b^iI\boldsymbol{\sigma}_i{}^2)=4\langle\underline{\phi}^1\underline{\chi}^2\rangle_{++}.
\end{array}
\tag{5.24}
$$

In this case, by writing $\mathit{\Psi}=\mathit{\Phi}\otimes X$, $\underline{\psi}=\underline{\phi}^1\underline{\chi}^2$ and $\psi=\phi^1\chi^2$, and using the following notations

$$
\mathrm{A}^1\mathit{\Psi}=(\mathrm{A}\Phi)\otimes X \quad \text{and} \quad \mathrm{A}^2\mathit{\Psi}=\mathit{\Phi}\otimes(\mathrm{A}X),
\tag{5.25}
$$

the action of operators is translated through

$$
\begin{array}{ccc}
\Sigma_j{}^a\mathit{\Psi} & \xmapsto{\alpha} & \boldsymbol{\sigma}_j{}^a\underline{\psi}=\boldsymbol{\sigma}_j{}^a\psi\boldsymbol{\sigma}_3{}^af^1f^2 \\
 & \searrow^{\beta\circ\alpha} & \downarrow\beta \\
 & & 4\langle\boldsymbol{\sigma}_j{}^a\underline{\psi}\rangle_{++}=\boldsymbol{\sigma}_j{}^a\psi\boldsymbol{\sigma}_3{}^a
\end{array}
\tag{5.26}
$$

and

$$
\begin{array}{ccc}
i^a\mathit{\Psi} & \xmapsto{\alpha} & I^a\underline{\psi}=\psi I^a\boldsymbol{\sigma}_3{}^af^1f^2 \\
 & \searrow^{\beta\circ\alpha} & \downarrow\beta \\
 & & 4\langle I^a\underline{\psi}\rangle_{++}=\psi I^a\boldsymbol{\sigma}_3{}^a.
\end{array}
\tag{5.27}
$$

It is worth mentioning that the above definition of an operator two-particle Pauli spinor can be identified with the representation introduced by Doran *et al.* (1993, 1996) for the case of non-relativistic two-particle states. In this representation, a basis for the state space is given by

$$
\{1, I\boldsymbol{\sigma}_i{}^1, I\boldsymbol{\sigma}_i{}^2, I\boldsymbol{\sigma}_i{}^1I\boldsymbol{\sigma}_j{}^2\}, \quad \text{where} \quad i,j\in\{1,2,3\},
\tag{5.28}
$$

and the shorthand notation $I\boldsymbol{\sigma}_i{}^a=I^a\boldsymbol{\sigma}_i{}^a$ was used. This basis has 16 elements, while a classical two-particle Pauli spinor can be expressed in terms of 4 complex or 8 real coefficients. This doubling is a consequence of the fact that the algebraic and operator representations of the spinors include a distinct substitute for the imaginary unit $i=\sqrt{-1}$. Indeed, from relations (5.27) it follows that $i\mathit{\Psi}$ is mapped by

$$
i\mathit{\Psi}=(i\mathit{\Phi})\otimes X \quad \overset{\alpha}{\mapsto} \quad I^1\underline{\psi} \quad \overset{\beta}{\mapsto} \quad \psi I\boldsymbol{\sigma}_3{}^1
\tag{5.29}
$$

and

$$i\mathit{\Psi} = \mathit{\Phi} \otimes (iX) \quad \overset{\alpha}{\mapsto} \quad I^2 \underline{\psi} \quad \overset{\beta}{\mapsto} \quad \psi \boldsymbol{I\sigma_3}^2. \tag{5.30}$$

This ambiguity can be eliminated by requiring that the algebraic and operator two-particle Pauli spinors satisfy

$$I^1 \underline{\psi} = I^2 \underline{\psi} \quad \text{and} \quad \psi \boldsymbol{I\sigma_3}^1 = \psi \boldsymbol{I\sigma_3}^2. \tag{5.31}$$

These requirements, whose ramifications can be analyzed simply from either the algebraic or the operator representation, imply in this later that

$$\psi = -\psi \boldsymbol{I\sigma_3}^1 \boldsymbol{I\sigma_3}^2, \tag{5.32}$$

which allows one to write

$$\psi = \psi \frac{1}{2}\Big(1 - \boldsymbol{I\sigma_3}^1 \boldsymbol{I\sigma_3}^2\Big). \tag{5.33}$$

Then, the considered state can be described by

$$\mathit{\Psi} = \mathit{\Phi} \otimes X \quad \overset{\alpha}{\mapsto} \quad \underline{\psi} = \underline{\phi}^1 \underline{\chi}^2 E \quad \overset{\beta}{\mapsto} \quad \psi = \phi^1 \chi^2 E, \tag{5.34}$$

where

$$E = \frac{1}{2}\Big(1 - \boldsymbol{I\sigma_3}^1 \boldsymbol{I\sigma_3}^2\Big). \tag{5.35}$$

The multivector E is an idempotent and its product on the right with a state represents a projection operation ($E^2 = E$ and $E(1-E) = 0$). The meaning of this projection operation can be understood by noting that the product of E with both $\boldsymbol{I\sigma_3}^1$ and $\boldsymbol{I\sigma_3}^2$ results in the bivector $\frac{1}{2}(\boldsymbol{I\sigma_3}^1 + \boldsymbol{I\sigma_3}^2)$, which implies that the product on the right of $\psi = \phi^1\chi^2 E$ with both $\boldsymbol{I\sigma_3}^1$ and $\boldsymbol{I\sigma_3}^2$ has the same result. Thus, the resulting effect of the projection operation is to halve the number of degrees of freedom in the algebraic and operator spinors. In this representation, the product on the right with the multivector

$$J = E\boldsymbol{I\sigma_3}^1 = E\boldsymbol{I\sigma_3}^2 = \frac{1}{2}\Big(\boldsymbol{I\sigma_3}^1 + \boldsymbol{I\sigma_3}^2\Big), \tag{5.36}$$

which satisfies

$$J^2 = -E, \tag{5.37}$$

is the representative for multiplication by the imaginary unit $i = \sqrt{-1}$ in the classical representation. The complex linear combinations of states are then expressed by sums of products on the right with multivectors of the form $a + bJ$, where a and b are real scalars. The four complex basic states of a system of two spin-$\frac{1}{2}$ particles are represented in this context by:

$$
\begin{aligned}
\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} &\overset{\alpha}{\mapsto} f^1 f^2 E &&\overset{\beta}{\mapsto} E,\\
\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} &\overset{\alpha}{\mapsto} -I\boldsymbol{\sigma}_2{}^2 f^1 f^2 E &&\overset{\beta}{\mapsto} -I\boldsymbol{\sigma}_2{}^2 E,\\
\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} &\overset{\alpha}{\mapsto} -I\boldsymbol{\sigma}_2{}^1 f^1 f^2 E &&\overset{\beta}{\mapsto} -I\boldsymbol{\sigma}_2{}^1 E,\\
\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} &\overset{\alpha}{\mapsto} I\boldsymbol{\sigma}_2{}^1 I\boldsymbol{\sigma}_2{}^2 f^1 f^2 E &&\overset{\beta}{\mapsto} I\boldsymbol{\sigma}_2{}^1 I\boldsymbol{\sigma}_2{}^2 E.
\end{aligned}
\tag{5.38}
$$

5.2.4 The N-Particle Case

The generalization of the above results for the case of a system of N spin-$\frac{1}{2}$ particles is simple, and essentially consists of introducing a factor algebra in the tensor product for each particle introduced in the system. More precisely, the algebra of operators in this case is given by

$$T^N \mathcal{C}\ell_{3,0} = \underbrace{\mathcal{C}\ell_{3,0}\otimes\cdots\otimes\mathcal{C}\ell_{3,0}}_{N \text{ factors}}, \tag{5.39}$$

corresponding to

$$\hat{T}^N \mathcal{C}\ell_{1,3}{}^+ = \underbrace{\mathcal{C}\ell_{1,3}{}^+\hat{\otimes}\cdots\hat{\otimes}\mathcal{C}\ell_{1,3}{}^+}_{N \text{ factors}}, \tag{5.40}$$

and included through the relations $\boldsymbol{\sigma}_i{}^a = \gamma_i{}^a\gamma_0{}^a$ in the N-particle spacetime algebra,

$$\hat{T}^N \mathcal{C}\ell_{1,3} = \underbrace{\mathcal{C}\ell_{1,3}\hat{\otimes}\cdots\hat{\otimes}\mathcal{C}\ell_{1,3}}_{N \text{ factors}}, \tag{5.41}$$

whose fundamental property is given by

$$\frac{1}{2}(\gamma_\mu{}^a\gamma_\nu{}^b + \gamma_\nu{}^b\gamma_\mu{}^a) = \delta^{ab}\eta_{\mu\nu}, \tag{5.42}$$

where $\gamma_\mu{}^1 = \gamma_\mu\hat{\otimes}\cdots\hat{\otimes}1,\ \ldots,\ \gamma_\mu{}^N = 1\hat{\otimes}\cdots\hat{\otimes}\gamma_\mu$.

The representation of a wave function given by a simple tensor product is represented with an extra factor corresponding to each new particle introduced in the system. A *classical N-particle Pauli spinor* is then defined in terms of tensor products of N single-particle Pauli spinors. The expression of such a classical spinor in terms of square matrices gives an element of the algebra of operators, leading naturally to the algebraic description for the spinor, guaranteed by the commutativity of elements of the algebra of operators from different factor algebras (this guarantees the commutativity of the idempotents with other elements of the algebra). In this way, an *algebraic N-particle Pauli spinor* can be defined as an element of the minimal left ideal

$$\mathcal{I}_N = \left\{ Af \;\middle|\; A \in \hat{T}^N \mathcal{C}\ell_{1,3}{}^+ \text{ and } f = \frac{1}{2}\big(1+\boldsymbol{\sigma}_3{}^1\big)\cdots\frac{1}{2}\big(1+\boldsymbol{\sigma}_3{}^N\big)\right\}. \tag{5.43}$$

The reduction of an algebraic N-particle Pauli spinor,

$$A^1\cdots B^N f^1\cdots f^N \quad\to\quad \underline{\phi}^1\cdots\underline{\chi}^N = \phi^1\cdots\chi^N f^1\cdots f^N \tag{5.44}$$

where the elements $A^1,\ldots,B^N$, each belonging to a copy of $\mathcal{C}\ell_{3,0}\simeq\mathcal{C}\ell_{1,3}{}^+$, are replaced by the elements $\phi^1,\ldots,\chi^N$, each belonging to a copy of $\mathcal{C}\ell_{3,0}{}^+\simeq\mathcal{C}\ell_{1,3}{}^{++}$, is straightforward from the commutativity of elements of different copies of $\mathcal{C}\ell_{3,0}\simeq\mathcal{C}\ell_{1,3}{}^+$ and from the property $\boldsymbol{\sigma}_3{}^a f^a = f^a$.

This reduction and the fact that the idempotent $f^1\cdots f^N$ is a fixed factor in the expression for an algebraic N-particle Pauli spinor leads to the definition of an *operator N-particle Pauli spinor* as an element of the subalgebra $T^N\mathcal{C}\ell_{3,0}{}^+\simeq\hat{T}^N\mathcal{C}\ell_{1,3}{}^{++}$. Given that a single-particle operator spinor ϕ is given in terms of its algebraic counterpart by $\phi = 2\langle\underline{\phi}\rangle_+ = 2\langle\underline{\phi}\rangle_-\boldsymbol{\sigma}_3$, there are 2^N ways to express the operator N-particle Pauli spinor $\phi^1\cdots\chi^N$ in terms of its algebraic counterpart, $\underline{\phi}^1\cdots\underline{\chi}^N$. The simplest is given by $\phi^1\cdots\chi^N = 2^N\langle\underline{\phi}^1\cdots\underline{\chi}^N\rangle_{+\cdots+}$, where $\langle A\rangle_{+\cdots+}$ denotes the extraction of the even grade part of all factors of an element A from $T^N\mathcal{C}\ell_{3,0}\simeq\hat{T}^N\mathcal{C}\ell_{1,3}{}^+$. This is an expression of the fact that an operator N-particle Pauli spinor is encoded 2^N times in its algebraic counterpart.

The generalization of the transformations (5.24) are immediate and, at the same time, too extensive to be given here.

