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EXACT SOLUTIONS IN EXTENDED GRAVITY: COSMOLOGY AND WORMHOLES

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To my daughter Cecília. May you see in the future much of what we dream together for the world.

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"O meu interesse pela ciência sempre se limitou essencialmente ao estudo dos princípios. O fato de ter publicado tão pouco se deve a essa mesma circunstância, pois a grande necessidade de entender os princípios fez com que eu passasse a maior parte do meu tempo em buscas infrutíferas."

— Albert Einstein

Resumo

O já estabelecido e bem-sucedido modelo padrão da Cosmologia (Modelo ΛCDM), cujas equações de campo derivam da Relatividade Geral, descreve relativamente bem o universo como sendo isotrópico e homogêneo em sua distribuição de matéria e energia. Algumas explicações físicas consistentes, porém, parecem estar longe de serem explicadas apenas por esse modelo, como os 96% que não são matéria bariônica, mas uma outra forma de energia e matéria que compõem o universo (Matéria e Energia Escuras). A Relatividade Geral de Einstein também se provou adequada em pequenas escalas ao explicar o avanço do periélio da órbita de Mercúrio em torno do Sol como também o desvio da luz e na Astrofísica Relatívistica ao descrever as estrelas de nêutrons e prevendo a existência de buracos negros. Nesta tese vamos explorar extensões da teoria da Relatividade Geral : o Modelo de Matéria Induzida – ou Modelo STM (sigla em inglês para Space-Time-Matter Model) e a teoria de gravidade $f(R,T)$ conservada, onde R é o escalar de curvatura de Ricci e T o traço do tensor energia-momentum. Aplicamos o modelo STM à Cosmologia e obtemos uma equação de estado única para as três eras do universo (radiação, matéria e energia escura). Esse modelo apresenta o nosso universo, 4-dimensional, e toda matéria existente nele, como uma manifestação geométrica na superfície de um vácuo espaço-temporal em 5 dimensões, com uma energia associada a este vácuo. Todos os parâmetros cosmológicos foram analisados e comparados com os dados observacionais do modelo ΛCDM. Também analisamos buracos de minhoca atravessáveis. Esse estudo foi feito à luz da teoria $f(R,T)$ conservada, ou seja, impondo-se a conservação do tensor energia-momentum na teoria. Neste segundo trabalho analisamos parâmetros que levam à obediência das condições de energia no interior do buraco de minhoca sem a necessidade de ser preenchido com matéria exótica, como ocorre se consideramos as soluções previstas pela Relatividade Geral, como condição para que sejam atravessáveis.

Abstract

The already established and successful standard model of Cosmology, the ΛCDM Model, (an acronym for Lambda Cold Dark Matter Model), whose field equations are derived from General Relativity, describes very well the universe as being isotropic and homogeneous in its distribution of matter and energy. Some solid physical explanations, however, seem to be far from being explained by this model alone, such as the 96% of energy and matter that fills the universe (Dark Matter and Dark Energy). Einstein's General Relativity has also proved adequate on small scales in explaining the advance of the perihelion of Mercury's orbit around the Sun as well as the bending of light, and the Relative Astrophysics in describing neutron stars and predicting the existence of black holes. In this thesis, we are going to explore extensions of the theory of General Relativity: the Induced Matter Model – or STM Model (Space-Time-Matter Model), and the $f(R,T)$ conserved theory of gravity, where R is Ricci´s scalar of curvature and T the trace of the energy-momentum tensor. We applied the STM model to Cosmology and obtained a unique equation of state for the three eras of the universe (radiation, matter, and dark energy). This model presents our universe, 4-dimensional, and all the matter in it, as a geometric manifestation on the surface of a 5-dimensional space-time vacuum, with the energy associated with that vacuum. All cosmological parameters were analyzed and compared with observational data from the ΛCDM model. We also analyzed traversable wormholes. This study was carried out in the light of the conserved $f(R,T)$ theory, that is, imposing the conservation of the energy-momentum tensor in the theory. In this second work, we analyze parameters that satisfied energy conditions inside the wormhole without the need to be filled with exotic matter, as occurs as a condition for them to be traversable in the solutions obtained from General Relativity,

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1 Introduction

For more than a century, since it was proposed by Einstein in 1915 (EINSTEIN, 1915), General Relativity has gone through several tests that established it as the standard gravitational theory. The first three of these experimental tests were proposed by Einstein himself still in 1915. They are the anomaly in Mercury perihelion, the gravitational redshift, and the bending of light in the vicinity of strong gravitational fields. (GUIDRY, 2019; D'INVERNO, 1992; RYDEN, 2017). The Hulse-Taylor´s pulsar (PSR 1913+16), a binary system with a pulsar and a neutron star orbiting a common barycenter was discovered in 1974. (HULSE; TAYLOR, 1975). This discovery became a powerful test of predictions for the time behavior as perceived by a distant observer, predicted by General Relativity.

Another prediction of the theory is the existence of gravitational waves. These waves are emitted when masses are accelerated, in Hulse-Taylor´s binary the masses are the pulsar and its companion, or by the coalescence of compact stars, black holes, and black holes and compact stars, to cite some examples. These ripples in space-time were experimentally detected for the first time in 2015 (ABBOTT *et al.*, 2016). Since then more than 50 gravitational-wave events were detected by the Ligo-Virgo Collaboration and most recently, by the Kagra detector. (CASTELVECCHI, 2020). Other tests that legitimate General Relativity as the standard gravity was carried out along the observational discoveries, both in astrophysical and cosmological scales.

However, there are regimes of very intense gravitational fields, such as in the interior of compact stars or structures of galactic/cosmological scales, singularities, in which apparently some deviations can appear thus lacking theoretical models of extended gravity. One of the main characteristics that these alternative models are able to properly describe gravity in these extreme regimes, with the analytical capacity to recover the structure of General Relativity, based on the definition of its own parameters. Thus, as it will be possible to verify, the two extended gravity models used in the applications presented in this thesis, namely the Space-Time-Matter model theory and the conservative $f(R,T)$ gravity have this extensive property for General Relativity.

In special, there are some open questions in Cosmology and Astrophysics that General Relativity itself has some difficulties explaining. In standard Cosmology, the ΛCDM model with a time-dependent scaling factor on the spatial components describes the geometry of space-time in the presence of matter in an expanding universe and its distinct eras in cosmological time. However, this model explains only 5% of the universe we know in the baryonic form but fails in explaining the nature of Λ (dark energy that accelerates the expansion of the Universe), that represents 68% of all energy in the Universe, and the origin of 'CDM', i.e., the Cold Dark Matter which is 27% of Universe energy. These percentages of dark matter and dark energy in the Universe were recently confirmed in 2018 by the Planck Collaboration, after observations made by the Planck satellite. (AGHANIM; et. al., 2018).

Another question related to standard cosmology concerns the equations of state for each of the three stages of the Universe's evolution. Eras of radiation, matter, and the current era of dark energy are described by separate Equations of State (EoS) in the form $p = \omega\rho$, with p being the pressure, ρ the density, and ω the cosmological parameter. This parameter assumes three discrete values along with the cosmological evolution. According to the ΛCDM model (RYDEN, 2017), this ω factor is $\frac{1}{3}$, in radiation era, equal to 0 in matter era and equal to -1 in current dark energy era. In the search for an EoS that is continuous in its analytical form, there are works in which some parameterizations are proposed, but as ansatz, especially for the ω factor. (MAOR *et al.*, 2002; SZYDŁOWSKI *et al.*, 2006; EFSTATHIOU, 1999; NAKAMURA; CHIBA, 1999).

In the Benchmark Model for the Universe (ΛCDM model) the scale factor $a(t)$ has different analytical solutions for each era. The transition from the radiation-dominated phase where $a(t) \propto t^{1/2}$ to the $a(t) \propto t^{2/3}$ matter-dominated era is not so abrupt. Nor is it the later transition from the matter-dominated phase to the lambda-dominated phase of exponential growth. However, to cure this transition behavior of the scale factor it is necessary to find numerical solutions to the Friedmann equation, as described in (RYDEN, 2017).

Obtaining analytic functions for both the ω EoS parameter and the scale factor a is something unusual in the literature. For example, in (LIMA *et al.*, 2013a) the authors propose a cosmological model based on a dynamical vacuum energy density which is dependent on a power series of the Hubble parameter. This inception gives a unified cosmological framework with a more complete description of the Universe's evolution.

Similarly, but using the extended $f(R, T^{\phi})$ gravity model (MORAES, 2016) also presents a complete cosmological scenario including inflationary and radiation-dominated eras in a self-consistent way, describing all the different stages of the dynamics of the universe. Here, R and T^{ϕ} are, respectively the Ricci curvature scalar and the trace of the energy-momentum tensor of a scalar field ϕ.

The question surrounding the widely assumed superiority of analytics solutions on numerical methods has been the object of argumentation mainly on the part of the philosophers of science, as in (ARDOUREL; JEBEILE, 2017).

The central result of this thesis is the proposal of a unique and continuous EoS for the Universe's evolution to describe its three eras. This EoS has been obtained analytically from a cosmological model, the Induced Matter Model (STM), based on field equations induced from a metric in 5 dimensions, in an empty space-time (LAPOLA *et al.*, 2021).

The STM model (Space-Time-Matter Model) was proposed and studied extensively by Paul Wesson and James Overduin (WESSON; OVERDUIN, 2018) - (OVERDUIN; WESSON, 1997) - (OVERDUIN *et al.*, 2013)

This model was the evolution of the initial works by Kaluza-Klein (KK) models (KALUZA, 1921)-(HOHM; SAMTLEBEN, 2013). Kaluza's original idea was to unify the known fundamental forces at that time, namely gravity and electromagnetism. Knowing that Einstein's description of gravity was made in terms of space-time warps and curves, Kaluza had to consider an extra space-like dimension in which electromagnetism could also manifest as "warps" and "curves" in space-time.

As astrophysical and cosmological observations have been progressed over time, KK theory applications were not restricted to the unification of forces. For instance, KK dark energy and dark matter alternatives were recently proposed (REDDY; LAKSHMI, 2015)-(SHARIF; KHANUM, 2011) and (CHENG *et al.*, 2002)-(KONG; MATCHEV, 2006). Even stellar models were constructed in 5D theory. (KARSAI *et al.*, 2016)-(BARNAFÖLDI *et al.*, 2010).

The existence and stability of strange stars in extra dimensions have been investigated by (ARBANIL *et al.*, 2019). Other recent interesting applications of KK 5D theory to cosmology can be seen in (SENGUPTA, 2020)-(MORAES; CORREA, 2019) and experimental tests of KK theory in (MBELEK, 2020). Measurements of geodesic precession from Gravity Probe B experiment constrained possible departures from Einstein's GR for a spinning test body in KK theory (OVERDUIN *et al.*, 2013). Wetterich's parametrization EoS was used to obtain cosmological solutions in a 5D Ricci-flat universe (ZHANG *et al.*, 2006). Some relations for the embedding of spatially flat Friedmann Lemaître Robertson Walker (FLRW) cosmological models in flat 5D manifolds were presented in (SEAHRA; WESSON, 2002). More applications and references of this Space-Time-Matter model are summarized and cited in Section 3.2 of this thesis.

The novelty of our work, presented in Chapter 3 is that by using this model we start from an empty Riemannian fifth-dimensional space-time but with associated constant vacuum energy. In this framework, all the energy and pressure of our 4D Universe appears as a geometric manifestation of the bulk surface of this empty 5D space-time.

In a previous work (MORAES; MIRANDA, 2012), which originated the one presented in this thesis, the authors have already used this model, applying a 5d metric like ours, but looking for a solution only for the time component of the Einstein tensor in order to find a form for the dark energy. We extend this study, including a cosmological constant and investigating the solution of all components of Einstein's equations.

An important result found concerns the scale factor obtained from the solutions of the cosmological field equations. The analytical form found is $a(t) = c_5\sqrt{\sinh\left(\sqrt{\frac{2}{3}|\Lambda|}\,t\right)}$, where $c5$ is a integration constant, the cosmological constant from the fifth dimension and the time t. This scale factor obtained as the solution of the 5-dimensional model is able to describe the main phases of the Universe's evolution.

Another important feature that this work brings is the comparison we make of the cosmological parameters obtained from the model with the data observed experimentally. In the end, comparing our results with the observational data (AGHANIM; et. al., 2018; ALAM *et al.*, 2017), the Hubble parameter obtained fits satisfactorily with the newest experimental measurements.

An alternative gravity theory in 4-dimensional space-time popularized in the second decade of the 2000s is the $f(R,T)$ theory, originally proposed in (HARKO *et al.*, 2011), to correct some incompleteness of $f(R)$ theory, with R being the Ricci curvature scalar, in studies on the galactic scale and the solar system as appointed, for example in (ERICKCEK *et al.*, 2006). Like the $f(R)$ gravity (SOTIRIOU; FARAONI, 2010), $f(R,T)$ theory modifies the Einstein-Hilbert action and includes in the original Lagrangian new terms dependent only on the trace of the energy-momentum tensor T.

Since Harko´s article, the $f(R,T)$ gravity theory has been successfully tested in many areas, as wormhole solutions of spherically symmetric spacetime via Noether symmetry (SHARIF; NAWAZISH, 2019), models corresponding to different relations for the pressure components in wormholes (radial and lateral), and several EoS, reflecting different matter content (ELIZALDE; KHURSHUDYAN, 2019), cosmological scenarios with some modifications in the field equations (TRETYAKOV, 2018), and a non-equilibrium picture of thermodynamics at the apparent horizon of FRW universe (SHARIF; ZUBAIR, 2012).

In the astrophysics context, the hydrostatic equilibrium of neutron stars and white dwarfs in $f(R,T)$ gravity was first investigated by (MORAES *et al.*, 2018) and (CARVALHO *et al.*, 2017). Recently (DEB *et al.*, 2019), also showed that $f(R,T)$ gravity is a good candidate for a suitable theory for explaining the observed massive stellar objects like super-Chandrasekhar stars, and magnetars, which remain difficult to describe at the standard framework of General Relativity unless one significantly changes the structure of the star as, for example, investigated in (LENZI; LUGONES, 2012). In this work, were presented hybrid stars in the General Relativity framework to explain massive pulsars.