The representation (5.34) of the algebraic and operator spinors, required to make the algebraic and operator descriptions compatible with the classical one, can be extended to systems of any number of spin-$\frac{1}{2}$ states as follows. Any multivector corresponding to a state for a system of N spin-$\frac{1}{2}$ states contains the multivector E_N as a factor on the right. For this general case, the multivector E_N is required to be idempotent and satisfy

$$E_N I\boldsymbol{\sigma}_3{}^1 = E_N I\boldsymbol{\sigma}_3{}^2 = \cdots = E_N I\boldsymbol{\sigma}_3{}^a = \cdots = E_N I\boldsymbol{\sigma}_3{}^N. \tag{5.45}$$

These conditions are equivalent to

$$E_N = -E_N I\boldsymbol{\sigma}_3{}^1 I\boldsymbol{\sigma}_3{}^2 = \cdots = -E_N I\boldsymbol{\sigma}_3{}^1 I\boldsymbol{\sigma}_3{}^a = \cdots = -E_N I\boldsymbol{\sigma}_3{}^1 I\boldsymbol{\sigma}_3{}^n, \tag{5.46}$$

which in turn can be expressed by

$$E_N = E_N \frac{1}{2}\Big(1 - I\boldsymbol{\sigma}_3{}^1 I\boldsymbol{\sigma}_3{}^2\Big) = \cdots = E_N \frac{1}{2}\Big(1 - I\boldsymbol{\sigma}_3{}^1 I\boldsymbol{\sigma}_3{}^a\Big) = \cdots = E_N \frac{1}{2}\Big(1 - I\boldsymbol{\sigma}_3{}^1 I\boldsymbol{\sigma}_3{}^n\Big). \tag{5.47}$$

The fact that each multivector in these equations is idempotent allows one to write

$$E_N = E_N \prod_{a=2}^{n} \frac{1}{2}\Big(1 - I\boldsymbol{\sigma}_3{}^1 I\boldsymbol{\sigma}_3{}^a\Big), \tag{5.48}$$

where the product in question is the geometric product. The idempotent E_N can then be defined by

$$E_N = \prod_{a=1}^{N} \frac{1}{2}\Big(1 - I\boldsymbol{\sigma}_3{}^1 I\boldsymbol{\sigma}_3{}^a\Big). \tag{5.49}$$

The complex structure is provided by the multivector

$$J_N = E_N I\boldsymbol{\sigma}_3{}^a, \quad \text{for} \quad a \in \{1, \ldots, n\}. \tag{5.50}$$

A basic example is provided by the case of $N = 3$, for which

$$\begin{aligned} E_3 &= \frac{1}{2}\Big(1 - I\boldsymbol{\sigma}_3{}^1 I\boldsymbol{\sigma}_3{}^2\Big)\frac{1}{2}\Big(1 - I\boldsymbol{\sigma}_3{}^1 I\boldsymbol{\sigma}_3{}^3\Big) \\ &= \frac{1}{4}\Big(1 - I\boldsymbol{\sigma}_3{}^1 I\boldsymbol{\sigma}_3{}^2 - I\boldsymbol{\sigma}_3{}^1 I\boldsymbol{\sigma}_3{}^3 - I\boldsymbol{\sigma}_3{}^2 I\boldsymbol{\sigma}_3{}^3\Big) \end{aligned} \tag{5.51}$$

and

$$J_3 = \frac{1}{4}\Big(I\boldsymbol{\sigma}_3{}^1 + I\boldsymbol{\sigma}_3{}^2 + I\boldsymbol{\sigma}_3{}^3 - I\boldsymbol{\sigma}_3{}^1 I\boldsymbol{\sigma}_3{}^2 I\boldsymbol{\sigma}_3{}^3\Big). \tag{5.52}$$

5.2.5 Inner Product of Two-Particle Pauli Spinors

The Hermitian adjoint of the classical two-particle Pauli spinor $\varPsi = \varPhi \otimes X$ can be defined to be

$$\varPsi^\dagger = \varPhi^\dagger \otimes X^\dagger. \tag{5.53}$$

It can be translated to the algebraic and operator forms as

$$\begin{array}{ccc} \varPsi^\dagger = \varPhi^\dagger \otimes X^\dagger & \xmapsto{\ \alpha\ } & \underline{\psi}^\dagger = \underline{\phi}^{1\dagger}\underline{\chi}^{2\dagger} = f^1 f^2 \phi^{1\dagger}\chi^{2\dagger} \\ & \searrow^{\beta\circ\alpha} & \big\downarrow \beta \\ & & \psi^\dagger = \phi^{1\dagger}\chi^{2\dagger}. \end{array} \tag{5.54}$$

The Hermitian inner product of $\mathit{\Psi} = \mathit{\Phi} \otimes X$ with a second classical two-particle Pauli spinor, $\mathit{\Psi}' = \mathit{\Phi}' \otimes X'$, is given by

$$\mathit{\Psi}^\dagger \mathit{\Psi}' = (\mathit{\Phi}^\dagger \mathit{\Phi}')(X^\dagger X'). \tag{5.55}$$

As a generalization of relations (4.68), in section 4.1.4, this Hermitian inner product can be mapped through the maps α and β, given by relations (5.24), to its algebraic and operator counterparts, as follows:

$$\begin{array}{ccc}
\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \mathit{\Psi}^\dagger \mathit{\Psi}' & \overset{\alpha}{\longmapsto} & \underline{\psi}^\dagger \underline{\psi}' = (\underline{\phi}^{1\dagger} \underline{\phi}'^{1})(\underline{\chi}^{2\dagger} \underline{\chi}'^{2}) = \Big(\langle \phi^{1\dagger}\phi' \rangle - \langle \phi^{1\dagger}\phi' \boldsymbol{I\sigma}_3{}^1 \rangle \boldsymbol{I\sigma}_3{}^1\Big)\Big(\langle \chi^{1\dagger}\chi' \rangle - \langle \chi^{1\dagger}\chi' \boldsymbol{I\sigma}_3{}^2 \rangle \boldsymbol{I\sigma}_3{}^2\Big) f^1 f^2 \\
& \overset{\beta\circ\alpha}{\searrow} & \Big\downarrow \beta \\
& & \langle \phi^{1\dagger}\phi^{1\prime} \rangle_q \langle \chi^{2\dagger}\chi^{2\prime} \rangle_q = \Big(\langle \phi^{1\dagger}\phi' \rangle - \langle \phi^{1\dagger}\phi' \boldsymbol{I\sigma}_3{}^1 \rangle \boldsymbol{I\sigma}_3{}^1\Big)\Big(\langle \chi^{1\dagger}\chi' \rangle - \langle \chi^{1\dagger}\chi' \boldsymbol{I\sigma}_3{}^2 \rangle \boldsymbol{I\sigma}_3{}^2\Big).
\end{array} \tag{5.56}$$

5.3 Relativistic Multi-Particle Spinors

As in the non-relativistic case, a possible state for a system of two relativistic spin-$\frac{1}{2}$ particles can be expressed by the tensor product $\mathit{\Phi} \otimes X$, where $\mathit{\Phi}$ and X are classical Dirac spinors describing states for the particles 1 and 2, respectively. In this case, an operator defined on the tensor product space is represented by a linear combination of tensor products of the form $\mathrm{A} \otimes \mathrm{B}$, where A and B are complex 4×4 matrices representing operators acting on the spaces of the particles 1 and 2, respectively. As shown in the previous chapter, the algebra of operators for each of the two particles is the Dirac algebra, which corresponds to the complexified geometric algebra of spacetime, $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$. In this way, the algebra of operators acting on the tensor product space of two relativistic spin-$\frac{1}{2}$ particles corresponds to $(\mathbb{C} \otimes \mathcal{C}\ell_{1,3}) \otimes (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})$. Analogously to the case of a single particle, this algebra can be used to represent states as well as operators for the system of two particles. However, as in the non-relativistic case, this does not correspond to a Clifford algebra, and an inclusion of it into a Clifford algebra is necessary to properly define multi-particle spinors. An adequate extension of the algebra $(\mathbb{C}\otimes\mathcal{C}\ell_{1,3})\otimes(\mathbb{C}\otimes\mathcal{C}\ell_{1,3})$ to a Clifford algebra can be realized only if a compatible extension of the Dirac algebra, $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$, could be performed.

5.3.1 The Dirac Algebra and its Inclusion in a Higher-Dimensional Algebra

As emphasized by Figueiredo *et al.* (1990), the usual Dirac algebra considered in physics, $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$, is isomorphic to the real Clifford algebra $\mathcal{C}\ell_{4,1}$ (cf. also, da Rocha and Vaz (2007)). If $\{\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ is an orthogonal basis of unit vectors for the vector space $\mathbb{R}^{4,1}$, the product of the Clifford algebra $\mathcal{C}\ell_{4,1}$ is defined by

$$\epsilon_0{}^2 = -1, \quad \epsilon_j{}^2 = \epsilon_4{}^2 = 1, \quad \text{and} \quad \epsilon_A\epsilon_B = -\epsilon_B\epsilon_A, \tag{5.57}$$

where $j \in \{1, 2, 3\}$ and $A, B \in \{0, 1, 2, 3, 4\}$, with $A \neq B$. It follows that, the unit pseudoscalar of $\mathcal{C}\ell_{4,1}$, given by $\epsilon_5 = \epsilon_0\epsilon_1\epsilon_2\epsilon_3\epsilon_4$, satisfies

$$\epsilon_5\epsilon_A = \epsilon_A\epsilon_5 \quad \text{and} \quad \epsilon_5{}^2 = -1, \tag{5.58}$$

for $A \in \{0, 1, 2, 3, 4\}$. In this way, the isomorphism $\mathbb{C} \otimes \mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{4,1}$ can be defined through the identifications

$$\gamma_\mu = \epsilon_\mu\epsilon_4 \quad \text{and} \quad i = \epsilon_5, \tag{5.59}$$

where $\mu \in \{0, 1, 2, 3\}$. Indeed, these identifications imply

$$\gamma_0{}^2 = (\epsilon_0\epsilon_4)(\epsilon_0\epsilon_4) = -\epsilon_0{}^2\epsilon_4{}^2 = 1 \quad \text{and} \quad \gamma_j{}^2 = (\epsilon_j\epsilon_4)(\epsilon_j\epsilon_4) = -\epsilon_j{}^2\epsilon_4{}^2 = -1, \tag{5.60}$$

for $j \in \{1, 2, 3\}$, and

$$\gamma_\mu\gamma_\nu = (\epsilon_\mu\epsilon_4)(\epsilon_\nu\epsilon_4) = -(\epsilon_\nu\epsilon_4)(\epsilon_\mu\epsilon_4) = -\gamma_\nu\gamma_\mu, \tag{5.61}$$

for $\mu, \nu \in \{0, 1, 2, 3\}$ and $\mu \neq \nu$. In addition, $i = \epsilon_5$ commutes with arbitrary elements of the algebra and its square is -1.

Note that the correspondences $\gamma_\mu = \epsilon_\mu\epsilon_4$ identify the vectors of the algebra $\mathcal{C}\ell_{1,3}$ with bivectors of the algebra $\mathcal{C}\ell_{4,1}$. In this way, products of vectors of $\mathcal{C}\ell_{1,3}$, which generate general elements of this Clifford algebra, correspond to products of bivectors of $\mathcal{C}\ell_{4,1}$, which generate even grade elements of this larger Clifford algebra. This shows that the real Clifford algebra $\mathcal{C}\ell_{1,3}$ is isomorphic to the even subalgebra $\mathcal{C}\ell_{4,1}{}^+$ of the Clifford algebra $\mathcal{C}\ell_{4,1}$: $\mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{4,1}{}^+$.

One might consider the even subalgebra $\mathcal{C}\ell_{4,1}{}^+ \simeq \mathcal{C}\ell_{1,3}$ as the subalgebra relevant to define multi-particle Dirac spinors. However, the imaginary unit, given by the pseudoscalar ϵ_5, is the product of the five generators of the algebra and is thus an element of $\mathcal{C}\ell_{4,1}$ but not of $\mathcal{C}\ell_{4,1}{}^+$. One can not construct the appropriate idempotents in an alternating tensor product of copies of $\mathcal{C}\ell_{4,1}$, since in this case the spin factors of the idempotents do not commute, that is,

$$\frac{1}{2}(1 + i^1\gamma_{12}{}^1)\frac{1}{2}(1 + i^2\gamma_{12}{}^2) \neq \frac{1}{2}(1 + i^2\gamma_{12}{}^2)\frac{1}{2}(1 + i^1\gamma_{12}{}^1), \tag{5.62}$$

where $\frac{1}{2}(1 + i^a\gamma_{12}{}^a) = \frac{1}{2}(1 + \epsilon_5{}^a\epsilon_{21}{}^a)$.

Now, the pattern in the sequence of inclusions

$$\mathcal{C}\ell_{3,0} \to \mathcal{C}\ell_{1,3} \to \mathcal{C}\ell_{4,1}, \tag{5.63}$$

wherein each Clifford algebra is included in the next as its even subalgebra, allows one to conclude that the Clifford algebra that will permit the construction of the correct idempotents is the next in the sequence, $\mathcal{C}\ell_{2,4}$. This is indeed the case. If $\{\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$ is an orthogonal basis of unit vectors for $\mathbb{R}^{2,4}$, the Clifford algebra $\mathcal{C}\ell_{2,4}$ can be defined such that its product is given by

$$\zeta_0{}^2 = \zeta_5{}^2 = 1, \quad \zeta_j{}^2 = \zeta_4{}^2 = -1, \quad \text{and} \quad \zeta_U\zeta_V = -\zeta_V\zeta_U, \tag{5.64}$$

where $j \in \{1, 2, 3\}$ and $U, V \in \{0, 1, 2, 3, 4, 5\}$, with $U \neq V$. In this case, the isomorphism $\mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{2,4}{}^+$ can be defined through the identifications

$$\epsilon_A = \zeta_A\zeta_5, \tag{5.65}$$

where $A \in \{0, 1, 2, 3, 4\}$. Note from these relations that

$$\epsilon_0{}^2 = -\zeta_0{}^2\zeta_5{}^2 = -1, \quad \epsilon_j{}^2 = -\zeta_j{}^2\zeta_5{}^2 = 1, \quad \text{and} \quad \epsilon_4{}^2 = -\zeta_4{}^2\zeta_5{}^2 = 1, \tag{5.66}$$

for $j \in \{1, 2, 3\}$, and

$$\epsilon_A\epsilon_B = (\zeta_A\zeta_5)(\zeta_B\zeta_5) = -(\zeta_B\zeta_5)(\zeta_A\zeta_5) = -\epsilon_B\epsilon_A, \tag{5.67}$$

for $A, B \in \{0, 1, 2, 3, 4\}$ and $A \neq B$.

The Clifford algebra $\mathcal{C}\ell_{2,4}$ is not the only one whose even subalgebra corresponds to the Dirac algebra $\mathcal{C}\ell_{4,1}$. A construction similar to the one above allows one to verify that another possibility is furnished by the Clifford algebra $\mathcal{C}\ell_{4,2}$. This is a simple consequence of the isomorphism $\mathcal{C}\ell_{2,4}{}^+ \simeq \mathcal{C}\ell_{4,2}{}^+$.

5.3.2 The Tensor Product Algebra Acting on Two-Particle States

As shown above, the Dirac algebra $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$ is isomorphic to the Clifford algebra $\mathcal{C}\ell_{4,1}$ through the identifications $\gamma_\mu = \epsilon_\mu\epsilon_4$ and $i = \epsilon_5$. This fact implies that the algebra of operators acting on relativistic two-particle spin-$\frac{1}{2}$ states, $(\mathbb{C} \otimes \mathcal{C}\ell_{1,3}) \otimes (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})$, can be identified with the algebra $\mathcal{C}\ell_{4,1} \otimes \mathcal{C}\ell_{4,1}$ through the relations

$$\gamma_\mu{}^a = \epsilon_\mu{}^a \epsilon_4{}^a \quad \text{and} \quad i^a = \epsilon_5{}^a, \tag{5.68}$$

where $\gamma_\mu{}^1 = \gamma_\mu \otimes 1$, $\gamma_\mu{}^2 = 1 \otimes \gamma_\mu$, $i^1 = i \otimes 1$, $i^2 = 1 \otimes i$, $\epsilon_A{}^1 = \epsilon_A \otimes 1$, $\epsilon_A{}^2 = 1 \otimes \epsilon_A$, $\epsilon_5{}^2 = 1 \otimes \epsilon_5$. According to the usual definition of the tensor product of algebras, it satisfies

$$\frac{1}{2}(\gamma_\mu{}^a \gamma_\nu{}^b + \gamma_\nu{}^b \gamma_\mu{}^a) = \eta_{\mu\nu}, \quad \text{for} \quad a = b, \tag{5.69}$$

$$\gamma_\mu{}^a \gamma_\nu{}^b = \gamma_\nu{}^b \gamma_\mu{}^a, \quad \text{for} \quad a \neq b, \tag{5.70}$$

and

$$i^a \gamma_\mu{}^b = \gamma_\mu{}^b i^a, \quad \text{for} \quad a, b \in \{1, 2\}. \tag{5.71}$$

These properties do not define a Clifford algebra. In addition, despite the fact that the alternating tensor product $\mathcal{C}\ell_{4,1} \hat{\otimes} \mathcal{C}\ell_{4,1}$ is a Clifford algebra, it does not allow one to construct the appropriate idempotent and corresponding ideal, as seen in the previous subsection. A way to circumvent the problem is to embed the algebra $\mathcal{C}\ell_{4,1} \otimes \mathcal{C}\ell_{4,1}$ in a Clifford algebra, and refer the desired constructions to this larger algebra.

Since, as seen above, the Clifford algebra $\mathcal{C}\ell_{4,1}$ is isomorphic to the even subalgebra $\mathcal{C}\ell_{2,4}{}^+$ of the Clifford algebra $\mathcal{C}\ell_{2,4}$, through $\epsilon_A = \zeta_A \zeta_5$, where $A \in \{0, 1, 2, 3, 4\}$, the algebra $\mathcal{C}\ell_{4,1} \otimes \mathcal{C}\ell_{4,1}$ can be identified with the algebra $\mathcal{C}\ell_{2,4}{}^+ \hat{\otimes} \mathcal{C}\ell_{2,4}{}^+$ through the relations

$$\epsilon_A{}^a = \zeta_A{}^a \zeta_5{}^a, \tag{5.72}$$

where $\zeta_U{}^1 = \zeta_U \hat{\otimes} 1$ and $\zeta_U{}^2 = 1 \hat{\otimes} \zeta_U$, for $U \in \{0, 1, 2, 3, 4, 5\}$. In this case, the algebra $\mathcal{C}\ell_{2,4}{}^+ \hat{\otimes} \mathcal{C}\ell_{2,4}{}^+$ is understood as the even sublagebra of the Clifford algebra $\mathcal{C}\ell_{2,4} \hat{\otimes} \mathcal{C}\ell_{2,4}$, given by an alternating tensor product. The product of this larger algebra is then given by

$$(\zeta_0{}^a)^2 = (\zeta_5{}^a)^2 = 1, \quad (\zeta_j{}^a)^2 = (\zeta_4{}^a)^2 = -1, \quad \text{and} \quad \zeta_U{}^a \zeta_V{}^b = -\zeta_V{}^b \zeta_U{}^a, \tag{5.73}$$

where $U, V \in \{0, 1, 2, 3, 4, 5\}$, with $U \neq V$, and $a, b \in \{1, 2\}$, with $a \neq b$. This properties can be summarized by

$$\frac{1}{2}(\zeta_U{}^a \zeta_V{}^b + \zeta_V{}^b \zeta_U{}^a) = \delta^{ab} \tau_{UV}, \tag{5.74}$$

where τ_{UV} are the components of the metric tensor for the space $\mathbb{R}^{2,4}$. This expression makes it clear that $\mathcal{C}\ell_{2,4} \hat{\otimes} \mathcal{C}\ell_{2,4}$ is in fact a Clifford algebra.

5.3.3 Two-Particle Dirac Spinors

A tensor product of the form $\varPhi \otimes X$, where $\varPhi$ and X are classical Dirac spinors, can be understood as a *classical two-particle Dirac spinor*. The corresponding tensor product in terms of square matrices, $\Phi \otimes \mathrm{X}$, can be understood as a matrix representation for an element of the algebra of operators $(\mathbb{C}\otimes\mathcal{C}\ell_{1,3})\otimes(\mathbb{C}\otimes\mathcal{C}\ell_{1,3}) \simeq \mathcal{C}\ell_{4,1}\otimes\mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{2,4}{}^{+}\hat{\otimes}\mathcal{C}\ell_{2,4}{}^{+}$. Such an element can be written as

$$\underset{\sim}{\phi}\hat{\otimes}\underset{\sim}{\chi} = \underset{\sim}{\phi}^1\underset{\sim}{\chi}^2 = \underset{\sim}{\chi}^2\underset{\sim}{\phi}^1, \tag{5.75}$$

where $\underset{\sim}{\phi}^1 = \underset{\sim}{\phi}\hat{\otimes}1$ and $\underset{\sim}{\chi}^2 = 1\hat{\otimes}\underset{\sim}{\chi}$, and $\underset{\sim}{\phi}$ and $\underset{\sim}{\chi}$ are the respective elements of the algebra $\mathbb{C} \otimes \mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{2,4}^{+}$ defining the algebraic Dirac spinors corresponding to $\varPhi$ and X. Given that the elements $\underset{\sim}{\phi}$ and $\underset{\sim}{\chi}$ correspond to elements of the minimal left ideal $\mathcal{I} = \{Af \mid A \in \mathbb{C} \otimes \mathcal{C}\ell_{1,3} \text{ and } f = \frac{1}{2}(1+\gamma_0)(1+i\gamma_{12})\}$, they can be written respectively as ϕf and χf, where ϕ and χ are the even grade elements defining the corresponding operator Dirac spinors. This, in addition to the fact that the product of elements of the algebra $(\mathbb{C} \otimes \mathcal{C}\ell_{1,3}) \otimes (\mathbb{C} \otimes \mathcal{C}\ell_{1,3}) \simeq \mathcal{C}\ell_{4,1} \otimes \mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{2,4}{}^{+}\hat{\otimes}\mathcal{C}\ell_{2,4}{}^{+}$ belonging to different copies of $\mathcal{C}\ell_{1,3}) \simeq \mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{2,4}{}^{+}$ is commutative, allows one to write

$$\underset{\sim}{\phi}\hat{\otimes}\underset{\sim}{\chi} = \underset{\sim}{\phi}^1\underset{\sim}{\chi}^2 = \phi^1\chi^2 f^1 f^2, \tag{5.76}$$

where $\phi^1 = \phi\hat{\otimes}1$ and $\chi^2 = 1\hat{\otimes}\chi$, and,

$$f^1 = \frac{1}{2}(1+{\gamma_0}^1)\frac{1}{2}(1+i^1{\gamma_{12}}^1) \quad \text{and} \quad f^2 = \frac{1}{2}(1+{\gamma_0}^2)\frac{1}{2}(1+i^2{\gamma_{12}}^2). \tag{5.77}$$

The expression (5.76) defines the *algebraic two-particle Dirac spinor* corresponding to $\varPhi\otimes X$ as an element of the minimal left ideal $\mathcal{I}_2 = \{Af \mid A \in \mathcal{C}\ell_{2,4}{}^{+}\hat{\otimes}\mathcal{C}\ell_{2,4}{}^{+} \text{ and } f = f^1 f^2\}$. The reduction of an algebraic two-particle Dirac spinor $A^1B^2f^1f^2$ to an element of the form (5.76), where A^1 and B^2 are replaced by elements from their respective copies of $\mathcal{C}\ell_{1,3}{}^{+} \simeq \mathcal{C}\ell_{4,1}{}^{++} \simeq \mathcal{C}\ell_{2,4}{}^{+++}$ (see Table 5.1), is straightforward from the commutativity of elements of different copies and from the known way in which the reduction is performed in the case of a single-particle (cf. subsection 4.2.2), which follows in this case from the property $i^a{\gamma_{12}}^a f^a = f^a$.

algebra	multiplier	generators
$\mathcal{C}\ell_{2,4}$	-	${\zeta_0}^2 = {\zeta_5}^2 = 1,\ {\zeta_1}^2 = {\zeta_2}^2 = {\zeta_3}^2 = {\zeta_4}^2 = -1$
$\mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{2,4}{}^{+}$	ζ_5	$(\zeta_1\zeta_5)^2 = (\zeta_2\zeta_5)^2 = (\zeta_3\zeta_5)^2 = (\zeta_4\zeta_5)^2 = 1,\ (\zeta_0\zeta_5)^2 = -1$
$\mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{4,1}{}^{+} \simeq \mathcal{C}\ell_{2,4}{}^{++}$	$\zeta_4\zeta_5$	$(\zeta_0\zeta_4)^2 = 1,\ (\zeta_1\zeta_4)^2 = (\zeta_2\zeta_4)^2 = (\zeta_3\zeta_4)^2 = -1$
$\mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,3}{}^{+} \simeq \mathcal{C}\ell_{4,1}{}^{++} \simeq \mathcal{C}\ell_{2,4}{}^{+++}$	$\zeta_4\zeta_0$	$(\zeta_1\zeta_0)^2 = (\zeta_2\zeta_0)^2 = (\zeta_3\zeta_0)^2 = 1$

TABLE 5.1 – Generators of even subalgebras of $\mathcal{C}\ell_{2,4}$.

This reduction and the fact that the idempotent $f^1 f^2$ is a fixed factor in the expression for an algebraic two-particle Dirac spinor imply that the even grade multivectors belonging to the subalgebra $\mathcal{C}\ell_{1,3}{}^{+} \otimes \mathcal{C}\ell_{1,3}{}^{+} \simeq \mathcal{C}\ell_{4,1}{}^{++} \otimes \mathcal{C}\ell_{4,1}{}^{++} \simeq \mathcal{C}\ell_{2,4}{}^{+++} \hat{\otimes} \mathcal{C}\ell_{2,4}{}^{+++}$ can suffice to describe the considered states. This allows one to define an *operator two-particle Dirac spinor* as an element of the subalgebra $\mathcal{C}\ell_{2,4}{}^{+++} \hat{\otimes} \mathcal{C}\ell_{2,4}{}^{+++}$. Given that the single-particle operator Dirac spinor ϕ in equation (5.76) can be expressed in terms of its algebraic counterpart by $\phi = 4\langle \mathrm{Re}(\underline{\phi})\rangle_{+} = 4\langle \mathrm{Re}(\underline{\phi})\rangle_{-}\gamma_0 = 4\langle \mathrm{Im}(\underline{\phi})\rangle_{+}\gamma_{21} = 4\langle \mathrm{Im}(\underline{\phi})\rangle_{-}\gamma_0\gamma_{21}$, and similar expressions hold for χ (cf. subsection 4.2.3), there are 16 ways to obtain the operator two-particle Dirac spinor $\phi^1\chi^2$ from its algebraic counterpart, $\underline{\phi}^1\underline{\chi}^2$. The simplest is given by $\phi^1\chi^2 = 16\langle \mathrm{Re}(\underline{\phi}^1)\mathrm{Re}(\underline{\chi}^2)\rangle_{++}$. This is an expression of the fact that an operator two-particle Dirac spinor is encoded 16 times in its algebraic counterpart.

It is worth mentioning that the above developments, specifically the identification and extension of the algebra of operators to the alternating tensor product of copies of the conformal spacetime Clifford algebra $\mathcal{C}\ell_{2,4}$, as well as the definitions of multi-particle Dirac spinors, have not been presented before this work.

The relation between the above definitions for two-particle spinors can be expressed through the following extensions of the maps α and β for Dirac spinors, introduced in the previous chapter (cf. subsection 4.2.3, equation (4.123)):

$$
\begin{array}{c}
\Psi \otimes X = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \\ b^0 + ib^3 \\ -b^2 + ib^1 \end{pmatrix} \otimes \begin{pmatrix} c^0 + ic^3 \\ -c^2 + ic^1 \\ d^0 + id^3 \\ -d^2 + id^1 \end{pmatrix} \\
\big\downarrow \alpha \\
\underline{\phi}^1\underline{\chi}^2 = \Big((a^0 + a^j I\boldsymbol{\sigma}_j{}^1) + (b^0 + b^j I\boldsymbol{\sigma}_j{}^j)\boldsymbol{\sigma}_3{}^1\Big)\Big((c^0 + c^j I\boldsymbol{\sigma}_j{}^2) + (d^0 + d^j I\boldsymbol{\sigma}_j{}^2)\boldsymbol{\sigma}_3{}^2\Big) f^1 f^2 = \phi^1\chi^2 f^1 f^2 \\
\big\downarrow \beta \\
\phi^1\chi^2 = \Big((a^0 + a^j I\boldsymbol{\sigma}_j{}^1) + (b^0 + b^j I\boldsymbol{\sigma}_j{}^j)\boldsymbol{\sigma}_3{}^1\Big)\Big((c^0 + c^j I\boldsymbol{\sigma}_j{}^2) + (d^0 + d^j I\boldsymbol{\sigma}_j{}^2)\boldsymbol{\sigma}_3{}^2\Big) = 16\langle \mathrm{Re}(\underline{\phi}^1)\mathrm{Re}(\underline{\chi}^2)\rangle_{++}.
\end{array}
\tag{5.78}
$$