It is known that one of the main criticisms of the $f(R,T)$ extended gravity theories is the fact that they are not conservative, i.e., the covariant derivative of the energy-momentum tensor is non-zero, i.e., $\nabla_\mu T^{\mu\nu} \neq 0$, indicating the non-conservation of energy. This question was commented by (BARRIENTOS; RUBILAR, 2014), that correct the conservation equation of the energy-momentum tensor since that in the original development of the theory in (HARKO *et al.*, 2011) the authors left aside an essential term that has consequences in the equation of motion of test particles. In this way, this theory can be actually conservative and is in better agreement with the assumption that all energy in the universe is conserved, without the "creation" of matter. In (ALVARENGA *et al.*, 2013) the authors shows that for a linear equation of state $p = \omega\rho$, for the theory to be conserved the function $f(T)$ needs to have a power-law form in the trace of the energy-momentum tensor. In particular, in a dust-dominated Universe ($\omega = 0$), the exponent is equal to $\frac{1}{2}$.

The implications of a conserved $f(R, T)$ gravity model in astrophysics, in particular for investigating neutron stars, were studied for the first time in (SANTOS *et al.*, 2019). The application for conservative $f(R, T)$ theory in quark stars was done by (CARVALHO *et al.*, 2020). Also in astrophysics, the specific form for the energy-momentum tensor function in conserved theory depends on the equation of state chosen for the neutron and quark star matter. In contrast with previous works of compact stars in the non-conservative version of the theory, these new results of conservative theory show that neutron and quark star masses can change considerably with an increase in stellar radii.

However, so far in the literature, the conserved $f(R, T)$ gravity has not been applied to wormhole metrics, only the non-conserved version. (SHARIF; NAWAZISH, 2019; ELIZALDE; KHURSHUDYAN, 2019; MORAES; SAHOO, 2018). Thus, in a work presented in Chapter 4, we investigate, in the light of the conservative f(R,T) theory, the field equations from a four-dimensional metric of space-time that describes a classical traversable wormhole, as proposed in (MORRIS; THORNE, 1988). Imposing the conservation of the energy-momentum tensor expression, the conservation equation results in a specific form for $f(R, T)$ function, depending on the form of the equation of state used, and the shape function factor assumed in the wormhole metric.

For the case of wormholes, starting with a well-known form for the shape function $b(r) = \frac{r_o}{r}$, where $r = r_0$ is the throat of the wormhole (ELIZALDE; KHURSHUDYAN, 2019). Assuming for both radial and transverse pressure a linear relation with the energy density, that the $f(R, T)$ needs to be linear in T for the theory to be conserved. Notably, $f(R, T) = R + \lambda T$ function. In this function, R the Ricci curvature scalar, λ is a constant parameter, and T is the trace of the energy-momentum tensor.

This particular linear form for $f(R, T)$ is the only one possible with an EoS linear in the energy density, for the theory to be conserved when applied to wormholes. In the sequence, we obtained the conditions for the wormhole to be traversable without the need for its throat to be filled with exotic matter.

This present work is organized as follows. In Chapter 2 there is a brief summary of the Einstein field equations for General Relativity in 4 dimensions and presents the Space-Time-Matter Model. Chapter 3, the principal results of the thesis, we reproduced our article entitled "Induced equation of state for the universe epochs constrained by the Hubble parameter" (LAPOLA *et al.*, 2021). A copy of this article is also included at the end of the thesis. Chapter 4 discuss for the first time the results of a wormhole model in the conservative $f(R,T)$ gravity with both radial and transverse pressure depending linearly on the energy density, and in Chapter 5 we present our main conclusions.

2 Space-Time-Matter

2.1 Matter as a geometric manifestation of a 5D manifold

2.1.1 Field Equations of General Relativity

The matter as energy changes spacetime geometry, and its existence curves the surrounding spacetime. This is the essence of Einstein's General Relativity (GR), originally formulated by him in 1915 (EINSTEIN, 1915) As a preamble let us remember, the original field equations given by the Einstein equation from General Relativity. The basic role of GR field Einstein equations is to calculate the space-time geometry around a gravitational field source, finding an expression for the metric tensor $g_{\mu\nu}$ (the indices μ and ν range from 0 to 3).

The Einstein equation is given by

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.1}$$

where the Einstein tensor is $G_{\mu\nu}$ and Λ is the cosmological constant. The Einstein tensor is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{2.2}$$

In Eq. (2.2) the Einstein tensor is written in terms of Ricci tensor $R_{\mu\nu}$ and the Ricci scalar curvature R. At the right-hand side of the Eq (2.1), G is the Newtonian gravitational constant, c is the speed of light and $T_{\mu\nu}$ is the energy-momentum tensor that describes the energy density and pressure in a certain locality in space-time. In all subsequent chapters and sections, we use the geometrized unit system $G = c = 1$. Taking the trace on both sides of the Eq. (2.1), i.e. multiplying each term by $g^{\mu\nu}$, remembering that in this 4-dimensional space-time we have $g^{\mu\nu} g_{\mu\nu} = 4$. After this operation, one multiplies the result of the trace of Eq.(2.1) by $1/2 g_{\mu\nu}$, and finally subtracting this new result from Eq. (2.1), one obtains another form to express the Einstein Equation as

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \tag{2.3}$$

.

In cosmology, for the case of our Universe, or even in astrophysics, for a compact star, for example, matter can be described as a perfect fluid, where $T = diag(\rho, -p, -p, -p)$. In this expression, ρ is the energy density, and p is the pressure of the universe or the star, respectively. For the case of wormholes, where we have anisotropy in the pressure as we will see, we have, still in the general relativity framework that $T = diag(\rho, -p_r, -p_t, -p_t)$, being ρ the energy density and p_r and p_t the radial and tangential pressures, respectively.

2.1.2 Space-Time-Matter Theory

Also called the Induced Matter Model, the Space-Time-Matter theory with five dimensions is primarily based on Einstein's idea of general relativity with 4 dimensions. General relativity is a theory that can reasonably be called complete as a theory of gravity in the four dimensions of space-time. But, analyzing the physics, it is possible to see that the gravitational field and their matter source are not to be regarded as separate things, but unified. In the same way, a space-time that presents curvature in a certain location indicates the presence of matter, i.e., the curvature is matter, and vice-versa. Or in the words of John A. Wheeler: "Space-time tells matter how to move. Matter tells space-time how to curve" (THORNE *et al.*, 2000). As Einstein's field equations can also be used in any number of dimensions, we can generalize GR increasing the space-time dimensions, in particular for 5D. The primal idea that the Universe may have more than four dimensions was introduced by Theodor Kaluza, who in 1921 realized that a 5D manifold could be used to unify Einstein's theory of general relativity with Maxwell's theory of electromagnetism (KALUZA, 1921). After this, Einstein endorsed the idea. In 1926 Oskar Klein proposed the connection to quantum theory by assuming that the extra dimension was microscopically small, with a size in fact connected via Planck's constant to the magnitude of the electron charge. (KLEIN, 1926). After these inaugural works, other studies related to the 5 dimensions were published over time, as mentioned in chapter 4. At the final of the 1980s and at beginning of the 1990s a series of papers and results using the called Space-Time-Matter (STM) theory took place (KALUZA, 1921; WESSON, 1985; WESSON, 1986; WESSON, 1992; SUNDRUM, 1999; WESSON; et al., 1996; LEON, 1988). This STM theory is based on a 5D Riemannian manifold, where the extra dimension intends to induce matter into 4D space-time (WESSON; OVERDUIN, 2018). This happens because the seemingly empty 5D field equations $R_{AB} = 0$, where R_{AB} is the 5D Ricci tensor, contains the 4D Einstein's equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ with an effective energy-momentum tensor that depends on derivatives of the metric coefficients with respect to the extra coordinate $x^4 = l$. This result is based on Campbell's embedding theorem of differential geometry (CAMPBELL, 1926). Briefly, Campbell‘s theorem says that any analytic Riemannian space with n dimensions is embedded in a Riemannian space with $n+1$ dimensions. Here, n is the dimension of the space. In this work, we are using this theorem in a 4D Riemannian manifold that is embedded in a 5D space, i.e., $n = 4$. Then this implies that the 4D equations of General Relativity $G_{\mu\nu} = 8\pi T_{\mu\nu}$ are embedded in a system of 5D equations like $G_{AB} = 0$, or still with a cosmological term, i.e.,$G_{AB} = \Lambda$. STM theory gives physical meaning to this theorem. If we take one of the components of the equation

$$G_{AB} = 0, \tag{2.4}$$

with $A, B = 0, 1, 2, 3, 4$, write out all of its components, it is always possible to identify them into those which do not involve the extra dimension and those that depend on it. Putting these on the left and right sides of an equals sign produces a relation between the field and matter. This method was applied using a Friedman-Lemaitre-Robertson-Walker metric tensor for the standard Cosmology, with an extra dimension and a scale factor as its fifth coordinate. So is possible to see in detail how the STM theory works and the results fit very well with observational data. Then, it will be possible to see in detail and in practice a complete 4D cosmological scenario induced from an empty 5D Riemannian space-time, which leads, among other results, to a unified equation of state for the three eras of the evolution of our 4D Universe (LAPOLA *et al.*, 2021).

3 Induced equation of state for the universe epochs constrained by the Hubble parameter

This chapter presents the main results of the article published in the Chinese Journal of Physics in its August 2021 edition. In the present work, the central theme of this thesis, we apply the STM model to standard ΛCDM cosmology and confront the theoretical with observational data. The article generalizes a model proposed in (MORAES, 2015) that considers the possibility of the existence of an extra dimension in space, which could explain not only the generation of energy and the consequent existence of matter in the Universe but also the origin of dark energy responsible for its current accelerated expansion. The research was developed from just three hypotheses: the first proposed the existence of a 5th dimension in the Universe, with an expansion rate different from the other dimensions, which expand in our 4D homogeneous and isotropic space-time. The second is the association of the energy density and pressure of the Universe in 4D to terms involving the rate of expansion of the extra dimension, in the components of Einstein's equation in five dimensions. Finally, the third hypothesis assumed the existence of a constant energy density associated with the creation of a vacuum in this five-dimensional space. Based only on these three assumptions, it was possible to obtain an expression for the Hubble parameter, which measures the expansion rate of our Universe and its evolution over time, that is, from the Big Bang to current times. The fitting with the observational data was in good agreement, as is possible to see in the final section. (www.doi.org/10.1016/j.cjph.2021.04.021)

3.1 Synthesis

We present a five-dimensional cosmological metric that reveals a four-dimensional energy-momentum tensor. We analyze three cases for the resulting field equations: null, positive, and negative cosmological constant Λ. For the case with a null cosmological constant, we obtain a solution able to describe the radiation-dominated era of the Universe. The positive five-dimensional cosmological constant case yields a bounce, cosmological model. In the negative Λ case, the scale factor for the line element is obtained as $a(t) = c_5\sqrt{\sinh\left(\sqrt{\frac{2}{3}|\Lambda|}\, t\right)}$, where c_5 is a constant. This solution can remarkably describe not only the late-time cosmic acceleration but also the non-accelerated stages of the cosmic expansion, namely the radiation and matter-dominated epochs in a continuous form. We obtain an analytical equation of state capable of describing the different epochs of the universe and it reads as $p = \omega(z)\rho$ with $\omega(z) = \frac{1}{3}\left[1 - \frac{4}{1+c_5^4(1+z)^4}\right]$, with p being the pressure of the universe, ρ its density and z is the redshift. In our model, the constant c_5 is related to the Hubble constant as $H_0 = \sqrt{\frac{|\Lambda|}{6}} \coth\left[arcsinh\left(\frac{1}{c_5}\right)^2\right]$ and this satisfactorily fits the observational data for the low redshift sample of the experimental measurements of the Hubble parameter, which results in $H_0 = 72.2^{+5.3}_{-5.5}$km s^{-1} Mpc^{-1} and $c_5 = 0.600^{+0.061}_{-0.058}$

3.2 Introduction

Since the late nineties of the last century, a lot of efforts have been made to describe the observable universe as a brane embedded in a higher-dimensional space (SUNDRUM, 1999; GOGBERASHVILI; SINGLETON, 2004; SAMI *et al.*, 2004; FABBRI *et al.*, 2004; BURGESS *et al.*, 2001; FLANAGAN *et al.*, 2000). Some results obtained from such a set up for the universe are remarkable. Braneworld models of dark energy were recently presented in references (JAWAD, 2015)-(RANI; JAWAD, 2016). In Reference (SAHNI *et al.*, 2005), the possibility of the ΛCDM cosmological model be a braneworld model in disguise was investigated. In the astrophysics of compact objects context, braneworld models can predict some deviations from standard General Relativity (GR) outcomes and fit with some peculiar observations (GERMANI; MAARTENS, 2001)-(LUGONES; ARBANIL, 2017). Recent literature on braneworld models applications can be found in (PRASETYO *et al.*, 2018)-(BARBOSA-CENDEJAS *et al.*, 2014).

The braneworld scenario was originally proposed as an alternative to the hierarchy problem, as it can be checked, for instance, in References (YANG *et al.*, 2012)-(DAS *et al.*, 2008). The concept of extra dimensions has also been used in attempts to unify the four fundamental forces of nature (HALL; NOMURA, 2002)-(APPELQUIST, 1984). On this regard, one should recall that braneworld models are a low energy limit of string theory (BURIKHAM, 2005; EL-NABULSI, 2009).

The extra dimensional universe configurations are not only related to brane scenarios. There are the Kaluza-Klein (KK) models too (KALUZA, 1921)-(HOHM; SAMTLEBEN, 2013). Kaluza's original idea was to unify the known fundamental forces at that time, namely gravity and electromagnetism. Knowing that Einstein's description of gravity was made in terms of space-time warps and curves, Kaluza had to consider an extra space-like dimension in which electromagnetism could also manifest as "warps" and "curves". In fact, by elevating the Einstein's field equations of GR to a five-dimensional (5D) manifold, Kaluza obtained the original four-dimensional (4D) Einstein's field equations of GR together with Maxwell electromagnetism formalism. Klein's contribution came a few years later motivated by the "invisible" feature of the extra dimension. Klein's idea was that we could not see or perceive the extra dimension because it was really tiny and curled up. In other words, it was compactified in a circular topology.