In this case, by writing $\Psi = \Phi \otimes X$, $\underline{\psi} = \underline{\phi}^1\underline{\chi}^2$ and $\psi = \phi^1\chi^2$, and using the notations

$$
\mathrm{A}^1\,\Psi = (\mathrm{A}\Phi) \otimes X \quad \text{and} \quad \mathrm{A}^2\,\Psi = \Phi \otimes (\mathrm{A}X), \tag{5.79}
$$

the action of operators is translated through

$$
\begin{array}{ccc}
\Gamma_\mu{}^a\,\Psi & \xrightarrow{\ \alpha\ } & \gamma_\mu{}^a\underline{\psi} = \gamma_\mu{}^a\psi\gamma_0{}^a f^1 f^2 \\
 & \searrow^{\beta\circ\alpha} & \big\downarrow \beta \\
 & & \gamma_\mu{}^a\psi\gamma_0{}^a,
\end{array}
\tag{5.80}
$$

$$
\begin{array}{ccl}
\Gamma_5{}^a\,\Psi & \xmapsto{\alpha} & \gamma_5^a\underline{\psi} = \gamma_5{}^a\psi f^1 f^2 = \psi\boldsymbol{\sigma}_3{}^a f^1 f^2 \\
& \searrow^{\beta\circ\alpha} & \qquad\qquad\qquad \downarrow \beta \\
& & \qquad\qquad\qquad \psi\boldsymbol{\sigma}_3{}^a
\end{array}
\tag{5.81}
$$

and

$$
\begin{array}{ccl}
i^a\,\Psi & \xmapsto{\alpha} & i^a\underline{\psi} = i^a\psi f^1 f^2 = \psi I\boldsymbol{\sigma}_3{}^a f^1 f^2 \\
& \searrow^{\beta\circ\alpha} & \qquad\qquad\qquad \downarrow \beta \\
& & \qquad\qquad\qquad \psi I\boldsymbol{\sigma}_3{}^a.
\end{array}
\tag{5.82}
$$

Note from the expressions for the algebraic and operator two-particle Dirac spinors in relations (5.78) that such spinors are determined by $8 \times 8 = 64$ coefficients, while the corresponding classical spinor is determined by 32 coefficients. As in the non-relativistic case, this doubling is a consequence of the fact that both the algebraic and operator representations has two representatives for the imaginary unit. This ambiguity can be eliminated by requiring that

$$
i^1\underline{\psi} = i^2\underline{\psi} \quad \text{and} \quad \psi I\boldsymbol{\sigma}_3{}^1 = \psi I\boldsymbol{\sigma}_3{}^2, \tag{5.83}
$$

which implies the following description for the considered state

$$
\Psi = \Phi \otimes X \quad \overset{\alpha}{\mapsto} \quad \underline{\psi} = \underline{\phi}^1\underline{\chi}^2 E \quad \overset{\beta}{\mapsto} \quad \psi = \phi^1\chi^2 E, \tag{5.84}
$$

where

$$
E = \frac{1}{2}\Big(1 - I\boldsymbol{\sigma}_3{}^1 I\boldsymbol{\sigma}_3{}^2\Big). \tag{5.85}
$$

Again, the multivector E is an idempotent and its product on the right with a state represents a projection operation, whose resulting effect is to halve the number of degrees of freedom in the algebraic and operator spinors. The product on the right with the multivector

$$
J = EI\boldsymbol{\sigma}_3{}^1 = EI\boldsymbol{\sigma}_3{}^2 = \frac{1}{2}\Big(I\boldsymbol{\sigma}_3{}^1 + I\boldsymbol{\sigma}_3{}^2\Big), \tag{5.86}
$$

for which

$$
J^2 = -E, \tag{5.87}
$$

is the representative for multiplication by the imaginary unit, and the complex linear combinations of states are performed by sums of products on the right with multivectors of the form $a + bJ$, where a and b are real scalars.

As a final remark on the definitions of multi-particle spinors, it should be noted that although the descriptions are based on simple tensor products, the fact that a general state is given by a sum of these simple ones does not imply that the definitions are incorrect.

The algebraic definition of a multi-particle spinor describes it as an element of an ideal of a Clifford algebra, which may be a sum of elements of the ideal. In the same way, the operator definition of a multi-particle spinor describes it as an element of a subalgebra of a Clifford algebra, which may be a sum of elements of the subalgebra.

5.3.4 The N-Particle Case

Similarly to the non-relativistic case, the generalization of the above results for the case of a system of N spin-$\frac{1}{2}$ particles is simple. The algebra of operators in this case is given by

$$T^N\mathcal{C}\ell_{4,1} = \underbrace{\mathcal{C}\ell_{4,1} \otimes \cdots \otimes \mathcal{C}\ell_{4,1}}_{N \text{ factors}}, \tag{5.88}$$

corresponding to

$$\hat{T}^N\mathcal{C}\ell_{2,4}{}^+ = \underbrace{\mathcal{C}\ell_{2,4}{}^+ \hat{\otimes} \cdots \hat{\otimes} \mathcal{C}\ell_{2,4}{}^+}_{N \text{ factors}}, \tag{5.89}$$

and included through the relations $\epsilon_A{}^a = \zeta_A{}^a\zeta_5{}^a$ in the N-particle algebra

$$\hat{T}^N\mathcal{C}\ell_{2,4} = \underbrace{\mathcal{C}\ell_{2,4} \hat{\otimes} \cdots \hat{\otimes} \mathcal{C}\ell_{2,4}}_{N \text{ factors}}, \tag{5.90}$$

whose fundamental property is given by

$$\frac{1}{2}(\zeta_U{}^a\zeta_V{}^b + \zeta_V{}^b\zeta_U{}^a) = \delta^{ab}\tau_{UV}, \tag{5.91}$$

where $\zeta_U{}^1 = \zeta_U \hat{\otimes} \cdots \hat{\otimes} 1$, $\ldots$, $\zeta_U{}^N = 1 \hat{\otimes} \cdots \hat{\otimes} \zeta_U$.

A *classical N-particle Dirac spinor* is defined in terms of tensor products of N single-particle Dirac spinors. The expression of such a classical spinor in terms of square matrices gives an element of the algebra of operators, leading naturally to the algebraic description, guaranteed by the commutativity of elements of the algebra of operators belonging to different factor algebras. An *algebraic N-particle Dirac spinor* can then be defined as an element of the minimal left ideal

$$\mathcal{I}_N = \left\{ Af \;\middle|\; A \in \hat{T}^N\mathcal{C}\ell_{2,4}{}^+ \text{ and } f = \frac{1}{2}\big(1+\gamma_0{}^1\big)\frac{1}{2}\big(1+i\gamma_{12}{}^1\big)\cdots\frac{1}{2}\big(1+\gamma_0{}^N\big)\frac{1}{2}\big(1+i\gamma_{12}{}^N\big) \right\}. \tag{5.92}$$

Again, the reduction of an algebraic N-particle Dirac spinor,

$$A^1 \cdots B^N f^1 \cdots f^N \quad \rightarrow \quad \underline{\phi}^1 \cdots \underline{\chi}^N = \phi^1 \cdots \chi^N f^1 \cdots f^N \tag{5.93}$$

where the elements $A^1, \ldots, B^N$, each belonging to a copy of $\mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{2,4}{}^+$, are replaced by the elements $\phi^1, \ldots, \chi^N$, each belonging to a copy of $\mathcal{C}\ell_{1,3}{}^+ \simeq \mathcal{C}\ell_{4,1}{}^{++} \simeq \mathcal{C}\ell_{2,4}{}^{+++}$, is straightforward from the commutativity of elements of different copies of $\mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{2,4}{}^{+}$ and from the property $i\gamma^a_{12} f^a = f^a$.

This reduction and the fact that the idempotent $f^1 \cdots f^N$ is a fixed factor in an algebraic N-particle Dirac spinor leads to the definition of an *operator N-particle Dirac spinor* as an element of the subalgebra $T^N\mathcal{C}\ell_{1,3}{}^{+} \simeq T^N\mathcal{C}\ell_{4,1}{}^{++} \simeq \hat{T}^N\mathcal{C}\ell_{2,4}{}^{+++}$. Since a single-particle operator spinor ϕ is given in terms of its algebraic counterpart by $\phi = 4\langle\mathrm{Re}(\underline{\phi})\rangle_{+} = 4\langle\mathrm{Re}(\underline{\phi})\rangle_{-}\gamma_0 = 4\langle\mathrm{Im}(\underline{\phi})\rangle_{+}\gamma_{21} = 4\langle\mathrm{Im}(\underline{\phi})\rangle_{-}\gamma_0\gamma_{21}$, the operator N-particle Dirac spinor $\phi^1 \cdots \chi^N$ can be obtained from its algebraic version, $\underline{\phi}^1 \cdots \underline{\chi}^N$, in 2^{2N} ways. The simplest is given by $\phi^1 \cdots \chi^N = 2^{2N}\langle\mathrm{Re}(\underline{\phi}^1) \cdots \mathrm{Re}(\underline{\chi}^N)\rangle_{+\cdots+}$. This is an expression of the fact that an operator N-particle Dirac spinor is encoded 2^{2N} times in its algebraic counterpart.

The representation (5.84) is extended to the case of N-particle Dirac spinors in a similar manner to the extension of two-particle to N-particle Pauli spinors, since the complex structure in the operator representation is similar in the non-relativistic and relativistic cases.

5.3.5 Relativistic Two-Fermion Wave Equation

A relativistic wave equation for two Dirac particles is important for many reasons, one of which is the case when dealing with bound states (GREINER; REINHARDT, 2008). The principal equation in this context can be written as

$$\left(i\hbar\Gamma^a \cdot \partial_a - m_a c\right)\left(i\hbar\Gamma^b \cdot \partial_b - m_b c\right) \mathit{\Psi}\left(x^a, x^b\right) = \int \mathrm{d}^4 x^{a\prime} \mathrm{d}^4 x^{b\prime}\, V\left(x^a, x^b; x^{a\prime}, x^{b\prime}\right) \mathit{\Psi}\left(x^{a\prime}, x^{b\prime}\right), \tag{5.94}$$

where all irreducible contributions to the interaction of the two particles are included into the interaction $V\left(x^a, x^b; x^{a\prime}, x^{b\prime}\right)$. This is one of the versions of the *Bethe-Salpeter equation* (SALPETER; BETHE, 1951; NAKANISHI, 1969). The point here is not to study this equation, but simply to express it in Clifford algebraic and operator terms, by recognizing the wave function as a classical two-particle Dirac spinor, recognizing the product of Dirac operators on the left-hand side of the equation as a tensor product, and then mapping all the elements of the equation to their algebraic and operator counterparts. The right-hand side involves an interaction term which is not analyzed here. In this way, the algebraic version of equation (5.94) can be written as

$$\left(i^a\hbar\gamma^{\mu a}\partial_\mu{}^a - m_a c\right)\left(i^b\hbar\gamma^{\mu b}\partial_\mu{}^b - m_b c\right)\underline{\psi}\left(x^a, x^b\right) = \int \mathrm{d}^4 x^{a\prime} \mathrm{d}^4 x^{b\prime}\, V\left(x^a, x^b; x^{a\prime}, x^{b\prime}\right) \underline{\psi}\left(x^{a\prime}, x^{b\prime}\right), \tag{5.95}$$

or,

$$\left(i^a\hbar\nabla^a - m_a c\right)\left(i^b\hbar\nabla^b - m_b c\right)\underline{\psi}\left(x^a, x^b\right) = \int \mathrm{d}^4x^{a\prime}\mathrm{d}^4x^{b\prime}\, V\left(x^a, x^b; x^{a\prime}, x^{b\prime}\right)\underline{\psi}\left(x^{a\prime}, x^{b\prime}\right), \tag{5.96}$$

where the wave function is now given by an algebraic two-particle Dirac spinor, that is, an element of the minimal left ideal $\mathcal{I}_2 = \{Af \mid A \in \mathcal{C}\ell_{2,4}{}^{+}\hat{\otimes}\mathcal{C}\ell_{2,4}{}^{+}$ and $f = f^1 f^2\}$. In this form, the equation is expressed in terms of the algebra $(\mathbb{C}\otimes\mathcal{C}\ell_{1,3})\otimes(\mathbb{C}\otimes\mathcal{C}\ell_{1,3}) \simeq \mathcal{C}\ell_{4,1}\otimes\mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{2,4}{}^{+}\hat{\otimes}\mathcal{C}\ell_{2,4}{}^{+}$, and a search for solutions should target those elements that belong to the ideal $\mathcal{I}_2$. If one requires the same action for multiplication by each of the two imaginary units, the representation (5.84) should be used. In this case, the above equation can be rewritten as

$$\left(\hat{p}^a - m_a c\right)\left(\hat{p}^b - m_b c\right)\underline{\psi}\left(x^a, x^b\right) = \int \mathrm{d}^4x^{a\prime}\mathrm{d}^4x^{b\prime}\, V\left(x^a, x^b; x^{a\prime}, x^{b\prime}\right)\underline{\psi}\left(x^{a\prime}, x^{b\prime}\right), \tag{5.97}$$

where now the wave function includes the projector E as a factor on the right, and the operator $\hat{p}^a$, and in the same manner $\hat{p}^b$, are defined by

$$\hat{p}^a(\underline{\psi}) = \hbar\nabla^a\underline{\psi}I\boldsymbol{\sigma}_3{}^a = \hbar\nabla^a\underline{\psi}J \tag{5.98}$$

(see the previous subsection for the definitions of E and J).

Finally, transforming $\underline{\psi} = \psi f^a f^b$ into its operator counterpart, ψ, which corresponds essentially to the removal of the idempotents f^a and f^b as factors, transforms the equation (5.97) to

$$\left(\hat{p}^a - m_a c\right)\left(\hat{p}^b - m_b c\right)\psi\left(x^a, x^b\right) = \int \mathrm{d}^4x^{a\prime}\mathrm{d}^4x^{b\prime}\, V\left(x^a, x^b; x^{a\prime}, x^{b\prime}\right)\psi\left(x^{a\prime}, x^{b\prime}\right), \tag{5.99}$$

where the operators $\hat{p}^a$ and $\hat{p}^b$ are now given by

$$\hat{p}^a(\psi) = \hbar\nabla^a\psi\gamma_0{}^a J, \tag{5.100}$$

and the wave function is now given by an element of the subalgebra $\mathcal{C}\ell_{1,3}{}^{+}\otimes\mathcal{C}\ell_{1,3}{}^{+} \simeq \mathcal{C}\ell_{4,1}{}^{++}\otimes\mathcal{C}\ell_{4,1}{}^{++} \simeq \mathcal{C}\ell_{2,4}{}^{+++}\hat{\otimes}\mathcal{C}\ell_{2,4}{}^{+++}$. In this form, the equation is written in terms of the algebra $\mathcal{C}\ell_{1,3}\otimes\mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{4,1}{}^{+}\otimes\mathcal{C}\ell_{4,1}{}^{+} \simeq \mathcal{C}\ell_{2,4}{}^{++}\hat{\otimes}\mathcal{C}\ell_{2,4}{}^{++}$, and the wave function is restricted to involve only even grade elements. Remember that a similar reduction is also present in the passage of the Pauli and Dirac equations to their operator forms (cf. subsections 4.1.5 and 4.2.2). This fact raises the possibility that the operator forms of the wave equations may be more fundamental. Even if this is not the case, they at least appear to be more economical.

6 Conclusions

In this work, the states for systems of multiple spin-$\frac{1}{2}$ particles in the context of quantum mechanics (both non-relativistic and relativistic) are described through the concept of a multi-particle spinor — in particular, multi-particle Dirac spinors are introduced for the first time in this work. This concept is introduced in chapter 5 as a generalization of single-particle Pauli and Dirac spinors, in their classical and Clifford algebraic and operator forms, which are presented in chapter 4. To prepare for these developments, the basic concepts of the algebra of the three-dimensional Euclidean space $\mathcal{C}\ell_{3,0}$ and of the algebra of Minkowski spacetime $\mathcal{C}\ell_{1,3}$ are introduced in chaper 2. In chapter 3, the deep relationship between the two algebras is used to develop the basics of relativistic kinematics and of the Lorentz group of transformations. To close the chapter, a brief development of Maxwell's equations and other aspects of electromagnetism in the context of the algebra $\mathcal{C}\ell_{1,3}$ is given. In chapter 4, the forms of single-particle Pauli and Dirac spinors are examined in detail. The transformations from classical to algebraic to operator spinors are rigorously defined here and are used to systematically define the corresponding single-particle spinors and observable expectation values. These explicit transformations are presented for the first time here. The different but equivalent forms of the Pauli equation and of the Dirac equation are also discussed. The rigorous definitions given here of the transformations to algebraic and operator spinors are essential for their extension to the transformations of multi-particle spinors studied in chapter 5.

The classical definition of a multi-particle Pauli spinor is straightforward, and is given in terms of tensor products of single-particle classical Pauli spinors. The fact that the algebra of operators acting on the multi-particle states, given by tensor products of copies of the Clifford algebra $\mathcal{C}\ell_{3,0}$, does not correspond to a Clifford algebra, motivated the search for an embedding of this algebra in a Clifford algebra, which would allow spinors to be defined in this context in a similar way to the case of a single particle. In view of the fact that $\mathcal{C}\ell_{3,0}$ is isomorphic to the even subalgebra ${\mathcal{C}\ell_{1,3}}^+$, the algebra of operators could be considered as the corresponding tensor product of copies of ${\mathcal{C}\ell_{1.3}}^+$, which, when interpreted as an alternating tensor product, was found to correspond to the even subalgebra of a Clifford algebra. This larger algebra is identical to the multi-particle spacetime algebra, first introduced by Doran *et al.* (1993). A multi-particle algebraic Pauli spinor could then be defined as an element of a minimal left ideal of the even subalgebra of the multi-particle spacetime algebra. From this definition, as in the single-particle case, a multi-particle Pauli spinor could be defined in operator form, as an element of the subalgebra given by a tensor product of copies of the subalgebra ${\mathcal{C}\ell_{1,3}}^{++}$, isomorphic to the subalgebra given by the corresponding tensor product of copies of ${\mathcal{C}\ell_{3,0}}^{+}$.

A formally consistent extension of the single-particle Dirac spinor to a multi-particle one has not been proposed previous to this work. To perform such an extension, it is first observed that the classical complex Clifford algebra $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$, used in chapter 4 to develop the classical, algebraic and operator forms of the single-particle Dirac spinor, is isomorphic to the real Clifford algebra $\mathcal{C}\ell_{4,1}$. The latter is taken to be the correct Clifford algebra for the single-particle Dirac spinor. In analogy to the case of the Pauli spinor, the embedding in a Clifford algebra of the commutative structure of the classical tensor product requires that the Dirac algebra $\mathcal{C}\ell_{4,1}$ be interpreted as the even subalgebra of a larger Clifford algebra, in this case, as either ${\mathcal{C}\ell_{4,2}}^{+}$ or ${\mathcal{C}\ell_{2,4}}^{+}$. The latter of the two, ${\mathcal{C}\ell_{2,4}}^{+}$, was adopted here, since it is the even subalgebra of the conformal spacetime Clifford algebra $\mathcal{C}\ell_{2,4}$ that serves as the basis of the twistor program developed by Penrose and collaborators (PENROSE; RINDLER, 1988). The "multi-particle Dirac algebra", including the algebra of operators as an even subalgebra, is then found to be given by an alternating tensor product of copies of $\mathcal{C}\ell_{2,4}$. This construction is proposed for the first time in this work. A multi-particle algebraic Dirac spinor could then be defined as an element of a minimal left ideal of the even subalgebra of such a multi-particle Dirac algebra. From this definition, an operator multi-particle Dirac spinor can be defined as an element of the subalgebra given by a tensor product of copies of the subalgebra ${\mathcal{C}\ell_{2,4}}^{+++}$. Finally, the Bethe-Salpeter two-fermion equation was briefly discussed.

Several avenues of research could be of interest in the future. The most obvious of these would be to look for deeper relations between the multi-particle Dirac algebra and the conformal spacetime algebra. At the moment, we have no interpretation for the additional degrees of freedom necessary to describe the multi-particle spinors. A better understanding of their relation to conformal spacetime could provide physical content to the embedding of the usual spacetime Clifford algebra $\mathcal{C}\ell_{1,3}$ in $\mathcal{C}\ell_{4,1}$ and its extension to $\mathcal{C}\ell_{2,4}$.

Other attempts to interpret the additional degrees of freedom could also be explored. Among these are theories of the Kaluza-Klein type, although such theories normally only introduce additional space coordinates, consistent with the extension to $\mathcal{C}\ell_{4,1}$. Inclusion of what could be interpreted as an additional time coordinate as well is more difficult, although there is some precedent in a proposal to include a proper time as well as local time in a relativistic formalism (SAAD *et al.*, 1989). The square of the proper time is in fact one of the coordinates of the usual null vector representation in the conformal algebra

$\mathcal{C}\ell_{2,4}$ of a coordinate vector in $\mathcal{C}\ell_{1,3}$.

A simple and, at the same time, more radical line of research would be based on the observation that, for both the Pauli and Dirac spinors, the embedding of the algebra in a larger one in order to reproduce the tensor product, would appear to be necessary only to perform the transformation from the classical to the algebraic and operator forms of the spinors. Once the spinors and wave equations have been expressed in operator form, the commutativity of their tensor products is a natural property, even within the initial multiparticle algebra, since the operator forms of the Pauli and Dirac spinors and equations involve only even elements of this algebra. The extent to which this alternative formalism can provide new or interesting insights and solutions is still to be determined.

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Appendix A - Published Work

At a certain stage of this work, a possible objective was to provide an extension of the Bohmian mechanics (BOHM, 1952) to spin-$\frac{1}{2}$ particles, and possibly to the relativistic domain, through a Clifford algebraic formulation, which appeared to be convenient for the purpose of this extension. In a preliminary study, the bipolar reduction of the Schödinger equation was used, in the context of nuclear scattering, to examine the effects of absorption on incoming and outgoing scattering waves. Through the Wigner transform, the bipolar incoming and outgoing waves could be interpreted in terms of incoming and outgoing trajectories (DA CONCEIÇÃO *et al.*, 2023). The resulting article is attached in the following pages.

Quantum trajectories and the nuclear optical model

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PAPER

Quantum trajectories and the nuclear optical model

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Abstract

In the context of nuclear scattering, we use the bipolar reduction of the Schrödinger equation to examine the effects of optical model absorption on incoming and outgoing scattering waves. We compare the exact solutions for these waves, obtained using a bipolar quantum trajectory-based formalism, with their approximate WKB counterparts. Aside from reducing the magnitudes of the incoming and outgoing waves, absorption smooths the variation of the potential at the turning point, reducing reflection in this region. This brings the incoming exact solution and WKB approximation into closer agreement, but tends to worsen the agreement between the outgoing solutions. Inside the turning point, the WKB approximation overestimates the inward decaying solution. The exact solution also possesses an outward going component, solely due to reflection, with no WKB counterpart.

Introduction

Due to the complexity of the nucleus, flux is often lost to reaction channels different from those of immediate interest in nuclear scattering. In early experimental and theoretical efforts, this effect was explicitly attributed to the formation and decay of neutron resonances at low energy [1] and to nucleon-nucleon collisions at higher energy [2]. It was first characterized as an average imaginary contribution to the nucleon-nucleus effective (optical) potential by Bethe, in an analysis of compound nucleus formation [3]. Such a complex potential was later used effectively to describe nucleon-nucleus scattering, first using semiclassical methods [4] and, then, in exact solutions of the Schrödinger equation [5, 6]. The formal definition of the potential provided by Feshbach [7] has served as a basis for further developments that continue to the present, as can be attested by reviews of the subject, both old [8] and new [9, 10]. At present, with few exceptions, an imaginary term $iW(\vec{r}),\ \ W < 0$, is automatically included in the nuclear optical potential in any analysis of nucleon-nucleus or nucleus-nucleus scattering. Although the absorptive potential is ubiquitous in nuclear scattering studies, investigations of the effect of this potential on the traveling wave components or associated quantum trajectories have only been performed in the context of the WKB approximation [11–15].

In chemical physics, absorptive potentials are often used to absorb outgoing flux in quantum scattering calculations of colliding molecular partners, encompassing elastic and inelastic as well as chemically reactive scattering [16–25]. In this context, the absorbing potentials [called 'absorbing boundary conditions' (ABC's)] are only applied in the asymptotic regions, to avoid artificial reflections off of the hard wall edges of the numerical grids that would otherwise occur. Of course, there is still some reflection, due to the flux that is not absorbed by time the edge of the grid is reached; moreover, the ABC itself influences the asymptotic dynamics unphysically. One can always 'turn on' the ABCs more slowly, or otherwise extend them further into the asymptotic regime, in order to mitigate these sources of error, but this can add considerably to the computational cost, especially in the limit of deep tunneling. Accordingly, significant effort has been spent attempting to optimize the form of the ABCs, so as to minimize the extent of the asymptotic absorbing regions [22–25], but this remains a considerable challenge.

More recently, the quantum trajectory approach [26–55] offers an avenue that is particularly appealing for scattering applications. In this approach, in addition to (or in some formulations, instead of [48–55]) the usual quantum wave function, one works with a quantum trajectory or ensemble of trajectories. The quantum trajectories satisfy their own Newton-like time evolution equation that includes a 'quantum force', in additional to the usual classical forces. The quantum trajectory approach has the great advantage that ABCs are not needed; instead, all scattering quantities may be extracted directly from the quantum trajectories themselves, once they first reach the asymptotic region. This approach has proven particularly effective for extremely deep tunneling, for which the requisite ABC regions would be many orders of magnitude larger than the scattering region itself, and thus numerically unfeasible [51, 53, 55].

One of the defining features of scattering is reflection, which necessarily leads to interference with the incident wave. In most quantum trajectory formulations, interference manifests as oscillations in the corresponding quantum trajectories (corresponding to probability density oscillations), which can be quite severe and difficult to model. However, in the so-called 'bipolar' formalism [42–47, 54], the exact scattering wave function solution is decomposed into incoming and outgoing waves, in analogy with the WKB approximation. In this fashion, the bipolar waves—as well as the bipolar quantum trajectories that come from them—become much less oscillatory and better behaved. Moreover, the bipolar formalism naturally encompasses the idea of *flux transfer during the scattering process*, between incoming and outgoing components [43, 44, 47].

For these reasons, in this paper, we make a first effort to extend the previous studies of the exact bipolar decomposition in the chemical physics context, to include the effects of dynamical absorption (i.e. as opposed to ABCs) so as to allow the method to be applied in the nuclear scattering context. In addition, we make a comparison with the WKB approximation, which is often used in this context, and is in effect, an approximation of the exact bipolar formulation presented here [40–44, 47, 56–58]. In any event, the literature on the WKB approximation is immense [40, 56–68].

In the following, we will first discuss the unipolar and bipolar quantum trajectory formalisms, of which the latter most easily permits a classical-like trajectory interpretation of solutions of the Schrödinger equation. We will then comment on the relation between the exact bipolar quantum trajectory method and the well known but approximate WKB method. We will then develop and discuss numerical solutions to the equations, which furnish exact solutions to the Schrödinger equations, and compare these to semiclassical WKB solutions. Finally, we will discuss our results and possible future directions before concluding.

Formalities

Unipolar treatment

When an imaginary potential is included explicitly, the single-channel time-dependent Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{r})\psi + iW(\vec{r})\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{1}$$

Using the standard 'unipolar' (Madelung-Bohm) [26–30] decomposition of the wave function,

$$\psi(\vec{r},\, t) = R(\vec{r},\, t)\exp[iS(\vec{r},\, t)/\hbar], \tag{2}$$

the Schrödinger equation can be separated into two equations, which suggest a natural trajectory implementation.

The first of these two equations is the quantum Hamilton-Jacobi equation,

$$\frac{(\vec{\nabla}S)^2}{2m} + V(\vec{r}) + Q(\vec{r},\, t) + \frac{\partial S}{\partial t} = 0, \tag{3}$$

which will be seen to govern the dynamics of the quantum trajectories. Here,

$$Q(\vec{r},\, t) = -\frac{\hbar^2}{2m}\frac{1}{R}\nabla^2 R \tag{4}$$

is the quantum potential [26, 27, 29], whose negative gradient is the aforementioned quantum force.