As astrophysical and cosmological observations have progressed over time, KK theory applications were not restricted to the unification of forces. For instance, KK dark energy and dark matter alternatives can be seen, respectively in (REDDY; LAKSHMI, 2015)-(SHARIF; KHANUM, 2011) and (CHENG *et al.*, 2002)-(KONG; MATCHEV, 2006). Even stellar models were constructed in KK theory (KARSAI *et al.*, 2016)-(BARNAFÖLDI *et al.*, 2010).

Moreover, the virtual effects of KK states on Higgs physics in universal extra-dimensional models were examined in (PETRIELLO, 2002). The Space-Time-Matter Model introduced a generalized gravitational conformal invariance in the context of non compactified 5D KK theory (DARABI; WESSON, 2002). The stability of strange stars in extra dimensions has also been investigated recently (ARBANIL *et al.*, 2019). Other recent interesting applications of KK 5D theory to cosmology can be seen in (SENGUPTA, 2020)-(MORAES; CORREA, 2019) and experimental tests of KK theory in (MBELEK, 2020).

Effective properties of matter in KK theory have been investigated in (LIU; WESSON, 1994) and outlined a Machian interpretation of KK gravity (MASHHOON *et al.*, 1994). Some classical tests were applied to the theory in (KALLIGAS *et al.*, 1995) and the referred equation of motion was derived in (WESSON; LEON, 1995).

Measurements of geodesic precession from Gravity Probe B experiment constrained possible departures from Einstein's GR for a spinning test body in KK theory (OVERDUIN *et al.*, 2013). The Wetterich's parametrization equation of state (EoS) was used to obtain cosmological solutions in a 5D Ricci-flat universe (ZHANG *et al.*, 2006). Some relations for the embedding of spatially flat Friedmann Lemaître Robertson Walker (FLRW) cosmological models in flat KK manifolds were presented in (SEAHRA; WESSON, 2002).

The cosmological constant problem, namely, the huge discrepancy between theoretical and observed values of the cosmological constant in standard ΛCDM cosmology, was investigated in KK gravity by (WESSON; LIU, 2001) and in a quantum cosmology bf model derived from KK theory with a non-compactified extra dimension in (DARABI *et al.*, 2000).

Particularly, regarding the interpretation of the extra dimension in the 4D observable universe, the References (LIU; WESSON, 1994), (MORAES, 2015)-(FUKUI *et al.*, 2001) have fundamental importance. They consist of the following concept. The KK field equations read

$$G_{AB} = 0, \tag{3.1}$$

with G_{AB} being the Einstein tensor and the indices A, B run from 0 to 4. From Eq.(3.1), it can be seen that the KK field equations depend only on the 5D metric g_{AB}. By collecting in Eq.(3.1) the terms that depend on the extra coordinate we can associate them to an induced energy-momentum tensor in 4D. In practice we can transpose the extra dimensional terms of the Einstein tensor to the *rhs* of (3.1) inducing the energy-momentum tensor of matter.

This physical model, known as Space-Time-Matter, Model is strongly supported by Campbell's theorem of Riemannian geometry (CAMPBELL, 1926).

The extra-dimensional components of this 5D Ricci tensor are being naturally related with matter. This unify the gravitational field with its sources, as idealized by Einstein. Further applications of this remarkable idea can be appreciated in Refs.(LEON, 2010)-(HALPERN, 2000).

In the present article we apply this idea to the Friedmann–Lemaître–Robertson –Walker (FLRW) metric including an extra spatial dimension. The field equations will be taken as (1) in the presence of Λ, that is (SAHNI; SHTANOV, 2003)

$$G_{AB} + \Lambda g_{AB} = 0. \tag{3.2}$$

In this way, we will investigate three cases, namely $\Lambda = 0, \Lambda > 0$, and $\Lambda < 0$.

We will be particularly concerned with the role of the scale factors $a(t)$ and $\xi(t)$ of the 5D FRLW metric (MORAES; MIRANDA, 2012)-(CAMERA, 2010)

$$ds^2 = dt^2 - a(t)^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] - \xi(t)^2 dl^2. \tag{3.3}$$

In (3.3), $a(t)$ is the scale factor of the observable universe, and $\xi(t)$ is the extra-dimension scale factor. Moreover, we are assuming the spatial curvature of the universe to be null, in accordance with recent observational data on the fluctuations of the temperature of the cosmic microwave background radiation (HINSHAW *et al.*, 2013). Still, in (3.3), t is the time coordinate, r, θ, and ϕ are the polar spherical coordinates and l is the extra spatial coordinate. Throughout this work, natural units will be assumed, unless otherwise advised.

In the present article, we shall investigate the cosmological solutions obtained from the substitution of (3.3) in the different cases of (3.2). Remarkably, the field equations (3.2), that contain only the Einstein tensor and a cosmological constant, both in five dimensions, will induce a particular energy-momentum tensor for the matter in the form of radiation, dust and dark energy in the four-dimensional universe, as a geometrical manifestation of this five-dimensional setup. In the negative Λ case, from the Friedmann-like equations the time dependent solutions of the scale factors allow us to derive a quite elegant and simple EoS. This EoS is able to describe, in an analytical and continuous form, the radiation, matter and dark energy eras of the universe. Thus, in our model it is possible to obtain a unique EoS for the universe evolution, in our knowledge, a remarkable novelty in extra dimensional models. It will be shown that our results are in accordance with low-redshift observational data of the Hubble parameter.

The section is organized as follows. After the Introduction, in Sec. 3.3, we present the foundations of our model and how we obtain the 4D dynamics from a 5D space in the case of null, positive and negative cosmological constants. In Sec. 3.4 we obtain the analytical and continuous EoS for the Universe evolution, deceleration factor, and density parameter. In Sec. 3.5, using observational data from the Hubble parameter we constrain our model.

3.3 4D dynamics from 5D space

In the present section we will substitute Eq.(3.3) in Eqs.(3.1) and (3.2). For all cases, we will consider that matter in the 4D observable universe is a manifestation of a 5D universe. The terms on the 5D Einstein tensor for (3.3) which depend on the extra coordinate are transposed to the *rhs* of Eqs. (3.1) and (3.2) to play the role of an induced energy-momentum tensor.

Throughout our discussions, the energy-momentum tensor of a perfect fluid will be assumed, that is,
$T_A^B = \mathrm{diag}(\rho, -p, -p, -p, 0)$, with ρ being the matter-energy density and p the pressure of the universe. Note that $T_4^4 = 0$ since we will consider, such as in braneworld models, that matter is restricted to the 4D universe.

3.3.1 Null cosmological constant model

The non-null components of the Einstein tensor obtained when substituting the metric (3.3) in the field equations (3.1) read

$$G_0^0 = 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\frac{\dot{\xi}}{\xi}\right], \tag{3.4}$$

$$G_1^1 = G_2^2 = G_3^3 = \left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\dot{a}}{a}\frac{\dot{\xi}}{\xi} + 2\frac{\ddot{a}}{a} + \frac{\ddot{\xi}}{\xi}, \tag{3.5}$$

$$G_4^4 = 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right], \tag{3.6}$$

where dots indicate time derivatives.

We simply identify the new terms in G_0^0 and G_1^1 due to the extra-dimensional scale factor with the energy density ρ and with the pressure p, respectively. From the isotropy, $G_2^2 = G_3^3$ and the new terms in these components are also identified with the pressure p. We collect the terms that depend on the extra-dimensional scale factor ξ and its derivatives $\dot{\xi}$ and $\ddot{\xi}$, as described in (WESSON; et al., 1996). This mechanism yields to

$$\rho = -\frac{3}{8\pi}\frac{\dot{a}}{a}\frac{\dot{\xi}}{\xi}, \tag{3.7}$$

$$p = \frac{1}{4\pi}\left(\frac{\dot{a}}{a}\frac{\dot{\xi}}{\xi} + \frac{1}{2}\frac{\ddot{\xi}}{\xi}\right). \tag{3.8}$$

From $G_{44} = 0$ we also obtain the constraint equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a} = 0. \tag{3.9}$$

By solving Eq.(3.9), we have

$$a(t) = c_1\sqrt{t}. \tag{3.10}$$

Throughout the article, c_i, with $i = 1, 2, 3, ...$, are constants.

It is worth to remark that Eq.(3.10) describes a radiation-dominated universe, since in standard cosmology $a \sim t^{1/2}$ occurs exactly for such a stage of the universe evolution (RYDEN, 2003). Remarkably, when Kaluza developed his extradimensional theory of gravity, today called KK gravity, he intended to describe from $G_{AB} = 0$ uniquely, both 4D Einstein's field equations with matter and Maxwell's equations for electromagnetism, as it can be checked in (OVERDUIN; WESSON, 1997).

We can substitute (3.10) in the G_{00} component of Eq.(3.1) for metric (3.3) and derive the solution for $\xi(t)$ in

$$\frac{\dot{a}}{a} + \frac{\dot{\xi}}{\xi} = 0, \tag{3.11}$$

resulting in

$$\xi(t) = \frac{c_2}{\sqrt{t}}. \tag{3.12}$$

It can be verified that Eqs.(3.10) and (3.12) are solutions of Eq.(3.1).

It is interesting to remark that the solution obtained for $\xi(t)$ may indicate a compactification of the extra coordinate as time passes by. This can be clearly verified by deriving the referred Hubble parameter $H_l = \dot{\xi}/\xi$, which reads

$$H_l(t) = -\frac{1}{2\,t}, \tag{3.13}$$

and a negative Hubble parameter would indicate compactification rather than expansion of the referred space.

Solutions (3.10) and (3.12) when substituted in (3.7) and (3.8) yield, respectively,

$$\rho(t) = \frac{3}{32\pi\, t^2}, \tag{3.14}$$

$$p(t) = \frac{1}{32\pi\, t^2}. \tag{3.15}$$

We can see from Eqs.(3.14) and (3.15) that ρ and p, in this model, have a quadratic term on t in the denominator. Such a behaviour can also be seen in braneworld models (SAHNI; SHTANOV, 2003; SZABÓ *et al.*, 2007).

We can also note that, remarkably, by dividing (3.15) by (3.14), one has $\omega = p/\rho = 1/3$, which is the EoS parameter of a radiation-dominated universe (RYDEN, 2003). This result can also be verified in the Friedmann-like equations (3.7)-(3.8).

3.3.2 Non-null cosmological constant models

In the following calculations, we take a step further, introducing a cosmological constant associated with the fifth dimension in the formalism. We study the cases in which it is positive and negative.

3.3.2.1 Case I: $\Lambda > 0$.

Let us now work with Eq.(3.2). By substituting metric (3.3) in (3.2), we can once again collect the terms that depend on the extra dimension in the Einstein tensor components and associate them with the matter content of the observable universe, by writing the

FRLW 4D equations with an energy density ρ and pressure p given by:

$$\rho = -\frac{3}{8\pi}\left(\frac{\dot{a}}{a}\frac{\dot{\xi}}{\xi} + \frac{\Lambda}{3}\right), \tag{3.16}$$

$$p = \frac{1}{8\pi}\left(2\frac{\dot{a}}{a}\frac{\dot{\xi}}{\xi} + \frac{\ddot{\xi}}{\xi} + \Lambda\right). \tag{3.17}$$

Also, by recalling that $T_{44} = 0$, we have

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a} = -\frac{\Lambda}{3}. \tag{3.18}$$

Eq.(3.18) can be solved for the scale factor, yielding

$$a(t) = c_3\sqrt{\left|\sin\left(\sqrt{\frac{2}{3}}\Lambda\, t\right)\right|}. \tag{3.19}$$

The evolution of the scale factor (3.19) in time can be appreciated in Fig.4.3 below.

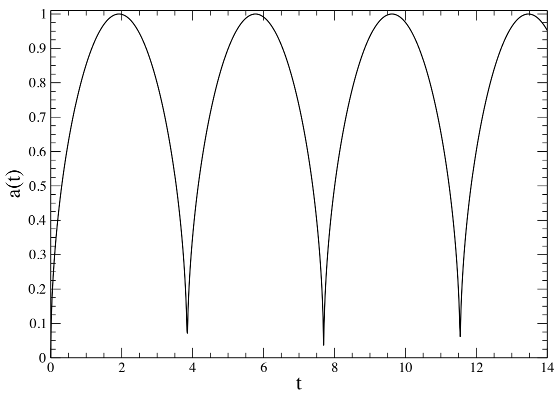

FIGURE 3.1 – Evolution of the scale factor as a function of time in natural units, for $c_3 = \Lambda = 1$.

By analysing Fig.4.3 we are led to conclude that a positive 5D cosmological constant yields a cyclic or bouncing universe (STEINHARDT; TUROK, 2002)-(BARROW; GANGULY, 2017) in the present model.

In possession of Eq.(3.19), we can use the non-null components of Eq.(3.2) to write

$$\xi(t) = c_4\frac{\left|\cos\left(\sqrt{\frac{2}{3}}\Lambda\, t\right)\right|}{\sqrt{\left|\sin\left(\sqrt{\frac{2}{3}}\Lambda\, t\right)\right|}}. \tag{3.20}$$

In Fig.3.2 we plot ξ against time. From Fig.3.2, we can see that ξ completes each cycle in the same time scale as a does, compactifying periodically. We can also see that, by keeping in mind that $a = 1$ at present, the length scale of the extra dimension is minimum today, which could justify the absence of shreds of evidence of extra dimensions in the Large Hadron Collider (CHATRCHYAN; et al., 2012)-(DATTA; RAYCHAUDHURI, 2013).

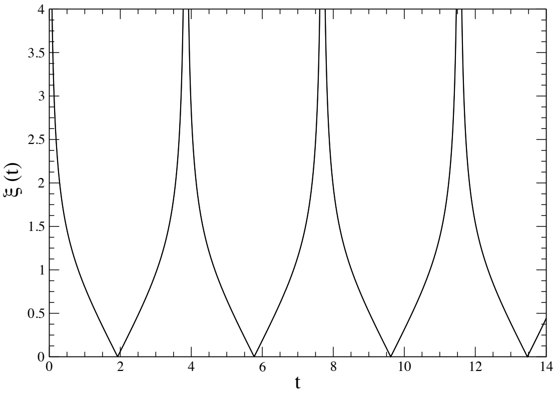

FIGURE 3.2 – Evolution of the extra-dimension scale factor as a function of time in natural units, for $c_4 = \Lambda = 1$.