The second equation is the continuity equation,

$$\frac{\partial\rho}{\partial t} + \frac{1}{m}\vec{\nabla}\cdot(\rho\vec{\nabla}S) = \frac{2}{\hbar}W(\vec{r})\rho, \tag{5}$$

with the density defined as

$$\rho(\vec{r},\,t) = (R(\vec{r},\,t))^2 = \psi^\dagger\psi. \tag{6}$$

The reduction of the wave equation defines trajectories when we associate the gradient of the action with the linear momentum,

$$\vec{p} = m\vec{v} = \vec{\nabla}S. \tag{7}$$

The continuity equation can then be written as

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = \frac{2}{\hbar}W(\vec{r}(t))\rho(\vec{r}(t)), \tag{8}$$

with the current density given by

$$\vec{j}(\vec{r}(t)) = \rho(\vec{r}(t))\vec{v}(\vec{r}(t)). \tag{9}$$

The continuity equation normally associates the time rate of change of the density ρ with the flux $\vec{j}$ into or out of the region. In trajectory terms, equation (8) would normally (i.e. if the right-hand side were zero) imply that probability is conserved along individual quantum trajectories. However, in the present absorbing context, the continuity equation is modified by the inclusion of an additional loss term on the right hand side, i.e. $2W\rho/\hbar$, which accounts for the loss of flux to reaction channels that are not being considered in the calculation. Such a continuity equation, including the absorptive potential, was proposed by Bethe in a study of compound nucleus formation [3].

In the time-independent case, the Hamilton-Jacobi equation becomes

$$\frac{\vec{p}\cdot\vec{p}}{2m} + V(\vec{r}) + Q(\vec{r}) = E, \tag{10}$$

while the continuity equation reduces to

$$\vec{\nabla}\cdot\vec{j} = \frac{2}{\hbar}W(\vec{r})\rho(\vec{r}) \tag{11}$$

The case in which $W = 0$ has been treated in many studies by using a family of quantum trajectories that adhere to both equations above [27, 31–39, 42–47, 49–51, 53–55]. It is not clear how the method might be extended to include the absorptive potential, as it is based on the invariance of the density under coordinate transformations. This will not be the case when absorption is included, as the density will no longer vary in accord with the flux as it would if the flux were conserved.

Bipolar treatment

An alternative that can be applied in the one-dimensional case, in particular, to the radial time-independent Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial r^2} + V(r) + iW(r) = i\hbar E\psi, \tag{12}$$

is the bipolar method, which expresses the wave function in terms of incoming and outgoing components,

$$\psi(r) = \psi_+(r) + \psi_-(r). \tag{13}$$

Using the continuous limit of transmission and reflection equations, these can be shown to satisfy the coupled equations [43, 44, 47],

$$\frac{d}{dt}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = \frac{p'}{2m}\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} + \frac{i}{\hbar}(E - 2[V + iW])\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}, \tag{14}$$

where we take the local momentum $p(r)$ to be,

$$p(r) = \sqrt{2m(E - V(r) - iW(r))} \quad \text{with} \quad p' = \frac{dp}{dr}. \tag{15}$$

A few comments are in order. First, note that the bipolar trajectories implied by the equations above are actually *classical* trajectories, unlike in the unipolar case. Classical trajectories are, of course, much more in line with the WKB approximation, even though the bipolar methodology above is exact. Second, note that the evolution equations above are couched in a time-dependent form, even though they are in fact designed to compute stationary or time-independent solutions of the Schrödinger equation. This was by design, in order that a trajectory-based or time-evolving theory could be developed (consult [44]).

On the other hand, for the present work, we find it useful to extract a fully equivalent time-independent equation, effectively replacing the coordinate t with r, through the following substitution:

$$\frac{d\psi_\pm}{dt} = \frac{\partial\psi_\pm}{\partial t} \pm \frac{p}{m}\frac{\partial\psi_\pm}{\partial r} \to -\frac{i}{\hbar}E\psi_\pm \pm \frac{p}{m}\frac{\partial\psi_\pm}{\partial r} \tag{16}$$

The evolution equations then become

$$\frac{d}{dr}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = \frac{p'}{2p}\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} + \frac{ip}{\hbar}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}, \tag{17}$$

which in terms of Pauli matrices reduces to

$$\Psi' = -\frac{p'}{2p}(1-\sigma_x)\Psi + \frac{ip}{\hbar}\sigma_z\Psi, \quad \text{with} \quad \Psi = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}. \tag{18}$$

For radial applications as considered here, the above equations determine an incoming solution ψ_- and an outgoing one ψ_+, whose local phase variations (which would be equal and opposite if W were zero) are determined by the momentum $p(r)$. Although neither the ψ_+ nor the ψ_- solution satisfies the Schrödinger equation in general, their sum always does, as we demonstrate below. Note that these bipolar equations were also derived by H Bremmer [57], as the continuous limit to scattering from a sequence of discrete steps, and used by Berry and Mount [58] in their extensive review of WKB-type approximations. Related, but more general, expressions were also derived by Fröman and Fröman [56].

An alternative but informative form of these equations can be obtained by projecting them onto equations for the sum $\psi = \psi_+ + \psi_-$ and difference $\psi_a = \psi_+ - \psi_-$ of the bipolar solutions. The scalar product with the row vector (1, 1) furnishes the equation for the sum,

$$\psi(r)' = (1,\,1)\cdot\Psi(r)' = -\tfrac{p'}{2p}(1,\,1)\cdot(1-\sigma_x)\Psi(r) + \tfrac{ip}{\hbar}(1,\,1)\cdot\sigma_z\Psi(r) = \tfrac{i}{\hbar}p(r)\psi_a(r) \tag{19}$$

while the scalar product with the row vector (1, − 1) gives the equation for the difference,

$$\psi_a(r)' = (1,\,-1)\cdot\Psi(r)' = -\tfrac{p'}{2p}(1,\,-1)\cdot(1-\sigma_x)\Psi(r) + \tfrac{ip}{\hbar}(1,\,1)\cdot\sigma_z\Psi(r) = -\tfrac{p'}{p}\psi_a(r) + \tfrac{i}{\hbar}p(r)\psi(r). \tag{20}$$

We rewrite these as

$$\begin{aligned}\psi(r)' &= \tfrac{i}{\hbar}p(r)\psi_a(r)\\ (p(r)\psi_a(r))' &= \tfrac{i}{\hbar}(p(r))^2\psi(r).\end{aligned} \tag{21}$$

The Schrödinger equation follows trivially from the two equations, as

$$\psi(r)'' = \frac{i}{\hbar}(p(r)\psi_a(r))' = -\frac{1}{\hbar^2}(p(r))^2\psi(r). \tag{22}$$

We note that the the coupled equations for ψ and ψ_a, when written as

$$\begin{pmatrix}\psi(r)\\ \frac{i}{\hbar}p(r)\psi_a(r)\end{pmatrix}' = \begin{pmatrix}0 & 1\\ \frac{-1}{\hbar^2}(p(r))^2 & 0\end{pmatrix}\begin{pmatrix}\psi(r)\\ \frac{i}{\hbar}p(r)\psi_a(r)\end{pmatrix}, \tag{23}$$

are identical to the standard reduction of the second-order Schrödinger equation to a first-order one,

$$\begin{pmatrix}\psi(r)\\ \psi(r)'\end{pmatrix}' = \begin{pmatrix}0 & 1\\ \frac{-1}{\hbar^2}(p(r))^2 & 0\end{pmatrix}\begin{pmatrix}\psi(r)\\ \psi(r)'\end{pmatrix}. \tag{24}$$

with the advantage that we can now identify the incoming and outgoing components as

$$\psi_\pm(r) = \frac{1}{2}\left(\psi(r) \pm \frac{1}{p(r)}\frac{\hbar}{i}\psi(r)'\right). \tag{25}$$

This association is general and can be applied to any solution of the radial Schrödinger equation. The bipolar components obtained in this manner will satisfy the coupled bipolar equations for $\psi_\pm$, but, in general, will not satisfy the Schrödinger equation. Note that a similar equation has been derived previously [47].

The analysis above confirms the choice of including the absorptive contribution to the optical potential directly in the local momentum, taking

$$p(r) = \sqrt{2m(E - V(r) - iW(r))}. \tag{26}$$

With this definition of the local momentum, the solutions of the bipolar equation, $\psi_\pm$, furnish the exact solution $\psi = \psi_+ + \psi_-$ to the Schrödinger equation. But we must than ask how one can associate $p(r)$ with a real momentum: in terms of the magnitude of the complex quantity, $|p(r)|$, as its real part, $\mathrm{Re}[p(r)]$, or in terms of the truncated quantity,

$$p_0(r) = \sqrt{2m(E - V(r))} \quad ? \tag{27}$$

Considering the fact that it is the real part that furnishes the oscillatory contribution to the action when the kinetic energy is positive, the most logical choice would seem to be $\mathrm{Re}[p(r)]$. We will return to this question shortly.

Relation to the WKB approximation

Neglecting the terms coupling $\psi_\pm$ in the bipolar equations, the latter reduce to the standard WKB equations,

$$\psi_{WKB\pm}(r)' = \left(\pm\frac{ip}{\hbar} - \frac{p'}{2p}\right)\psi_{WKB\pm}(r), \tag{28}$$

when we make the Langer modification to the centrifugal potential in the Hamiltonian [58], taking

$$\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} \to \frac{\hbar^2}{2m}\frac{(l+1/2)^2}{r^2}. \tag{29}$$

The solutions to these equations solutions can be written as

$$\psi_{WKB\pm}(r) = \frac{A_\pm}{\sqrt{p(r)}}\exp\left[\pm\frac{i}{\hbar}S(r)\right], \tag{30}$$

with

$$S(r) = \int_{r_0}^{r} p(r')dr'. \tag{31}$$

Since the terms neglected in the WKB equations are those that produce reflection of one component to the other, the WKB solutions are the reflectionless counterpart to the exact bipolar solutions.

The principal difficulty with the WKB solutions is determination of the constant coefficients $A_\pm$, which, strictly, should not be constants. This is often done by analyzing the values expected near a turning point, where the real part of $p(r)^2$ changes sign and p'/p has a peak. However, the imaginary contribution to the momentum $p(r)$ smooths this transition, making it less important when compared to neighboring points, even when the energy is high. Other regions of space can also introduce large variations in $A_\pm$, depending on the local variations in the potentials. In any event, a fairly rigorous, 'refined WKB' theory has been developed, which computes the first-order contribution to the reflection (and thereby changes to $A_\pm$) through the use of 'Stokes' and 'anti-Stokes' lines in the complex plane, emanating from real- and complex-valued turning points [23, 24, 56, 58–60]. The refined WKB approach is still an approximation, however.

The Wigner transform [69] of the bipolar solutions furnishes additional information about their physical content. For simplicity, we analyze their WKB approximations in a 'semiclassical Wigner'-type context [61, 65], calculating

$$F_{WKB\pm}(R, P) = \int_{-\infty}^{\infty} ds\, e^{-iPs} \times \psi_{WKB\pm}(R + s/2)\psi^*_{WKB\pm}(R - s/2). \tag{32}$$

Expanding the wave functions to first order in s before integrating, we find

$$F_{WKB\pm}(R, P) = 2\pi\frac{|A_\pm|^2}{|p(R)|}\exp\left[\mp\frac{2}{\hbar}\mathrm{Im}[S(R)]\right] \times \delta\left(P \mp \mathrm{Re}[p(R)] - \mathrm{Im}\left[\frac{\hbar\, p(R)'}{2p(R)}\right]\right). \tag{33}$$

Consistent with our discussion above, we find that the momentum is determined principally by the real part of $p(R)$, with the incoming/outgoing solutions corresponding to incoming and outgoing trajectories in the Wigner transform. However, here the momentum is modified by the imaginary part of the logarithmic variation in the momentum, which is nonzero only when an absorptive potential is included. Numerically, we have found the contribution of this additional factor to be quite small, except in the region of a turning point, where $\mathrm{Re}[p(r_t)^2] = 0$.

The Wigner function also contains an attenuation factor $\exp\left[\mp\frac{2}{\hbar}\mathrm{Im}[S(R)]\right]$, when the potential is absorptive. When the magnitude of the imaginary potential is small compared to the kinetic energy, we can approximate it as

$$\exp\left[\mp\frac{2}{\hbar}\mathrm{Im}[S(R)]\right] \approx \exp\left[\pm\frac{2}{\hbar}\int_{R_0}^{R}\frac{m}{p_0(r)}W(r)dr\right]$$
$$\approx \exp\left[\frac{2}{\hbar}\int_{t(R_0)}^{t(R)}W(r(t))dt\right], \quad (34)$$

where we have used the trajectory to convert the integral in r to one in t by taking $\pm m\, dr/p_0 = \pm\, dr/v_0 = dt$. Note that in this expression, the association between dr and dt depends on the direction of propagation. Taking this into account, we can associate the attenuation factor with the absorption term in the continuity equation, equation (8).

Numerical results

Here we will analyze solutions to the bipolar equations for protons incident on the nucleus ^{56}Fe. We use a fairly standard form for the real part of the potential, consisting of centrifugal term and a Woods-Saxon nuclear potential, together with a Coulomb potential modified to take into account a constant charge density in the nuclear interior. For the nuclear potential, we adopt the phenomenological optical potential of Becchetti and Greenlees [70]. We have

$$V(r) = V_l(r) + V_N(r) + V_C(r), \quad (35)$$

where the centrifugal potential is,

$$V_l(r) = \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}, \quad (36)$$

the real nuclear potential is given by

$$V_N(r) = \frac{V_0}{1+\exp[(r-R_V)/a_V]}, \quad (37)$$

with a radius of

$$R_V = r_V A_T^{1/3} \quad (38)$$

and the Coulomb potential is

$$V_C(R) = \begin{cases} \frac{Z_P Z_T e^2}{2R_C}\left(3-\left(\frac{r}{R_C}\right)^2\right) & r < R_C \\ \frac{Z_P Z_T e^2}{r} & r \geqslant R_C \end{cases}, \quad (39)$$

with a charge density radius of

$$R_C = r_C A_T^{1/3}, \quad (40)$$

where Z_P and Z_T are the charge numbers of the projectile and target, respectively, and A_T is to the mass number of the target.