From (3.19) and (3.20), we can write the explicit solutions for $\rho(t)$ and $p(t)$ as

$$\rho(t) = \frac{\Lambda}{16\pi} \cot^2 \left(\sqrt{\frac{2}{3}} \Lambda \, t \right) , \tag{3.21}$$

$$p(t) = \frac{\Lambda}{48\pi} \left[\cot^2 \left(\sqrt{\frac{2}{3}} \Lambda \, t \right) + 4 \right] . \tag{3.22}$$

Although bouncing models have their importance specially because they evade the Big-Bang singularity, we should discard the present model due to the impossibility of predicting the late-time accelerated expansion regime of the universe (RIESS, 1998; PERLMUTTER *et al.*, 1999) from Eq.(3.19).

3.3.2.2 Case II: $\Lambda < 0$.

Following the same approach of the previous section now for $\Lambda < 0$, we obtain the scale factors as

$$a(t) = c_5\sqrt{\sinh\left(\sqrt{\frac{2}{3}|\Lambda|}\,t\right)}, \tag{3.23}$$

$$\xi(t) = c_6\frac{\cosh\left(\sqrt{\frac{2}{3}|\Lambda|}\,t\right)}{\sqrt{\sinh\left(\sqrt{\frac{2}{3}|\Lambda|}\,t\right)}}. \tag{3.24}$$

The evolution of those scale factors can be appreciated in Figures 3.3-3.4 below.

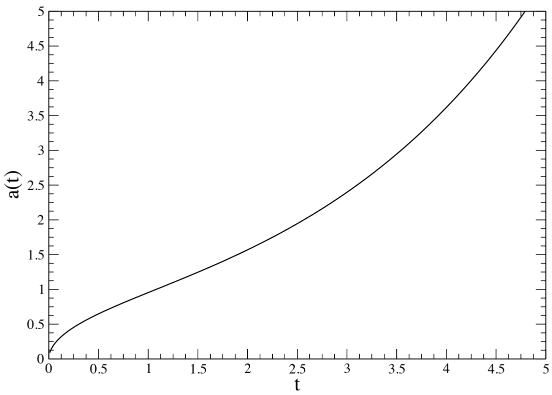

FIGURE 3.3 – Evolution of the scale factor as a function of time in natural units, for $c_5 = 1$ and $\Lambda = -1$.

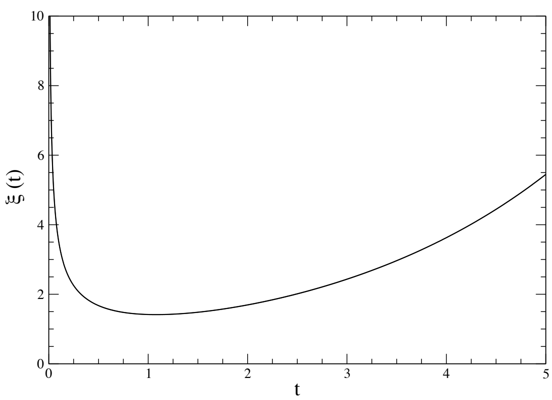

FIGURE 3.4 – Evolution of the extra-dimension scale factor as a function of time in natural units, for $c_6 = 1$ and $\Lambda = -1$.

We can see from Fig.3.3 that $a(t)$ assumes an exponential behavior as time grows, which may be an indication of the recent cosmic acceleration (RIESS, 1998; PERLMUTTER *et al.*, 1999). This will be clarified in Fig.3.5 below.

From Figure 3.4, we can see that the extra dimension is large for the primordial stages of the universe ($t \ll 1$). Then, it naturally suffers a process of compactification, assuming its minimum value for $t \sim 1$. After that, it maximizes its length scale once again.

It is possible to derive a relation between the scale factors $a(t)$ and $\xi(t)$. Starting from (3.4) and (3.6) for $\Lambda < 0$ we obtain the system of equations

$$\frac{\dot{a}}{a}\frac{\dot{\xi}}{\xi} + \left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3}, \tag{3.25}$$

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3}. \tag{3.26}$$

Subtracting (25) from (26) leads to

$$\frac{\dot{\xi}}{\xi} = \frac{\ddot{a}}{\dot{a}}, \tag{3.27}$$

which yields to the relation

$$\xi = K\dot{a}, \tag{3.28}$$

with constant K.

Therefore,

$$\frac{\xi}{a} = KH = \frac{c_6}{c_5}\sqrt{\frac{6}{|\Lambda|}}H, \tag{3.29}$$

where $H = H(t) = \frac{\dot{a}}{a}$ is the Hubble parameter.

The solutions for the induced matter content read

$$\rho(t) = \frac{|\Lambda|}{16\pi}\coth^2\left(\sqrt{\frac{2}{3}|\Lambda|}\,t\right), \tag{3.30}$$

$$p(t) = \frac{|\Lambda|}{48\pi}\left[\coth^2\left(\sqrt{\frac{2}{3}|\Lambda|}\,t\right) - 4\right] \tag{3.31}$$

The induced density can be rewritten as

$$\rho = \frac{|\Lambda|}{16\pi}\left(\frac{c_5}{c_6}\frac{\xi}{a}\right)^2 \tag{3.32}$$

or

$$\rho = \frac{3H^2}{8\pi}, \tag{3.33}$$

where

$$H(t) = \sqrt{\frac{|\Lambda|}{6}} \coth\left(\sqrt{\frac{2}{3}\,|\Lambda|\,t}\right). \tag{3.34}$$

The Hubble parameter has its evolution in time shown in Figure 3.5. We see from Fig.3.5 that the predicted Hubble parameter starts evolving as $\sim 1/t$, which is, indeed, expected from standard model predictions (RYDEN, 2003). After a period of time, $H(t) \sim$ *constant*. It is known that an exponential scale factor describes the cosmic acceleration. From the definition of the Hubble parameter, an exponential scale factor yields a constant Hubble parameter. In this way, the constant behavior that $H(t)$ assumes for high values of time is an indication of the recent cosmic acceleration.

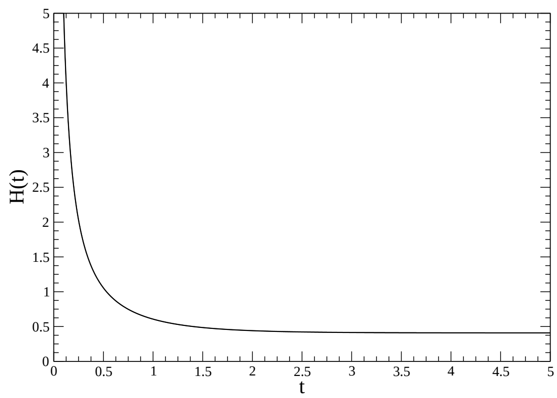

FIGURE 3.5 – Evolution of the Hubble parameter as a function of time in natural units, for $\Lambda = -1$.

3.4 Equation of state for the Universe evolution and deceleration parameter

In this section we investigate the solutions obtained in the previous section for the negative 5D cosmological constant case. It is shown that the induced matter-energy density and pressure can be related through a unique analytical EoS for the Universe evolution (CHODOS *et al.*, 1974; WEBER *et al.*, 2007). We also derive, from the scale factor solution, the deceleration parameter of the model.

3.4.1 Unified equation of state for the Universe evolution

Considering the case in which $\Lambda < 0$, the expressions for the density and pressure can be written in a unified form as

$$p = \omega(t)\rho, \tag{3.35}$$

where the EoS parameter can be written as

$$\omega(t) = -1 + \frac{4}{3} sech^2 \left(\sqrt{\frac{2}{3}|\Lambda|} t \right). \tag{3.36}$$

The evolution of the above EoS in time can be appreciated in Figure 6 below. From Figure 6 it is possible to realize some interesting cosmological features predicted by the model. One can note that for small values of time, $\omega \sim 1/3$. According to the standard model, the primordial value of ω is true 1/3, as the primordial universe dynamics is dominated by radiation, such that $p = \rho/3$ (RYDEN, 2003). As the universe expands and cools down, it allows a pressureless matter to be formed. This stage represents the matter-dominated stage of the universe, for which $\omega \sim 0$, which is also depicted in Fig.3.6. Last, but definitely not least, Fig.3.6 indicates that for high values of time, $\omega \sim -1$. According to recent observations on the cosmic microwave background radiation temperature fluctuations, $\omega = -1.073^{+0.090}_{-0.089}$ (HINSHAW *et al.*, 2013; AGHANIM; et. al., 2018). This negative pressure fluid is responsible for the cosmic acceleration in the standard model. Therefore, our present approach reveals a dominant negative pressure fluid for high values of time. Remarkably, it has also predicted other stages of the universe evolution, named radiation and matter-dominated eras, in a continuous and analytical form.

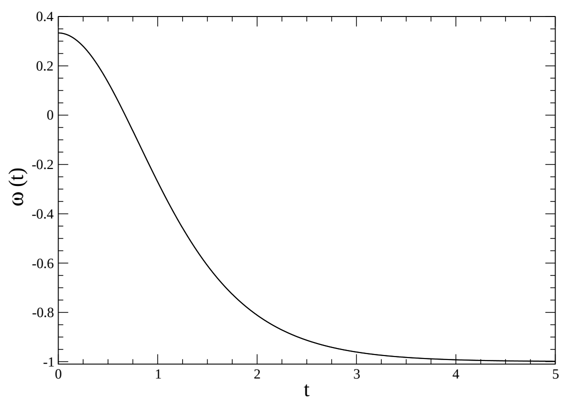

FIGURE 3.6 – Evolution of the equation of state parameter as a function of time in natural units, for $\Lambda = -1$.

It is important to show that the expression (35) for the density satisfies the continuity equation in 4D. Starting from the continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0, \tag{3.37}$$

substituting (3.35) and integrating on both sides leads to

$$\rho(t) = \frac{|\Lambda|}{16\pi}\left[\left(\frac{c_5}{a(t)}\right)^4 + 1\right] \tag{3.38}$$

proving that the continuity equation in 4D is satisfied in our model. Note for $a(t) << 1$, the energy density $\rho \propto a^-4$, denoting the radiation era.

3.4.2 The deceleration parameter

The deceleration parameter is defined as

$$q(t) = -\frac{\ddot{a}\, a}{\dot{a}^2}, \tag{3.39}$$

so that $q > 0$ indicates a decelerated expansion and $q < 0$ indicates an accelerated expansion.

In the present model, it can be shown that

$$q = -\frac{\dot{\xi}}{\xi}\frac{a}{\dot{a}} = -\frac{H_l}{H}. \tag{3.40}$$

Therefore, remarkably the deceleration factor in our model is the negative ratio between the Hubble parameter of the extra-dimension scale factor and the Hubble parameter in 4D.

Explicitly, the deceleration parameter for $\Lambda < 0$ reads

$$q(t) = 1 - 2\tanh^2\left(\sqrt{\frac{2}{3}|\Lambda|}\, t\right). \tag{3.41}$$

3.5 Cosmological parameters as functions of redshift and confrontation with observational data

To confront our solutions with observational data we should investigate the behavior of the Hubble and other cosmological parameters in terms of the redshift rather than time. We will concentrate our attention in the $\Lambda < 0$ case.

Taking into account the scale factor obtained in (3.23), the redshift can be written as

$$z(t) = -1 + \frac{1}{c_5\sqrt{\sinh\left(\sqrt{\frac{2}{3}|\Lambda|}\, t\right)}}. \tag{3.42}$$

The Hubble parameter is then expressed in terms of redshift as

$$H(z) = \sqrt{\frac{|\Lambda|}{6}} \coth \left[arcsinh \left[\frac{1}{c_5(1+z)} \right]^2 \right\} . \quad (3.43)$$

The above equation relates the constant c_5 of the 4D scale factor to the present value of the Hubble constant as

$$H_0 = \sqrt{\frac{|\Lambda|}{6}} \coth \left[arcsinh \left(\frac{1}{c_5} \right)^2 \right] . \quad (3.44)$$

3.5.1 Observational constraints

Hubble parameter data as a function of redshift yields one of the most straightforward cosmological tests today. It consists of constraining the cosmological models with values of the expansion rate as a function of redshift. It is even more interesting when the Hubble parameter data come from estimates of differential ages of objects at high redshifts, because it is inferred from astrophysical observations alone, not depending on any background cosmological models (check References (STERN *et al.*, 2010; LIMA *et al.*,)).

The data we use here comes from the 51 $H(z)$ data compilation from Magaña *et al.* (MAGA *et al.*, 2018). This compilation consists of 20 clustering (from Baryon Acoustic Oscillations and Luminous Red Galaxies) and 31 differential age $H(z)$ data.

We choose to work here only with the 31 differential age $H(z)$ data[1], because it does not depend on any background cosmological model. The age estimates depend only on models of the chemical evolution of objects at high redshifts. $H(z)$ estimates from clustering like Baryon Acoustic Oscillations usually assume a standard cosmological model to obtain the data from surveys.

In all analyses here, we have written a χ^2 function for parameters, with the likelihood given by $\mathcal{L} \propto e^{-\chi^2/2}$. The χ^2 function for the $H(z)$ data is given by the following:

$$\chi_H^2 = \sum_{i=1}^{31} \frac{[H_{obs,i} - H(z_i, \mathbf{s})]^2}{\sigma_{H_i,obs}^2}, \quad (3.45)$$

where $\mathbf{s}$ is the parameter vector, which we choose to be $\mathbf{s} = (c_5, H_0)$. Λ can be related to these parameters through Eq.(3.44).

In Figure 3.7 below, we can see the 31 $H(z)$ data used here and the best $H(z)$ fit we have found by minimizing χ_H^2.

[1]Marked as "DA" in Table 1 of Ref. (MAGA *et al.*, 2018).

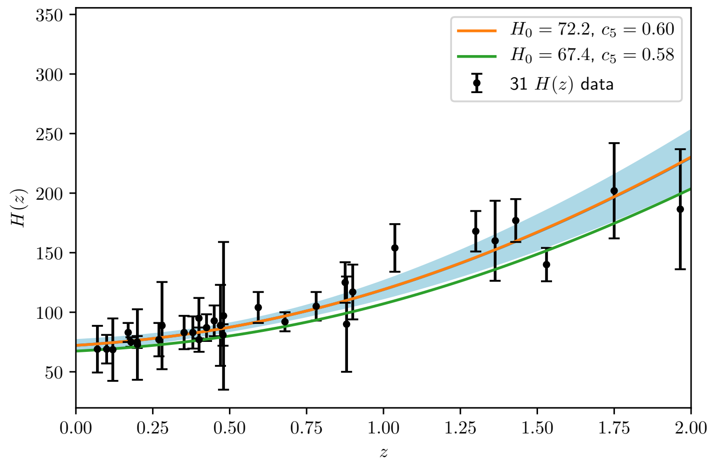

FIGURE 3.7 – Hubble parameter as a function of redshift for the best fit parameters from the 31 $H(z)$ data (H_0 = 72.2 km/s/Mpc, c_5 = 0.60). It is shown a curve with H_0 = 67.4 km/s/Mpc, in agreement with Planck data (AGHANIM; et. al., 2018; ALAM *et al.*, 2017) for the Hubble parameter and for a universe age of 13.8 Gyr, that corresponds to $c_5 = 0.58$. The blue region corresponds to a 2σ (95.4%) c.l. around the best fit.

To find the constraints over the free parameters, we have assumed flat priors for c_5 and H_0 and have sampled the posteriors with the so-called Affine Invariant Monte Carlo Markov Chain Ensemble Sampler by (GOODMAN; WEARE, 2010), which was implemented in Python language with the emcee software by (FOREMAN-MACKEY *et al.*, 2013). In order to plot all the constraints on each model, we have used the freely available software getdist[2], in its Python version.

The results of this analysis can be seen in Fig.3.8 and Table 3.1.

Parameter	95% limits
$\boldsymbol{c_5}$	$0.600^{+0.061}_{-0.058}$
$\boldsymbol{H_0}$	$72.2^{+5.3}_{-5.5}$
t_0	$12.59^{+0.69}_{-0.62}$

TABLE 3.1 – Mean value and 95% limits of the model parameters. In boldface are the free parameters and t_0 is a derived parameter. H_0 is in units of km/s/Mpc and t_0 in Gyr.

Substituting the central values for c_5 and H_0 shown in table 3.1, in Eq. (3.44) with some algebra one finds the value for the 5D negative cosmological constant with a magnitude as $|\Lambda| = 2.907\ 10^{-35} s^{-2}$. The magnitude of Λ, which has a geometrical origin coming from the fifth dimension, is in good agreement with the experimental ΛCDM value $\Lambda = 2.046\ 10^{-35}$s , as shown in (PERLMUTTER *et al.*, 1999)

[2] getdist is part of the great Affine Invariant Monte Carlo Markov Chain Ensemble Sampler, COSMOMC (LEWIS; BRIDLE, 2002).

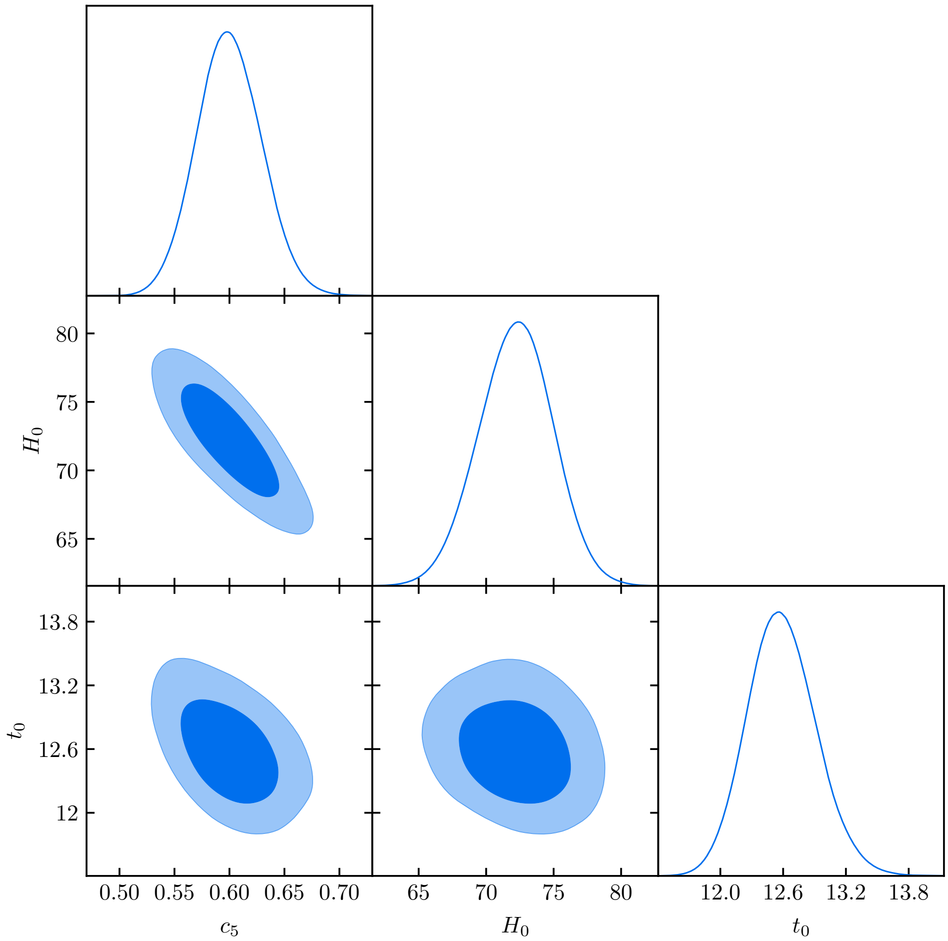

FIGURE 3.8 – Confidence contours from 31 $H(z)$ data analysis of the free parameters of the model, c_5 and H_0. We also show the constraints over the total age, t_0, which is a derived parameter (H_0 is in km/s/Mpc and t_0 in Gyr). The contours correspond to 68% and 95% c.l..

3.5.2 Extra dimensional scale factor, deceleration, and equation of state parameters as functions of redshift

In Fig. 3.9 we present the scale factor of the extra dimension as a function of redshift, which can be written as

$$\xi(z) = c_5\, c_6\, (1+z)\sqrt{1+\frac{1}{c_5^4\,(1+z)^4}}\,. \tag{3.46}$$

It is interesting to note that the extra-dimension scale factor has a free constant c_6, which is not fixed by the cosmological analysis. This happens because the extra dimensional dependence of the cosmological parameters only appears through the fraction $\dot{\xi}/\xi$. This means that, in this model, even if the scale of the extra dimension is very small, the cosmological effects would still be measurable. As a consequence, the extra dimensional length scale can be arbitrarily small.

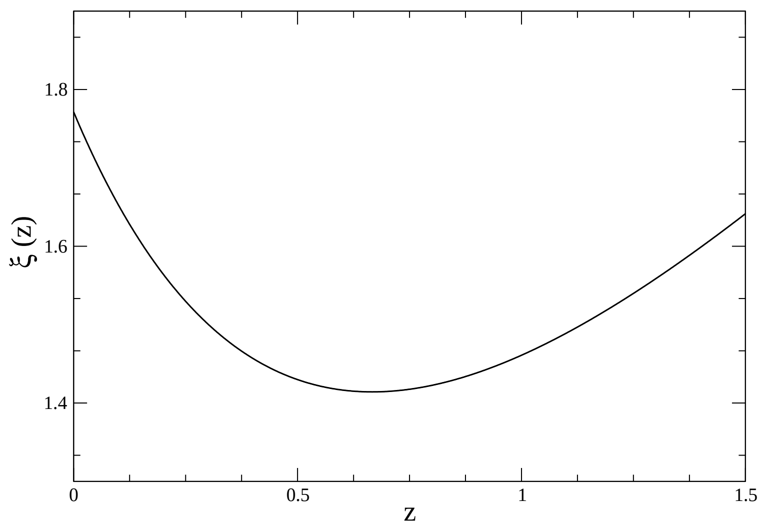

FIGURE 3.9 – Evolution of the extra-dimension scale factor as a function of redshift in natural units, for $c_5 = 0.60$ and $c_6 = 1$.

The deceleration parameter as a function of the redshift is given by

$$q(z) = 1 - \frac{2}{1 + c_5^4\,(1+z)^4}\,, \tag{3.47}$$

whose behavior can be seen in Fig.3.10. We can see that the model gives an accelerated expansion of the universe ($q < 0$) for the present epoch. Also, the obtained solution gives a transition from a decelerated to an accelerated universe expansion (RIESS, 2001). Our model prediction, $z \sim 0.66$, is compatible with the results found in (FAROOQ; RATRA, 2013).

The analytical expression for the EoS parameter as a function of redshift is

$$\omega(z) = \frac{1}{3}\left[1 - \frac{4}{1 + c_5^4(1+z)^4}\right], \tag{3.48}$$

whose pattern is shown in Fig.3.11. It depicts the EoS parameter evolution for different epochs of the universe. As expected by the standard cosmological model (TANABASHI *et al.*, 2018), for recent redshifts the parameter is $< -1/3$ (dark energy era), for past times the EoS parameter presents a null value, which is compatible with the matter-dominated phase and for larger values of z, is equal to 1/3 (radiation energy era).

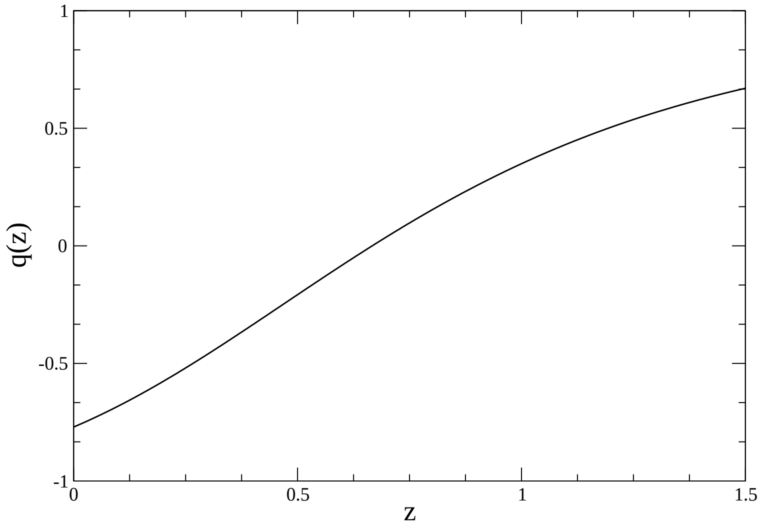

FIGURE 3.10 – Evolution of the universe deceleration parameter as a function of the redshift, for $c_5 = 0.60$.

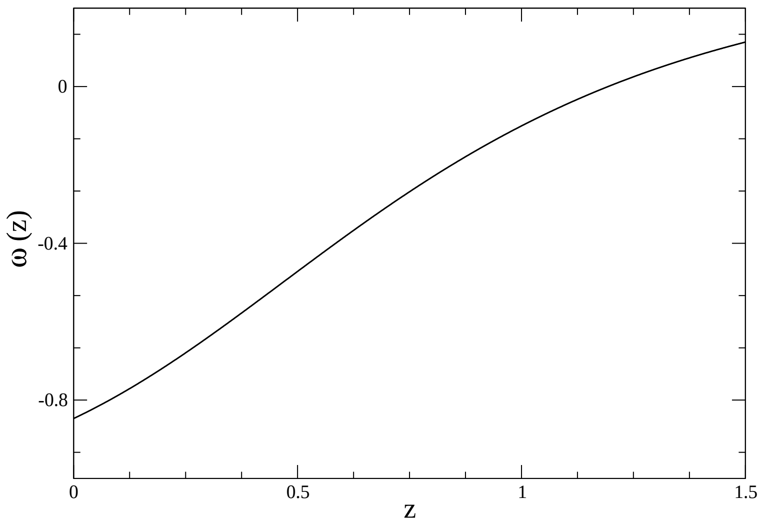

FIGURE 3.11 – Evolution of the EoS parameter as a function of the redshift in natural units, for $c_5 = 0.60$.

4 Wormholes in energy-momentum conserved $f(R,T)$ Gravity

Wormholes are shortcuts in the topology of space-time that connect two distant regions in the universe. The wormholes theory was proposed for the first time by Einstein and Rosen in 1935 when they established a solution based on the Schwarzschild metric, that describes bridges between two different areas of space-time modeled with Einstein's General Relativity vacuum field equations. In this work, Einstein and Rosen showed that only a few coordinate systems describe two asymptotically flat regions of the Schwarzschild space-time maximally extended, and the event horizon of a black hole is the key ingredient of bridge construction (EINSTEIN; ROSEN, 1935). After a period of about 27 years of numbness in the area, in 1962, Wheeler and Fuller published a paper showing that the Einstein-Rosen bridges were unstable and that they would collapse instantly as soon as they are formed, preventing even light from being able to cross them (FULLER; WHEELER, 1962).

The revival of the theory occurred in 1988 with Morris and Thorne (MORRIS; THORNE, 1988) when they found a metric for a “traversable” wormhole, i.e. a wormhole whose matter inside keeps its throat open. Solutions of the Einstein's field equations for humanly traversable wormholes, as shown by Morris and Thorne violate the null energy condition. By solving its field equations, we find that the throats of wormholes must be filled with exotic matter.

In any case, the fluid that permeates the inner of a wormhole must be anisotropic, as in the case of many stars configurations, i.e. with an energy-momentum tensor that, in addition to the energy density in its temporal component, also has radial pressure and tangential pressure, where the latter corresponds to its angular components.

In particular, the laws of General Relativity (GR) combined with the laws of quantum field theory tell us how to construct a wormhole and what kind of matter is needed to hold it open, so that things can pass through it (BLÁZQUEZ-SALCEDO *et al.*, 2021)-(KONOPLYA; ZHIDENKO, 2022)

Anyhow, some alternative theories of gravity have been capable of describing non-exotic matter in the wormhole interior, such as the $f(R)$ theory (SOTIRIOU; FARAONI, 2010), with R being the Ricci scalar, Gauss-Bonnet theory (MEHDIZADEH *et al.*, 2015) and Kaluza-Klein gravity (LEON, 2010), among others.