The imaginary part of the potential is normally composed of a Woods-Saxon plus a Woods-Saxon derivative term, commonly called volume and surface terms,

$$W(r) = W_v(r) + W_s(r), \quad (41)$$

where the volume term is

$$W_v(r) = \frac{W_{v0}}{1+\exp[(r-R_{W_v})/a_{W_v}]}, \quad (42)$$

with a radius of

$$R_{W_v} = r_{W_v} A_T^{1/3} \quad (43)$$

and the surface term is given by

$$W_s(r) = \frac{4W_{s0}\exp[(r-R_{W_s})/a_{W_s}]}{(1+\exp[(r-R_{W_s})/a_{W_s}])^2}, \quad (44)$$

with

$$R_{W_s} = r_{W_s} A_T^{1/3}. \quad (45)$$

The surface term of the imaginary potential is usually associated with absorption due to coupling to low-energy collective excitations of the nucleus, while the volume term is associated with higher energy single-particle excitations. The surface term is normally dominant at low incident energy, while the volume mode grows in importance as the energy increases. In a phenomenological potential such as the Becchetti-Greenlees one, the reduced radii of each of the terms in the potential, r_V, r_C, r_{W_v} and r_{W_s}, as well as the diffuseness parameters, a_V, a_{W_v} and a_{W_s}, and the potential strengths, V_0, W_{v0} and W_{s0}, are obtained by fitting to experimental data.

To solve the bipolar equations, we integrate outward from $r = 0$ and match to Coulomb wave functions (or to spherical Bessel functions, when the product of the charges is zero) at a radius R_m at which the nuclear potential $V_N + iW = 0$. At the matching radius, taken to be $R_m = 14.6$ fm for the system studied here, we define the internal wave function normalization A_l and the S-matrix S_l by requiring, for each value of the angular momentum l, that

$$A_l\psi(R_m) = \frac{i}{2}(H_l^{(-)}(R_m) - S_l H_l^{(+)}(R_m))$$
$$A_l\psi'(R_m) = \frac{i}{2}(H_l^{(-)\prime}(R_m) - S_l H_l^{(+)\prime}(R_m)), \tag{46}$$

where

$$\psi(R_m) = \psi_+(R_m) + \psi_-(R_m) \tag{47}$$

is the wave function obtained by integrating the coupled equations from $r = 0$ to $r = R_m$ and $H_l^{(\pm)}$ are the outgoing/incoming Coulomb/spherical Bessel wave functions.

To integrate the equations, we begin by requiring that the solution be regular at $r = 0$. For the case of $l = 0$, we take

$$\begin{pmatrix}\psi_+(0)\\ \psi_-(0)\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix} \tag{48}$$

and, at the first integration point $r = h$,

$$\begin{pmatrix}\psi_+(h)\\ \psi_-(h)\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix} + \begin{pmatrix}-\frac{p'(h)}{2p(h)} + \frac{ip(h)}{\hbar} & \frac{p'(h)}{2p(h)}\\ \frac{p'(h)}{2p(h)} & -\frac{p'(h)}{2p(h)} - \frac{ip(h)}{\hbar}\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}$$
$$= \begin{pmatrix}1 + \frac{ip(h)}{\hbar} - \frac{p'(h)}{p(h)}\\ -1 + \frac{ip(h)}{\hbar} + \frac{p'(h)}{p(h)}\end{pmatrix}. \tag{49}$$

For $l > 0$, when $r \to 0$, we have

$$\frac{p(r)}{\hbar} \to i\frac{\sqrt{l(l+1)}}{r} \quad \text{and} \quad \frac{p'(r)}{2p(r)} \to -\frac{1}{2r}. \tag{50}$$

In this limit, the coupled equations become

$$\frac{d}{dr}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = \frac{1}{2r}\begin{pmatrix}1 - 2\sqrt{l(l+1)} & -1\\ -1 & 1 + 2\sqrt{l(l+1)}\end{pmatrix}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}, \tag{51}$$

with eigenvalues

$$\lambda = 2l + 2 \quad \text{or} \quad -2l, \tag{52}$$

furnishing solutions of the form

$$\begin{pmatrix}\psi_+(r)\\ \psi_-(r)\end{pmatrix} = A\begin{pmatrix}-1/(1 + 2l + 2\sqrt{l(l+1)})\\ 1\end{pmatrix}r^{l+1}$$
$$+ B\begin{pmatrix}1\\ 1/(1 + 2l + 2\sqrt{l(l+1)})\end{pmatrix}\frac{1}{r^l}. \tag{53}$$

For the solution to be regular, we must have $B = 0$.

To take into account the contributions of other terms in the momentum at a finite value of $r = h$, for $l > 0$, we take the more general form of the eigenvector corresponding to the positive eigenvalue,

$$\begin{pmatrix}\psi_+(h)\\ \psi_-(h)\end{pmatrix} = \begin{pmatrix}\frac{ip'(h)/2p(h)}{p(h)/\hbar + \sqrt{(p(h)/\hbar)^2 - (p'(h)/2p(h))^2}}\\ 1\end{pmatrix}. \tag{54}$$

We have also tested and compared solutions by solving the equations for ψ and ψ_a. More explicitly, we solved for ψ and ψ' and then determined ψ_a and $\psi_\pm$. The matching condition at the radius R_m is the same. At small r, we use the fact that for angular momentum l, we have $\psi(r) \approx Ar^{l+1}$, to take

$$\begin{pmatrix} \psi(h) \\ \psi(h)' \end{pmatrix} = \begin{pmatrix} h \\ l+1 \end{pmatrix} \tag{55}$$

We have solved both sets of equations using an adaptive stepsize Runge-Kutta algorithm and obtained identical results from the two sets of equations for the incoming/outgoing waves $\psi_\pm$ and the total wave function ψ. We have also compared our results to those of a standard optical scattering code using a modified Numerov method to solve the standard Schrödinger scattering problem, theSCAT2 code [71]. We have made comparisons for the scattering of neutrons, protons and alpha particles incident on ^{56}Fe at energies between 10MeV and 100 MeV, using the Becchetti-Greenless phenomenological optical potentials for neutrons and protons [70] and the McFadden-Satchler potential for alpha particles [72]. We obtain agreement between our results and those of SCAT2 for the S-matrix elements and the cross sections of better than one part in 10^5.

We have also solved the WKB equations in the same manner, integrating from $r = 0$, but taking into account the Langer modification of the centrifugal potential. In the limit $r \to 0$, we find for all values of the angular momentum l,

$$\begin{pmatrix} \psi_{WKB+}(r) \\ \psi_{WKB-}(r) \end{pmatrix} = \begin{pmatrix} B/r^l \\ Ar^{l+1} \end{pmatrix}, \tag{56}$$

which requires that $B = 0$ for the solution to be regular. The outgoing WKB wave ψ_{WKB+} is thus identically zero in the region inside the turning point at $\mathrm{Re}[p(r_t)^2] = 0$. To continue the integration into the oscillatory region, just outside the turning point, at r_{t+}, we set

$$\psi_{WKB-}(r_{t+}) = \psi_{WKB-}(r_{t-})/\sqrt{2} \tag{57}$$

and

$$\psi_{WKB+}(r_{t+}) = -i\psi_{WKB-}(r_{t-})/\sqrt{2}\,, \tag{58}$$

in order to reproduce the usual WKB matching conditions at the turning point[58]. We note that we have also tested this expression without the factor of $\sqrt{2}$. Given the fact that we normalize the solutions at the matching radius, the only effect of this change would be to reduce the WKB wave function by a factor of $\sqrt{2}$ in the region interior to the turning point.

We have found that, outside the range of the nuclear potential, the inward/outward bipolar solutions oscillate in phase with the Coulomb/spherical Bessel solutions to the Schrödinger equation. However, the bipolar solutions, which are not solutions to the Schrödinger equation, maintain magnitudes more consistent with the semiclassical WKB approximation than to those of the incoming/outgoing solutions to the Schrödinger equation.

In figure 1, we compare the behavior with and without absorption of the exact incoming and outgoing waves ψ_+ and ψ_-, (solid green and red curves, respectively) as well as their WKB approximations (dashed green and red curves, respectively) for the values of the angular momentum, $l = 0, 3, 5$ for a proton incident on ^{56}Fe at an energy of 10 MeV. Rescaled real and imaginary parts of the potential (solid and dashed black curves, respectively) are also shown in order to correlate the variations in the wave functions with those of the potentials.

Several general features of the results are clear at a glance. In the case of no absorption, outside the turning point, the magnitudes of the exact incoming and outgoing waves are identical, as are those of the WKB solutions. In contrast, when absorption occurs outside the turning point, the magnitude of the incoming wave is always larger than that of the outgoing one, both for the exact and for the WKB solutions. This is a direct result of the absorption. Inside the turning point, the incoming WKB solution is identical to the total WKB one, as the outgoing WKB solution is zero in this region. In this region, the exact incoming wave dominates its outgoing component, but is smaller than the WKB incoming wave, in part due to reflection to the outgoing wave. The exact outgoing solution inside the turning point is nonzero only as a result of reflection from the rapidly changing potential, although it can also suffer absorption.

For $l = 0$, both exact incoming and outgoing waves suffer strong absorption as the pass through the imaginary potential. The absorption effectively cancels the effects of the reflection from the surface of the real potential seen in the nonabsorptive scattering, when compared to the nonabsorptive WKB solution. With absorption, the incoming WKB wave closely follows the exact one, while the outgoing WKB wave deviates substantially from the exact one. The relatively constant magnitude of the outgoing WKB component is due to compensation of the attenuation by a decrease of the $1/\sqrt{p(r)}$ flux factor as the outgoing component leaves the potential well. The summed wave functions differ both in magnitude and phase as a result.

For $l = 3$, the real potential dominates the scattering, with or without absorption, with both the incoming and outgoing components varying significantly in the region of the Coulomb barrier. The incoming exact and WKB components again lie fairly close to one another in the absorptive case, both decreasing strongly in the absorptive region. The outgoing components again differ greatly in this case. The increase in the exact outgoing component ψ_+ is due to strong reflection from the Coulomb barrier, before the absorptive region is reached,

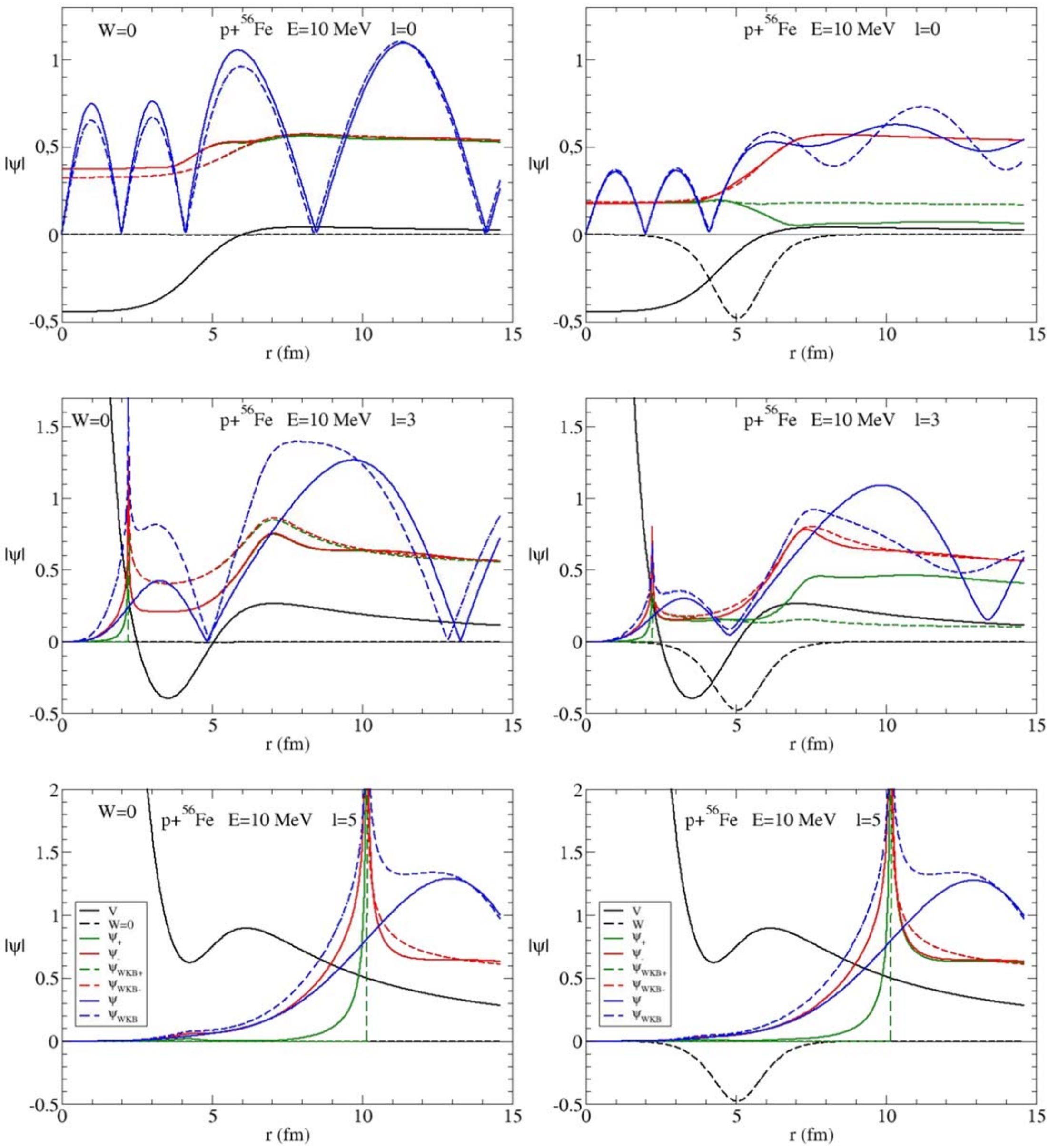

Figure 1. Incoming/outgoing waves and total wave functions for protons incident on ^{56}Fe at 10 MeV for the partial waves l = 0, 3, and 5. The left column corresponds to scattering with no absorption while absorption is included in the calculations shown in the right column. Exact results are shown as solid lines and WKB approximations to these as dashed lines. The incoming waves are given in red and the outgoing waves in green. The total exact and WKB wave functions are shown in blue. The real potential (solid black line) and the imaginary potential (dashed black line) are included to show their variation and their range of action but are not to scale.

while the WKB outgoing wave displays a similar compensation of attenuation by variation in the flux factor as seen at $l = 0$. Note that both with and without absorption, the exact and the WKB incoming/outgoing waves have a strong peak at the classical turning point, where $\mathrm{Re}[p(r_t)^2] = 0$. The contributions of the exact ψ_+ and ψ_- are equal and opposite at the peak and cancel in the total wave function.