An extension of the $f(R)$ theories of gravity was proposed by Harko and collaborators by further inserting a dependence on the trace of the energy-momentum tensor T, denoting the $f(R,T)$ gravity (HARKO *et al.*, 2011). Since then, the theory has been successfully tested in many areas (SHARIF; NAWAZISH, 2019; ELIZALDE; KHURSHUDYAN, 2019; TRETYAKOV, 2018; SHARIF; ZUBAIR, 2012; MORAES *et al.*, 2018; CARVALHO *et al.*, 2017; LIN *et al.*, 2017; SHAIKH, 2018).

In the present article, starting from the metric of a static wormhole, we will calculate the field equations using the $f(R,T)$ formalism, as doing in (HARKO *et al.*, 2013; MORAES; SAHOO, 2018; LOBO; OLIVEIRA, 2009; PAVLOVIC; SOSSICH, 2015; VISSER; WORMHOLES, 1995; SHARIF; NAWAZISH, 2019). However, in our approach we impose the conservation of energy-momentum tensor. In previous works, this assumption also was adopted. For example in (CARVALHO *et al.*, 2020), the conservation of the energy-momentum tensor was assumed. Starting from an $f(R,T)$ in the form $f(R,T) = R+2h(T)$, this generic function of the energy tensor momentum trace ($h(T)$), takes on a specific form. This form satisfies the conservation of energy-momentum tensor. In (CARVALHO *et al.*, 2020) the authors use the energy-momentum conservation in $f(R,T)$ formalism to solve the TOV equation for strange stars, using linear equations of state (EoS).

We assume a well-known ansatz for the shape function $b(r)$ in the wormhole metric and obtain the density expression as a function of the radius r. We then investigate which form of $h(T)$ satisfies the energy-momentum conservation, and investigate if the energy conditions can be satisfied, looking for the presence of usual matter in the throat of the wormhole.

It is interesting to quote that although so far wormholes have not been detected, attempts to do so have been constantly proposed, as one can check, for instance in (SHAIKH, 2018)-(NANDI *et al.*, 2017).

This chapter is organized as follows: in Section 4.1, we present the $f(R,T)$ theory and the energy-momentum tensor conservation equation. In Section 4.2 we present the wormhole metric and asymmetric energy-momentum tensor. Section 4.3 details the energy conditions that must be obeyed for the wormhole to be traversable. The results to combine the energy-momentum conserved $f(R,T)$ gravity with linear Equations of State (EoS) for the radial and tangential pressures, and the choice for the shape function $b(r)$, are presented in section 4.4. In section 4.5 is shown the non-exotic conditions for the wormhole to be traversable. Section 4.6 is dedicated to showing that our results are a particular case of the results of a previous work (MORAES; SAHOO, 2017) in which a more general shape function is presented to construct a wormhole but in the non-conservative $f(R,T)$ framework. We show, in fact, that this general shape function conserves the $f(R,T)$ energy-momentum tensor, when linear Equations of State (EoS) for the radial and tangential pressures are considered, implying that these pressures are proportional. The derivation of the covariant derivative of the energy-momentum tensor is shown in detail in Appendix A.

4.1 The $f(R,T)$ theory and the energy-momentum tensor conservation

As described in (HARKO *et al.*, 2011), the action for $f(R,T)$ gravitational theory reads

$$S = \int \left[\frac{f(R,T)}{16\pi} + \mathcal{L}_m \right] \sqrt{-g} d^4x, \tag{4.1}$$

with g being the determinant of the metric, $f(R,T)$ a function of the argument, expressed in terms of the Ricci curvature scalar R and the trace of the energy-momentum tensor T. In our case, the matter lagrangian are $\mathcal{L}_m = \rho$, with ρ being the density.

Varying this action with respect to the metric components $g_{\mu\nu}$, we obtain the general form of the field equations (HARKO *et al.*, 2011) given by

$$f_R(R,T)R_{\mu\nu} - \frac{1}{2}f(R,T)g_{\mu\nu} + (g_{\mu\nu} - \nabla_\mu \nabla_\nu) f_R(R,T) = 8\pi T_{\mu\nu} + f_T(R,T)(T_{\mu\nu} - \rho g_{\mu\nu}), \tag{4.2}$$

where $f_R(R,T) = \partial f/\partial R$, and $f_T(R,T) = \partial f/\partial T$.

The general expression for the covariant derivative of the energy-momentum tensor, as expressed in (HARKO *et al.*, 2011) is given by

$$\nabla^\mu T_{\mu\nu} = \frac{f_T(R,T)}{8\pi - f_T(R,T)} \left[(T_{\mu\nu} + \rho g_{\mu\nu}) \nabla^\mu ln f_T(R,T) + \nabla^\mu \rho g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \nabla^\mu T \right]. \tag{4.3}$$

As can be seen, the covariant derivative of the energy-momentum tensor, unlike in Einstein's general relativity, is not null. This indicates the non-conservation of energy and can be interpreted as the creation or destruction of matter. But, if we impose this conservation, i.e., $\nabla^\mu T_{\mu\nu} = 0$, we find a form to the function $f(R,T) = h(T)$ that satisfies this condition, avoiding the freedom to choose any function, as it is the case in $f(R,T)$ gravity. A summarized derivation of Eq. (4.3) obtained from the field equations (4.2) is ´resented in Appendix A, at the end of this thesis.

4.2 Wormhole metric and energy-momentum tensor

The static traversable wormholes, as shown by Morris and Thorne in (MORRIS; THORNE, 1988) , in Einstein's General Relativity context, violate the weak, strong, and dominant energy conditions, i.e., somewhere near the throat of the wormhole, someone must be able to encounter some negative energy density. Assuming that this traversable wormhole are time-independent, non-rotating, and spherically symmetric bridges, in Schwarzschild coordinates, without loss of generality, the spacetime metric of the wormhole has the form

$$ds^2 = e^{2\varphi(r)}dt^2 - \left[\frac{1}{1-\frac{b(r)}{r}}\right]dr^2 - r^2\left[d\theta^2 + sin^2\theta d\phi^2\right], \tag{4.4}$$

where $\varphi(r)$ is the redshift function, $b(r)$ is the shape function and $r = r_0$ is the throat of the wormhole, i.e., the minimum value that r can assume. By comparison with the Schwarzschild metric, this implies that the mass of the wormhole can be given by $b = 2GM$, assuming the same mass (positive) in the two mouths of the wormhole. In order for the spacetime geometry to tend to the appropriate flat limit, i.e. $\lim\limits_{r\to\infty}\varphi(r) = \varphi_0$, where φ_0 is constant. That is, this limit must exist and be finite, then the redshift function tends to the appropriate flat limit. There is no a priori requirement that $\varphi_-(\infty) = \varphi_+(\infty)$. This would only imply that time would pass differently in the two regions, or universes, connected by the wormhole. So for simplicity, we adopt the equality $\varphi_-(\infty) = \varphi_+(\infty) = \varphi(r) = 0$. This assumption implies zero tidal force in the wormhole as seen by stationary observers (VISSER; WORMHOLES, 1995; MORRIS; THORNE, 1988).

The energy-momentum tensor is given by

$$T^{\mu}_{\nu} = diag\left(\rho, -p_r, -p_t, -p_t\right), \tag{4.5}$$

with ρ being the density and p_r and p_t being the radial and tangential pressures, respectively. Then the wormhole is filled by an anisotropic fluid such as

$$T_{\mu\nu} = \left(\rho + p_t\right)u_\mu \mu_\nu + p_t g_{\mu\nu} + \left(p_{r-}p_t\right)x_\mu x_\nu \tag{4.6}$$

where $x_\mu$is the space-like direction vector orthogonal to the 4-velocity u_μ.

4.3 Energy Conditions

We investigate in this section the existence of traversable wormholes in this conservative construction that can be traversable, i.e, wormholes that obey the classical energy conditions.

As mentioned in Introduction, Wormholes in General Relativity lead to a violation of causality with the mathematical prediction of the occurrence of exotic matter inside them, i.e., matter that violates the energy conditions as described in (MORRIS; THORNE, 1988) and (VISSER; WORMHOLES, 1995). Fundamentally, they are described as the following:

• The strong energy condition (SEC) says that gravity should be always attractive, or in terms of energy-momentum tensor it reads $\rho + \Sigma_j p_j \geq 0$, $\forall j$.

• The dominant energy condition (DEC) is an indication that the energy density measured by any observer should be non-negative, which leads to $\rho \geq |p_j|$, $\forall j$.

• The weak energy condition (WEC) shows that the energy density measured by any observer should be always non-negative, i.e., $\rho \geq 0$ and $\rho + p_j \geq 0$, $\forall j$.

• The null energy condition (NEC) is usually a minimum requirement from SEC and WEC, i.e. $\rho + p_j \geq 0$, $\forall j$. It is possible, however, scenarios where p_r or p_t do not obey the NEC, and yet the WEC is partially satisfied (TAYDE *et al.*, 2022; HASSAN *et al.*, 2022; MANDAL *et al.*, 2020).

4.4 Wormhole Field Equations with linear EoS, and ansatz for $b(r)$

In our specific case for wormholes, p_j can be the radial and tangential pressures, p_r and p_t. So, in the following, considering an ansatz for the shape function $b(r)$, we will investigate the null Energy Condition mentioned above. After that, we will check the others with a help of an EoS.

Let us assume the general form $f(R, T) = R + h(T)$, with $h(T)$ being a function that depend only on the trace of the energy-momentum tensor. In this way, if $h(T) = 0$ we recover the Einstein-Hilbert action for general relativity (Eq. (4.1)). Then the derivatives are given by $f_R(R, T) = 1$, and $f_T(R, T) = h_T$.

Imposing the conservation of the energy-momentum tensor in $f(R, T)$ gravity, the left-side of Eq. (4.3) is equal to zero, and since $f_T(R, T) = h_T$, we obtain the equation

$$\left[(T_{\mu\nu} + \rho g_{\mu\nu}) \nabla^\nu ln(h_T) + \nabla^\nu \rho g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \nabla^\mu T\right] = 0, \tag{4.7}$$

which in its mixed form is written as

$$\left[(T^\mu_\nu + \rho \delta^\mu_\nu) \nabla^\nu ln(h_T) + \nabla^\nu \left(\rho - \frac{1}{2} T\right) \delta^\mu_\nu\right] = 0. \tag{4.8}$$

Then, to find an expression for $h(T)$ that satisfies Eq. (4.8) considering the trace of the energy-momentum tensor as $T = \rho - p_r - 2p_t$, Eq. (4.8) can be rewritten for $\mu = \nu = r$ as

$$\left[(p_r + \rho)\frac{\partial}{\partial r} ln\left(h_T\right) + \frac{1}{2}\frac{\partial}{\partial r}\left(\rho + p_r + 2p_t\right)\right] = 0, \tag{4.9}$$

We need to solve this differential equation to find the expression for h_T and $h(T)$ that satisfies this Eq (4.9). A viable way to solve this equation, and find $h(T)$ is to choose an equation of state relating the energy density to the radial and tangential pressures.

In this work, we consider linear equations of state with the energy density for the between pressures

$$p_r = \beta\rho, \tag{4.10}$$

$$p_t = \gamma\rho, \tag{4.11}$$

with β and γ being constants. With this choice, the conservation Eq. (4.9) becomes

$$\rho\left(1+\beta\right)\frac{\partial}{\partial r}\left[ln(h_T)\right] + \frac{1}{2}\frac{\partial}{\partial r}\left[\rho\left(1+\beta+2\gamma\right)\right] = 0 \tag{4.12}$$

From this conservation equation (4.12), since T is a linear function in ρ, we can rewrite it as follow

$$\left[\frac{\partial}{\partial T} ln(h_T) + \frac{1}{T}\frac{(1+\beta+2\gamma)}{2(1+\beta)}\right]\frac{\partial T}{\partial r} = 0, \tag{4.13}$$

which leads to

$$T\frac{1}{h_T}\frac{\partial h_T}{\partial T} = -\frac{(1+\beta+2\gamma)}{2(1+\beta)}. \tag{4.14}$$

Then,

$$T\frac{\partial h_T}{\partial T} = \alpha h_T, \tag{4.15}$$

where

$$-\frac{(1+\beta+2\gamma)}{2(1+\beta)} = \alpha = constant. \tag{4.16}$$

The solution of the Eq. (4.15) is

$$h_T = \lambda T^{\alpha}, \tag{4.17}$$

and by integrating the previous equation we find

$$h(T) = \frac{\lambda}{\alpha+1} T^{\alpha+1}, \tag{4.18}$$

i.e, a power law expression in the trace of the energy-momentum T.

Hence, the general function $f(R,T) = R + h(T)$ that satisfies the conservation law of $T^{\mu\nu}$, given by Eq. (4.3), for a linear EoS with asymmetry Δ, linear with ρ, i.e., $p_r = \beta\rho$, $p_t = \gamma\rho$ $(\Delta = p_t - p_r = (\gamma - \beta)\rho)$, is given by

$$h(T) = \frac{\lambda T^{\left(\frac{1+\beta-2\gamma}{2(1+\beta)}\right)}}{\left[\frac{1+\beta-2\gamma}{2(1+\beta)}\right]}. \tag{4.19}$$

The field equations given in their general form by Eq. (4.2) can be expressed in their mixed form by

$$G^{\mu}_{\nu} = 8\pi T^{\mu}_{\nu} + \frac{1}{2}h(T)\delta^{\mu}_{\nu} + h_T(T)\left(T^{\mu}_{\nu} - \rho\delta^{\mu}_{\nu}\right). \tag{4.20}$$

Then, the non-null components of this field equation (4.23) are given by

$$\frac{b^{'}}{r^2} = 8\pi\rho + \frac{1}{2}h(T), \tag{4.21}$$

$$\frac{b}{r^3} = -8\pi p_r + \frac{1}{2}h(T) - h_T(T)\left[p_r + \rho\right], \tag{4.22}$$

$$\frac{1}{2r^2}\left(b' - \frac{b}{r}\right) = -8\pi p_t + \frac{1}{2}h(T) - h_T(R,T)\left[p_t + \rho\right]. \tag{4.23}$$

The ansatz for the shape function is defined as in (ELIZALDE; KHURSHUDYAN, 2019)

$$b(r) = r_0\left(\frac{r_o}{r}\right), \tag{4.24}$$

with constant r_0 being the throat of the wormhole. The derivative of this shape function in respect to r are given by $b' = -b(r)/r$

This shape function obeys the conditions explained in (LOBO; OLIVEIRA, 2009; PAVLOVIC;

SOSSICH, 2015; VISSER; WORMHOLES, 1995), as follow

$$b(r) < r, \tag{4.25}$$

$$1 - \frac{b(r)}{r} \geq 0, \tag{4.26}$$

$$b^{'}(r) < \frac{b(r)}{r}, \tag{4.27}$$

$$\lim_{r \to \infty} \frac{b(r)}{r} = 0. \tag{4.28}$$

Using the field equations with $b(r)$ defined by Eq.(4.24) we obtain that $p_t = -\rho$, so

$$\gamma = -1. \tag{4.29}$$

Using the EoS (4.10) and (4.11), the pressure asymmetry Δ can be written as

$$\Delta = p_t - p_r = (-1 - \beta)\rho \tag{4.30}$$

Again, using the field equations (4.21), (4.22) and (4.23) it is possible to show that the pressure asymmetry can be written as:

$$\Delta(r) = -\left(1 + \beta\right)\rho = \frac{\frac{2b}{r^3}}{8\pi + f_T(R,T)}. \tag{4.31}$$

Thus, with Eq. (4.20) for the energy density with $b(r)$ given by Eq.(4.23), we arrive at

$$\frac{1+\beta}{2} = \frac{\left(1 + \frac{f(R,T)}{16\pi\rho}\right)}{\left(1 + \frac{f_T(R,T)}{8\pi}\right)}. \tag{4.32}$$

Since the left-hand side of Eq.(4.32) is a constant, we conclude that

$$f_T = \frac{f(R,T)}{2\rho}, \tag{4.33}$$

and

$$\frac{1+\beta}{2} = 1, \tag{4.34}$$

so

$$\beta = 1. \tag{4.35}$$

Since the trace is $T = 2\rho$, and $f(R,T) = R + h(T)$, Eq.(4.33) implies that

$$h_T(T) = \frac{h(T)}{T}, \tag{4.36}$$

which solution is $h(T) = \lambda T$.

In fact, using Eq.(4.19) with $\beta = 1$, and $\gamma = -1$ we obtain exactly the same solution and, as a consequence $h_T(T) = \lambda$ is a constant. Thus, we have proved that the only expression for $f(R,T)$ that obeys the conservation of the energy-momentum tensor, with linear dependence in ρ for the radial and tangential pressure, is a function linear in the trace T of the energy-momentum.

4.5 Non-exotic conditions for the wormhole to be traversable

Summarizing the results, for the solution $\beta = 1$, and $\gamma = -1$, and then $f(R,T) linear with T, wehav$

$$p_r = \rho, \tag{4.37}$$

$$p_t = -\rho, \tag{4.38}$$

$$T = 2\rho = -\Delta(r). \tag{4.39}$$

Then, substituting this results in the field equation (4.21) lead to an expression for the energy density given by

$$\rho(r) = \frac{-r_o^2}{(8\pi + \lambda)\, r^4}. \tag{4.40}$$

$$(\rho + p_r) = 2\rho = \frac{-2r_0^2}{(8\pi + \lambda)\, r^4}, \tag{4.41}$$

$$(\rho + p_t) = 0. \tag{4.42}$$

Then, in Eq. (4.40), for $\rho(r) \geq 0 \Rightarrow \lambda\ < -8\pi$,, i.e, the Dominant Energy Condition (DEC) can be respected at the wormhole's throat $(r = r_0)$, . In this condition, $\lambda\ < -8\pi$, the Null Energy Condition (NEC) also can be respected, as can be seen in Eqs.(4.41) and (4.42). It is important to stress that in GR where $\lambda = 0$ the energy density is negative and all the energy conditions are violated.

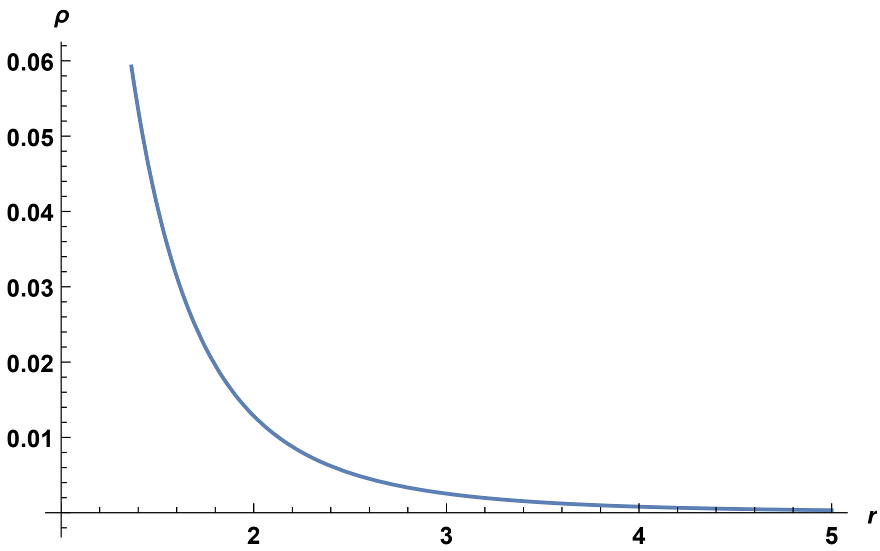

FIGURE 4.1 – The energy density versus r for $r_0 = 1$ and $\lambda = -30$.

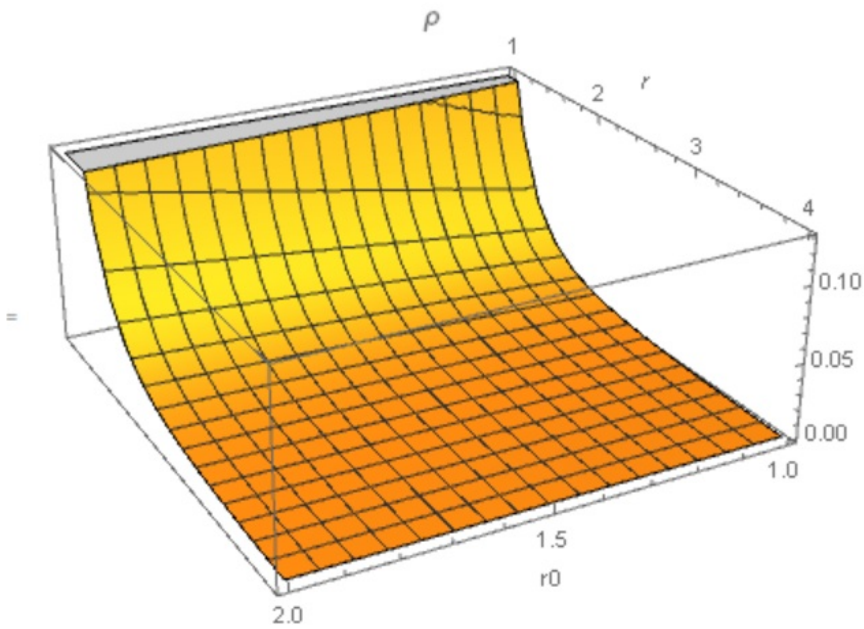

FIGURE 4.2 – The energy density ρ profile with $r_0 = 1$ to 2, $r = 1$ to 4 and $\lambda = -30$.

So, for these solutions, as seen in Figures (4.1), (4.2), and (4.3), in the limit $\lambda < -8\pi$, the Null Energy Condition (NEC), are also respected, no need to fill the wormhole with exotic matter, i.e, $p_r = \rho$, as we found above in the Eq. (4.40). As we already discussed, using only General Relativity framework, as in the Morris-Thorne work (MORRIS; THORNE, 1988), they found that $-p_r > \rho$, leading to the exotic matter solution as the

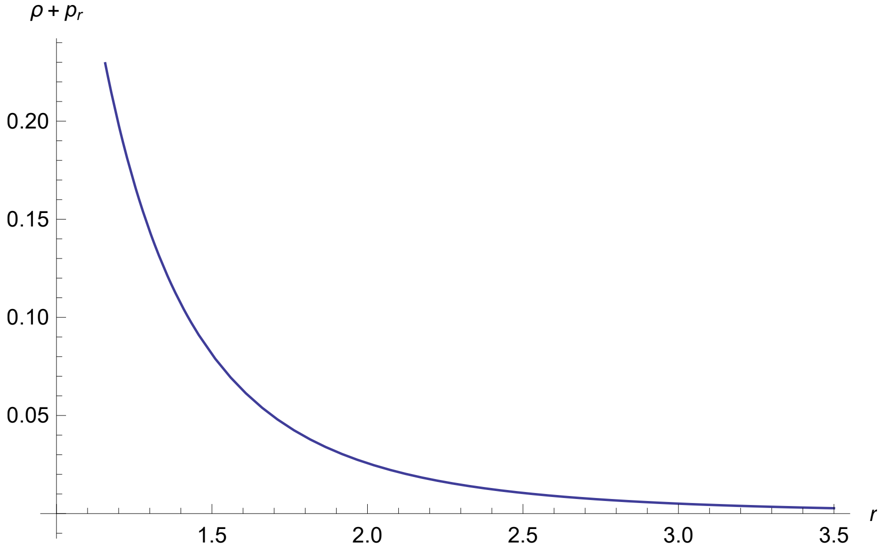

FIGURE 4.3 – The null energy condition (NEC) is obeyed for $r_0 = 1$ and $\lambda = -30$.

condition to maintain the throat of the wormhole opened. Finally, the Strong Energy Condition (SEC) are still obeyed for linear conservative $f(R,T)$ model:

Strong energy condition (SEC), is also obeyed for $\lambda < -8\pi$, with the solutions given by Eqs. (4.40) and (4.41)

$$\rho + \sum_j p_j = \rho + p_r + p_t = \rho > 0, \tag{4.43}$$

4.6 General form for shape function as the solution of the field equations

With the linear function $f(R,T) = R + \lambda T$, the field equations (4.24), (4.25), and (4.26) admits general solutions to ρ, p_r and p_t, given by

$$\rho = \frac{b'}{(8\pi + \lambda)\, r^2}, \tag{4.44}$$

$$p_r = -\frac{b}{(8\pi + \lambda)\, r^3}, \tag{4.45}$$

$$p_t = \frac{b - b'r}{(8\pi + \lambda)\, 2r^3}, \tag{4.46}$$

as was also found in (MORAES; SAHOO, 2017). In this article, the authors present the modeling of wormholes in the $f(R,T)theoryinitsmostgeneralform, withoutconsideringtheconservationo$

momentumtensor.

Assuming a relation between pressures as $p_t = np_r$, with n being an arbitrary constant, substituting it on Eqs (4.45) and (4.46), they found that the shape function which is the solution of the field equation can be written as

$$b(r) = Ar^{1+2n}, \tag{4.47}$$

with A being an integration constant.

Note that for $n = -1$, and $A = r_0$ we recover the more particular case for shape function assumed as given by Eq.(4.24) which was used in this thesis to deduce the energy density and wormhole pressures in the conserved $f(R,T)$ extended gravity model.

Considering Eq.(4.47), the factor n must be negative to satisfy the preliminary conditions to $b(r)$ shown in Eqs. (4.25)-(4.28).

For any value of n, deriving Eq. (4.47) with respect to r, substituting and combining in the Eqs. (4.44), (4.45), and (4.46), the radial and tangential pressure $p_r = \beta\rho$ and $p_t = \gamma\rho$, respectively, can be written with β and γ as a function of n, as

$$\beta = -\frac{1}{1+2n}, \tag{4.48}$$

$$\gamma = -\frac{n}{1+2n}, \tag{4.49}$$

where $p_t = np_r$ with $n = \frac{\gamma}{\beta}$. The trace of the energy-momentum tensor is

$$T = \rho - p_r - 2p_t = \left(1 + \frac{1}{1+2n} + \frac{2n}{1+2n}\right)\rho, \tag{4.50}$$

Thus,

$$T = 2\rho, \tag{4.51}$$

does not depend on n and is exactly the expression we obtained before for $n = -1$.

Taking in account the Eq.(4.8) of the conservation of the energy-momentum tensor we have

$$\left[\left(T^{\mu}_{\nu} + \rho\delta^{\mu}_{\nu}\right)\nabla^{\nu} ln(h_T) + \nabla^{\nu}\left(\rho - \frac{1}{2}T\right)\delta^{\mu}_{\nu}\right] = 0. \tag{4.52}$$

and for $T = 2\rho$, it is clear that $h(T) = \lambda T$ and $h_T = \lambda$ is the solution of the above equation. Thus, $f(R,T) = R + \lambda T$ is the unique form for wormholes in energy-momentum conservative $f(R,T)$ theory, for radial and tangential pressures depending linearly in the energy density ρ, with the general solution to $b(r)$ for traversable wormholes expressed by Eq. (4.47), with $n < 0$.

5 Conclusions

Extended gravity models appear in the literature with the purpose of evading GR problems. This thesis presented two different forms to treat gravitation, namely Space-Time-Matter Model and conservative $f(R,T) gravity$ and used each one of them to get, first, a complete cosmological scenario, and second to construct traversable wormholes without the need to be filled with exotic matter.

In this thesis, there are two unprecedented works in the literature, to which we apply two extended theories of gravitation. With the first described from Section 3, the Space-Time-Matter model, we investigate the evolution of the universe in which the emergence of matter is a purely geometric manifestation on the 4D surface contained in a fifth-dimensional manifold with associated vacuum energy.

The other work, which composes this thesis, in Section 4, shows the possibility of the existence of traversable wormholes without the need to be filled with exotic matter, from a model in which there is a function with a linear dependence with the trace of the energy-momentum tensor.

The imposition of conservation of the energy-momentum tensor for the theory $f(R,T)$ gave rise to a linear function with the energy-momentum tensor trace, with a region in which the energy conditions are respected.

Considering only a general 5D FLRW metric with scale factors acting in the usual three space coordinates and the extra spatial coordinate, and a five-dimensional Einstein equation with a negative 5D bulk cosmological constant, we have obtained analytical solutions for the scale factors involving square roots of hyperbolic functions of the time, as well as an induced EoS for the universe evolution capable of describing the different epochs of the universe in a continuous and analytical form.