For $l = 5$, the turning point is outside the range of the absorptive potential and the scattering is dominated by reflection from the barrier. Here the solutions with and without absorption are almost identical. The exact incoming and outgoing solutions lie atop one another, as do the WKB solutions.

In figure 2, we show similar results for the $l = 0, 5, 10$ partial wave in the case of a proton incident on ^{56}Fe at an energy of 30 MeV. Here, the additional kinetic energy greatly reduces the effects of the variations in the real potential. For $l = 0$, the exact and WKB incoming and outgoing waves lie close to one another at all radii, attesting to the limited importance of reflection in the scattering. The variation in magnitude of the incoming/outgoing waves in the case of no absorption is due to the variation in the flux factor $1/\sqrt{p(r)}$. In the case of an absorptive potential, both the incoming and outgoing waves suffer significant attenuation. The behavior of the incoming and outgoing waves with $l = 5$ is similar to those for $l = 0$, although some reflection can be observed in the slight bumps in the exact outgoing wave with absorption, as well as the solutions without absorption, in the

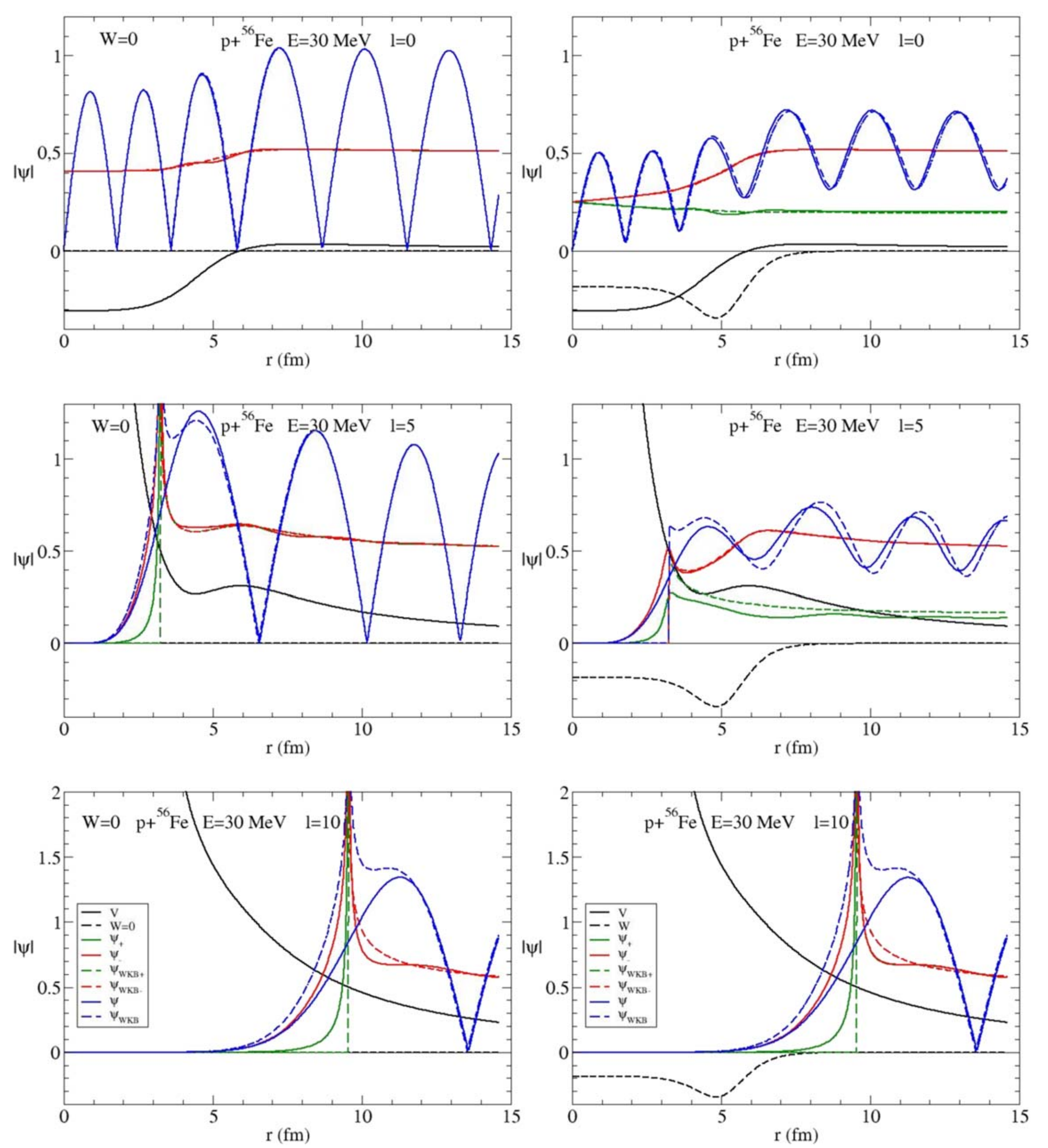

Figure 2. Incoming/outgoing waves and total wave functions for protons incident on ^{56}Fe at 30 MeV for the partial waves $l = 0, 5$, and 10. The left column corresponds to scattering with no absorption while absorption is included in the calculations shown in the right column. Exact results are shown as solid lines and WKB approximations to these as dashed lines. The incoming waves are given in red and the outgoing waves in green. The total exact and WKB wave functions are shown in blue. The real potential (solid black line) and the imaginary potential (dashed black line) are included to show their variation and their range of action but are not to scale.

region of the Coulomb barrier. Finally, at $l = 10$, the scattering is again dominated by the turning point, which lies outside the absorptive region. The behavior of the incoming and outgoing solutions is similar to that observed for $l = 5$ at 10 MeV and is the same with or without absorption. As also seen at 10 MeV for the analogous peripheral partial wave, the exact solutions approach the turning point more abruptly than the WKB ones.

Conclusions

We have studied the effects of absorption on the incoming and outgoing solutions to the bipolar equations, as well as on their WKB counterparts. As we have seen, through the Wigner transform, the bipolar incoming/outgoing waves can be interpreted in terms of incoming and outgoing trajectories.

As discussed in the Numerical Results section above, several trivial observations can be made immediately concerning our results. In the case of no absorption, outside the turning point, the exact incoming and outgoing waves are equal in magnitude, as are the WKB solutions. In contrast, when absorption occurs outside the turning point, the magnitude of the incoming wave is always larger than the outgoing one, both for the exact and for the WKB solutions. This is simply a result of the absorption. Inside the turning point, the incoming WKB solution is identical to the total WKB one, as the outgoing WKB solution is zero in this region. In this region, the exact incoming wave dominates its outgoing component, but is smaller than the WKB incoming wave, in part due to reflection to the outgoing wave. The exact outgoing solution inside the turning point is nonzero only as a result of reflection from the rapidly changing potential, although it can suffer absorption as well.

A comparison of the incoming bipolar solution and its WKB approximation, which neglects reflection, show that absorption tends to diminish the importance of reflection on the incoming wave, as the exact incoming wave and the WKB solution are, in general, quite similar in this case. This contrasts with the behavior of the outgoing wave, where reflection can produce large differences between the exact solution and the WKB one at low incident energy. As a result, at low energy, the WKB approximation succeeds in providing a better approximation to the solution without absorption than to the more physical solution with absorption. The discrepancies between the two decrease fairly quickly with energy, as can be seen in the comparison at 30 MeV.

Both the exact incoming/outgoing solutions and their WKB counterparts have sharp peaks in magnitude at a turning point. The peak is smoothed somewhat when the turning point occurs in the absorptive region, but does not disappear. The large contributions to the exact bipolar waves are almost equal and opposite in sign, as their sum furnishes the total wave function, which, as can be seen in the figures, is smooth in the region of the turning point.

Singularities (or near-singularities) in the vicinity of turning points (and more generally, caustics) are a notorious feature of semiclassical methods, and certainly nothing new. Various 'tricks' have been developed for dealing with them [56, 58–60], such as the method of comparison equations, used to obtain connection formulae. Expanding our scope a bit more broadly, the Fröman approach [56] yields a WKB-like approximation designed to work in conjunction with *arbitrary* trajectories, i.e. not necessarily classical trajectories. In this formalism, trajectories may easily be chosen that exhibit no turning points—and therefore no sharp features. Moreover, an exact, bipolar version has also been developed, that can easily incorporate even deep tunneling [44].

We have shown how a trajectory interpretation of the incoming and outgoing waves can be made evident through the Wigner transform. However, we have based this interpretation on the phase of the WKB approximation rather than on that of the exact bipolar waves. A comparison of the oscillatory behavior of the exact and WKB wave functions permits us to conclude that the local momentum obtained from the derivative of the phase factor of the exact solution will furnish a momentum very similar to that of the WKB solution. We thus consider the trajectory interpretation to be just as valid for the exact solution as for the WKB ones.

Considering the numerical methods used here, we conclude that the modified Numerov method is faster for solving the Schrödinger equation than the adaptive stepsize Runge-Kutta method but is not as precise. Solving for the wave function is more precise and faster than solving for the bipolar waves because of the turning points. Outside the turning points, the bipolar solutions are more stable. The advantage of the bipolar waves is their connection to classical physics and the deeper insight this furnishes into the dynamics. In any case, for the equations discussed here, all solutions were obtained at the stroke of a key.

Going forward, it is of course important to generalize the bipolar theory for multidimensional applications. To this end, there are two primary challenges that have been previously identified [47, 51]. The first is ensuring that the number of separate wave function components remains at just two (or at least some small number) and does not grow exponentially with d, the number of dimensions. In particular, bifurcating along each dimension separately would lead to 2^d separate multipolar components to content with. The second challenge is ensuring that both (or all) of the bipolar (or multipolar) components are themselves fairly smooth and interference-free. Satisfying both conditions at once is indeed nontrivial.

In the context of reactive scattering in chemical physics, substantial progress has been made by recognizing that within the space of internal coordinates, one dimension in particular—the 'reaction coordinate', describing the overall progress from reactant to product molecules—can be singled out as special [60]. Thus, previous multidimensional bipolar approaches [47, 51] have exploited this situation by bifurcating ψ *only* along the reaction coordinate—which can be done despite this coordinate being highly curvilinear, as is typical.

Of course, many scattering applications, including those in nuclear physics, require a more general treatment, not limited to internal coordinates with a single primary reaction coordinate. Even for the simplest cases where one or both colliding partners are treated as point particles (e.g., nucleons), unless central forces are in play and exploited to reduce the problem to 1D as is the case in this paper, the above reaction coordinate strategy will not be applicable. Thus, other ideas are needed.

One such idea for generalizing the bipolar equations to three Cartesian dimensions is naturally suggested by the present work. In particular, the association of the bipolar components with a wave function and its derivative suggests the following:

$$\frac{\hbar}{i}\vec{\nabla}\psi(\vec{r}) = \vec{p}(\vec{r})\psi_a(\vec{r});$$
$$\frac{\hbar}{i}\vec{\nabla}\cdot(\vec{p}(\vec{r})\psi_a(\vec{r})) = p(\vec{r})^2\psi(\vec{r}). \tag{59}$$

We can then write

$$\psi_{\pm}(\vec{r}) = \frac{1}{2}\left(\psi(\vec{r}) \pm \frac{1}{p(\vec{r})^2}\vec{p}(\vec{r})\cdot\frac{\hbar}{i}\vec{\nabla}\psi(\vec{r})\right). \tag{60}$$

Note that equation (60) was derived previously [47], except with $\vec{p}$ an arbitrary vector momentum field, chosen to correspond to the aforementioned reaction coordinate. Here, we propose to investigate other choices for $\vec{p}(r)$, better suited to more general scattering situations. Although these are four equations rather than two, our results here suggest that they could be worth the time and effort necessary to understand them better. We plan to work in this direction in the future.

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Data availability statement

No new data were created or analysed in this study.

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5. TÍTULO E SUBTÍTULO:
Clifford Algebras and Multi-Particle Spinors

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8. PALAVRAS-CHAVE SUGERIDAS PELO AUTOR:
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11. RESUMO:
Clifford, or geometric, algebras are introduced by presenting important particular cases. The introduction to the geometric algebra of the three-dimensional Euclidean space and the geometric algebra of spacetime shows how these algebras provide a synthetic and efficient way to describe geometric objects and rotations in three-dimensional Euclidean space and Minkowski spacetime, respectively. It is shown how the former algebra is included in the later, and how this algebra provides an elegant way to describe Lorentz transformations, the electromagnetic field and Maxwell's equations. The emergence of these algebras in the quantum mechanics of spin-1/2 particles is outlined, and a systematic study of Pauli and Dirac spinors is performed by transforming from the classical to the algebraic description of the spinors, which leads naturally to the operator definition of such spinors. These transformations are developed systematically for the first time in this work. At this point, the transformations are applied to obtain the corresponding versions of the Pauli and Dirac equations. The corresponding transformations for the adjoint spinors are also obtained and applied to express inner products and observables. This study concerning a single spin-1/2 particle is then extended to the context of systems of multiple spin-1/2 particles. In this new study, the Clifford algebra appropriate for description of non-relativistic multi-particle spinors is found to be identical to the so-called multi-particle spacetime algebra, introduced less formally in previous studies. Multi-particle algebraic and operator Pauli spinors are then defined for the first time, starting from the classical ones, in an analogous manner to the single-particle case. In order to properly define relativistic multi-particle spinors, the extension of the Dirac algebra from the usual complex algebra of Minkowski spacetime to a six-dimensional conformal space algebra is found to be necessary. In terms of this algebra, an extension of the algebra of operators to a Clifford algebra is performed, and multi-particle algebraic and operator Dirac spinors are defined for the first time, in terms of this extended algebra. Finally, the algebraic and operator versions of the Bethe-Salpeter equation are obtained. The different versions of spinors and their corresponding wave equations raise the possibility that the simpler operator versions could be more fundamental than the classical ones.

12. GRAU DE SIGILO:
(X) OSTENSIVO ( ) **RESERVADO** ( ) **SECRETO**