We have collected the extra-dimensional dependent terms in the 5D Einstein tensor and "moved" them to the *rhs* of the field equations to play the role of an induced energy-momentum tensor. (SCIAMA, 1953; LIU; MASHHOON, 1995).

From a quite general approach we have obtained some cosmological features particularly interesting. We have shown in Section 3.3.1 that general KK models with a null cosmological constant are restricted to a radiation-dominated universe - which evolves as $a \sim t^{1/2}$. We have shown that the extra-dimension scale factor yields a negative Hubble parameter for the extra coordinate, i.e., the extra coordinate length (naturally) compactifies.

In Section 3.3.2.1, we have inserted a positive 5D cosmological constant in the field equations. The approach has led to a cyclic or bouncing universe, i.e., a universe that goes from a collapsing era to an expanding era without displaying the singularity that standard model carries. Bouncing cosmological models are well-known alternatives to inflation and also provide the cosmological perturbations we see today. For a deeper understanding of bouncing cosmological models, besides (STEINHARDT; TUROK, 2002)-(BATTEFELD; PETER, 2015), we refer the reader to (BRANDENBERGER; PETER, 2017).

In Section 3.3.2.2, we considered $\Lambda < 0$. It is interesting to remark here that usually braneworld models contain negative bulk cosmological constant as a consequence of the appearance of terms $\sim \sqrt{-\Lambda}$ in their Friedmann-like equations (IDA, 2000; BAJC; GABADADZE, 2000).

Our negative cosmological constant model has shown to be able to uniquely describe the radiation, matter, and dark energy eras of the universe evolution in a continuous and analytic form, which can be seen, for instance in Fig. 3.6.

This is a quite non-trivial result. Cosmological models able to describe from a single analytic EoS the whole history of the Universe's evolution are rarely obtained in the literature (MORAES; SANTOS, 2016; LIMA *et al.*, 2013b). References (MORAES; SANTOS, 2016; LIMA *et al.*, 2013b) show cosmological scenarios obtained from $f(R, T^{\phi})$ gravity, with R being the Ricci scalar and T^{ϕ} the trace of the energy-momentum tensor of a scalar field ϕ, and decaying vacuum models, respectively. This interesting feature is a consequence of the remarkable hyperbolic solution obtained here for the scale factor. While we have obtained such a feature from the model, some other approaches use this solution as a prior *ansatz* (CHAWLA *et al.*, 2012)-(NAGPAL *et al.*, 2019). This kind of hyperbolic solution is also found in the flat ΛCDM concordance model, by neglecting radiation (WEINBERG, 1989)-(PIATTELLA, 2018). However, in this case, the time dependence of the scale factor is $a(t) \sim [\sinh(t)]^{2/3}$, departing from the $a(t) \sim [\sinh(t)]^{1/2}$ solution found here. Our solution for small values of time where $\sinh(t) \sim t$ incorporates the correct limit of the radiation era, namely $a(t)^{\frac{1}{2}}$.

Furthermore, from our solutions given by Eqs.(3.30) and (3.31) it is clear that $\rho \rightarrow$ *constant* for high values of time and $p = -\rho$. Moreover, from Eq.(3.34), the Hubble parameter is constant $H = \sqrt{\frac{|\Lambda|}{6}}$ which is also in agreement with ΛCDM model in the dark energy era. Here, the constant value for ρ reads $|\Lambda|/16\pi$, while in standard model it is $\Lambda_4/8\pi$, which implies that $|\Lambda| = 2\Lambda_4$ in this epoch of the Universe. This divergence in the values of the 4D and 5D cosmological constants may be an indication that if measured in 5D, the cosmological constant will have a greater value. This is somehow similar to what was presented in (MCWILLIAMS, 2010), that shows that the very existence of a 5D bulk in the Randall-Sundrum braneworld setup implies in an enhancement in the Hawking radiation emitted by black holes.

Our model satisfactorily fits the observational data for the experimental measurement of the Hubble parameter, as shown in Section 3.5. The adopted method resulted in $H_0 = 72.2^{+5.3}_{-5.5}$ km/s/Mpc, which is in agreement with the most recent estimate from local observations, $H_0 = 74.03 \pm 1.42$ km/s/Mpc (RIESS *et al.*, 2019) and also in agreement with the Planck collaboration estimate, $H_0 = 67.4 \pm 0.5$ km/s/Mpc (AGHANIM; et. al., 2018), in the context of flat ΛCDM cosmology. As a derived parameter, we have obtained the total age of the Universe as $t_0 = 12.59^{+0.69}_{-0.62}$ Gyr, which is in agreement with most of the age estimates today. Jimenez *et al.* (JIMENEZ *et al.*, 2019), for instance, have obtained a weighted average of $t_0 = 13.0 \pm 0.4$ Gyr from 22 globular clusters (O'MALLEY *et al.*, 2017), which is in agreement with our superior limit ($t_0 = 13.28$ Gyr at 95% c.l.). Our result is also in agreement with estimates of absolute ages of very-low-metallicity stars, in the range of 13.0 – 13.535 Gyr, as explained in (JIMENEZ *et al.*, 2019) and references therein.

In Section 4, is presented a static traversable wormhole within the $f(R,T)$ theory in its conservative form, i.e., assuming the conservation of the energy-momentum tensor. For the conservation equation to be satisfied, using linear EoS for the radial and tangential pressures (p_r and p_t) inside the wormhole, we find a $f(R,T)$ linear in the trace of the energy-momentum tensor. In this way, p_r has the highest pressure value possible (stiffest fluid) and $p_t = -1$, which suggests a dark matter halo surrounding the wormhole, something to be investigated in future work. Then, one obtained for p_r the casual limit, i.e, the sound speed in the wormhole interior is given by

$$\frac{dp_r}{d\rho} = 1 = c, \tag{5.1}$$

where c is the speed of light. We found a fluid that is not exotic in the same way as obtained in a recent similar wormhole work in linear f(R, T) gravity (ROSA; KULL, 2022). In this article, the authors analyzed traversable wormhole solutions in the linear form of $f(R,T) = R + \lambda T$ gravity satisfying the energy conditions, but without taking into account the energy-momentum tensor conservation. As we know in its original model, the theory of gravitation $f(R,T)$ does not conserve the energy-momentum tensor, which is very far from Einstein's general relativity, in which the conservation of the energy-momentum tensor is a fundamental condition. The novelty of our work is to assume conservation of the energy-momentum tensor, thus being in greater agreement with reality.

Unlike a Morris-Thorne wormhole, in which the exoticism of the matter is shown with $p_r < -\rho$, using General Relativity, we find $p_r = \rho$, denoting an ultra-relativistic (non-exotic) matter and whose radial pressure is positive,

As we have mentioned in the previous sections all the conditions for the shape function $b(r)$, established for the present traversable wormholes, are respected by the ansatz that we adopted (see Eqs. (4.25 to 4.28)). Furthermore, we have assumed the redshift function is a constant set to zero ($\varphi = 0$), implying in null tidal gravitational forces through the wormhole.

It is important to remark that obtaining wormhole solutions satisfying the WEC is not a trivial task. Here, the extra degrees of freedom provided by $f(R,T)$ gravity formalism allowed the material solutions to obey WEC. Thus, the wormhole can is filled by non-exotic matter, recalling that Morris and Thorne defined "exotic matter" as matter violating WEC. In our case, we find a wormhole that obeys all the energy conditions when $\lambda < -8\pi$. We also show that a shape function in the form expressed by $b(r) = Ar^{1+2n}$, with A being a constant. is the general case of the conserved $f(R,T)$ theory, with a function $f(R,T)$ linear in T. In this case, it is possible to have different values for the proportional radial and tangential pressures, $p_t = np_r$ linear in the energy density, and not only the case we investigated with $n = -1$.

Among the future perspectives that may derive from this work, we can highlight the idea of further analyzing the evolution of the cosmological parameters related to the extra scale factor $\xi(t)$.

An analysis can also be made for the possible influence of a dust term in our cosmological model, in order to improve the concordance with the observations.

We can also try to better constrain the constant c_5 considering more cosmological data sets, and also by the actual values of the equation of state parameter ω and deceleration q. Furthermore, we can consider a fifth-dimension cosmological constant that depends on time.

For the wormholes in the conserved $f(R,T)$ we can investigate the energy conditions for the most general form of the shape function. Finally, verify the possibility of traversable wormholes with more general $f(R,T)$ functions, which are non-linear in T, and still conserve the energy-momentum tensor.

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Appendix A - Covariant Derivative of $T_{\mu\nu}$ in $f(R,T)$ gravity

For the work in Chapter 4, we use the conservative version of the $f(R,T)$ theory. Then we use the covariant derivative equation of energy-momentum tensor on $f(R,T)$ gravity as one see in (BARRIENTOS; RUBILAR, 2014)

To deduce the covariant derivative expression for the energy-momentum tensor in Eq. (4.3), we use the following mathematical identity

$$(\nabla_\nu - \nabla_\nu)\,\phi \equiv R_{\mu\nu}\nabla^\mu\phi, \tag{A.1}$$

which is valid for any field ϕ. In our case, the field can be substituted by $f(R,T)$. Applying the covariant derivative in the left side of field equation (Eq. (4.2)), and use this identity above.

$$\nabla^\mu\left[f_R(R,T)R_{\mu\nu} - \frac{1}{2}f(R,T)g_{\mu\nu} + (g_{\mu\nu} - \nabla_\mu\nabla_\nu)\,f_R(R,T)\right] \tag{A.2}$$

$$= (\nabla^\mu f_R(R,T))\,R_{\mu\nu} + f_R(R,T)\nabla^\mu R_{\mu\nu} - \frac{1}{2}\nabla^\mu f(R,T)g_{\mu\nu} + \nabla^\mu\,(g_{\mu\nu} - \nabla_\mu\nabla_\nu)\,f_R(R,T) \tag{A.3}$$

$$= \nabla^\mu R_{\mu\nu} - \frac{1}{2}\nabla^\mu f(R,T)g_{\mu\nu} + (R_{\mu\nu}\nabla^\mu f_R(R,T))\,. \tag{A.4}$$

Assuming since the begining that $f_R(R,T) = 1$, then $\nabla^\mu(f_R(R,T)) = 0$, and $\nabla^\mu g_{\mu\nu} = 0$, as in general relativity framework.

Finally, using the covariant derivative of $f(R,T)$ in general form, given by

$$\nabla^\mu f(R,T) = f_R(R,T)\nabla^\mu R + f_T(R,T)\nabla^\mu T, \tag{A.5}$$

where $f_R(R,T)$=1.

Then we have the final expression for the covariant derivative of Eq. (4.2) left side

$$\nabla^{\mu}\left[f_R(R,T)R_{\mu\nu}-\frac{1}{2}f(R,T)g_{\mu\nu}+(g_{\mu\nu}-\nabla_{\mu}\nabla_{\nu})\,f_R(R,T)\right]$$

$$=\nabla^{\mu}R_{\mu\nu}-\frac{1}{2}\nabla^{\mu}Rg_{\mu\nu}-\frac{1}{2}g_{\mu\nu}f_T(R,T)\nabla^{\mu}T$$

$$=-\frac{1}{2}g_{\mu\nu}f_T(R,T)\nabla^{\mu}T. \tag{A.6}$$

In the last step above we apply the Bianchi identity for the Einstein tensor,

$$\nabla^{\mu}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)=\nabla^{\mu}G_{\mu\nu}=0. \tag{A.7}$$

This non-vanishing term on the lhs covariant derivative of field equations plays an essential role in $f(R,T)$ gravity.

Then we can write the covariant derivative of field equation (4.2) as

$$\nabla^{\mu}\left[8\pi T_{\mu\nu}-f_T(R,T)T_{\mu\nu}-f_T(R,T)\rho g_{\mu\nu}\right]+\frac{1}{2}g_{\mu\nu}f_T(R,T)\nabla^{\mu}T=0. \tag{A.8}$$

Isolating $\nabla^{\mu}T_{\mu\nu}$, and apllying the ∇^{μ} operator in the other terms results

$$\nabla^{\mu}T_{\mu\nu}=\frac{f_T(R,T)}{8\pi-f_T(R,T)}\left[(T_{\mu\nu}+\rho g_{\mu\nu})\,\nabla^{\mu}ln f_T(R,T)+\nabla^{\mu}\rho g_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\nabla^{\mu}T\right]. \tag{A.9}$$

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7. INSTITUIÇÃO(ÕES)/ÓRGÃO(S) INTERNO(S)/DIVISÃO(ÕES):
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11. RESUMO:

The already established and successful standard model of Cosmology, the Lambda CDM, (an acronym for Lambda Cold Dark Matter Model), whose field equations are derived from General Relativity, describes very well the universe as being isotropic and homogeneous in its distribution of matter and energy. Some solid physical explanations, however, seem to be far from being explained by this model alone, such as the 96\% of energy and matter that fills the universe (Dark Matter and Dark Energy). Einstein's General Relativity has also proved adequate on small scales in explaining the advance of the perihelion of Mercury's orbit around the Sun as well as the bending of light, and the Relative Astrophysics in describing neutron stars and predicting the existence of black holes. In this thesis, we are going to explore extensions of the theory of General Relativity: the Induced Matter Model – or STM Model (Space-Time-Matter Model), and the f(R,T) conserved theory of gravity, where R is Ricci´s scalar of curvature and T the trace of the energy-momentum tensor. We applied the STM model to Cosmology and obtained a unique equation of state for the three eras of the universe (radiation, matter, and dark energy). This model presents our universe, 4-dimensional, and all the matter in it, as a geometric manifestation on the surface of a 5-dimensional space-time vacuum, with the energy associated with that vacuum. All cosmological parameters were analyzed and compared with observational data from the ΛCDM model. We also analyzed traversable wormholes. This study was carried out in the light of the conserved f(R, T) theory, that is, imposing the conservation of the energy-momentum tensor in the theory. In this second work, we analyze parameters that satisfied energy conditions inside the wormhole without the need to be filled with exotic matter, as occurs as a condition for them to be traversable in the solutions obtained from General Relativity.

12. GRAU DE SIGILO:
(**X**) **OSTENSIVO** ( ) **RESERVADO** ( ) **SECRETO**