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KLEIN-GORDON OSCILLATOR IN COSMIC STRING SPACETIME IN THE PRESENCE OF ELECTRIC AND MAGNETIC FIELD WITH A COULOMB AND CORNELL POTENTIAL

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Pablo de Deus Silva

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I dedicate this work to my mother, Ana Claudia Brandão de Deus. I hope to make her proud after all the hard work she has put in for me to be here.

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"Living is worthwhile if one can contribute
in some small way to this endless chain of progress."
— Paul A.M. Dirac

Resumo

Neste trabalho, analisamos um Oscilador de Klein-Gordon (KGO) em um espaço-tempo produzido por um defeito topológico unidimensional, uma corda cósmica, na presença de campos elétricos e magnéticos com potenciais de Coulomb e Cornell. Para desenvolver a análise, separamos os campos de interação em cinco configurações diferentes, considerando: *i*) partícula livre, *ii*) campo elétrico radial estático, *iii*) campo elétrico axial estático, *iv*) campo magnético angular estático e *v*) campo magnético axial estático na presença de um potencial de Coulomb, um potencial de Cornell e um fluxo magnético que apresenta um sistema análogo ao efeito Aharonov-Bohm para estados ligados. As equações de movimento são resolvidas analiticamente para todos os cenários, com exceção do campo magnético na direção angular, em que uma solução numérica é apresentada para a densidade de probabilidade da posição da partícula. Seu comportamento é estudado alterando-se a intensidade do campo de interação, a densidade de massa linear da corda e a energia. Também é apresentada uma interpretação dos campos e fluxos magnéticos como não comutativos no espaço de momento. Os observáveis físicos, como a energia e o momento linear, são quantizados em diferentes sistemas físicos e sua dependência do parâmetro da corda e dos campos de interação é apresentada.

Abstract

In this work, we analyze a Klein-Gordon Oscillator (KGO) in a spacetime produced by a one-dimensional topological defect, a cosmic string, in the presence of electric and magnetic fields with Coulomb and Cornell potentials. To develop the analysis, we separate the interaction fields into five different configurations, considering: *i*) free particle, *ii*) static radial electric field, *iii*) static axial electric field, *iv*) static angular magnetic field and *v*) static axial magnetic field in the presence of a Coulomb potential, a Cornell potential and a magnetic flux that presents a system analogous to the Aharonov-Bohm effect for bound states. The equations of motion are solved analytically for all scenarios, with the exception of the magnetic field in the angular direction, where a numerical solution is presented for the probability density of the particle's position. Its behavior is studied by changing the intensity of the interaction field, the linear mass density of the string and the energy. An interpretation of magnetic fields and fluxes as non-commutative in momentum space is also presented. Physical observables such as energy and linear momentum are quantized in different physical systems and their dependence on the string parameter and the interaction fields is presented.

Résumé

Dans ce travail, nous analysons un oscillateur de Klein-Gordon (KGO) dans un espace-temps produit par un défaut topologique unidimensionnel, une corde cosmique, en présence de champs électriques et magnétiques avec des potentiels de Coulomb et de Cornell. Pour développer l'analyse, nous séparons les champs d'interaction en cinq configurations différentes, en considérant : *i*) particule libre, *ii*) champ électrique radial statique, *iii*) champ électrique axial statique, *iv*) champ magnétique angulaire statique et *v*) champ magnétique axial statique en présence d'un potentiel de Coulomb, d'un potentiel de Cornell et d'un flux magnétique qui présente un système analogue à l'effet Aharonov-Bohm pour les états liés. Les équations du mouvement sont résolues analytiquement pour tous les scénarios, à l'exception du champ magnétique dans la direction angulaire, où une solution numérique est présentée pour la densité de probabilité de la position de la particule. Son comportement est étudié en changeant l'intensité du champ d'interaction, la densité de masse linéaire de la corde et l'énergie. Une interprétation des champs et des flux magnétiques comme non-commutatifs dans l'espace des quantités de mouvement est également présentée. Les observables physiques telles que l'énergie et la quantité de mouvement linéaire sont quantifiées dans différents systèmes physiques et leur dépendance au paramètre de la corde et aux champs d'interaction est présentée.

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List of Abbreviations and Acronyms

CS	Cosmic String
GR	General Relativity
KG	Klein-Gordon
KGO	Klein-Gordon Oscillator
ODE	Ordinary differential equation
PDE	Partial differential equation
FDM	Finite-difference method
SOR	Successive Overrelaxation
BCH	Biconfluent Heun equation

List of Symbols

$\mathcal{L}$	Lagrangian density
m	Mass of the scalar field
ω_0	Oscillator's frequency
$\mathcal{E}$	Energy
L	Angular momentum
k	Angular momentum
α	Cosmic String parameter
η_C	Lagrangian density
η_L	Lagrangian density
Φ_B	Lagrangian density
ξ_C	Einstein gravitational constant
$T_{\mu\nu}$	Energy-Momentum Tensor
$g_{\mu\nu}$	Metric Tensor
$\eta_{\mu\nu}$	Minkowski metric
$h_{\mu\nu}$	Perturbation
χ	Einstein gravitational constant
p	Pressure
μ	String Linear Mass Density
G	Green's function
A_μ	Gauge potential
$F_{\mu\nu}$	Field Strength Tensor
g	Coupling constant
$\hat{e}_j$	Unit vector in the j-direction
δ	Delta operator or variation operator

Contents

1 Introduction

One of the most studied topological defects in the last decades is the cosmic string (BEZERRA, 1991; MARQUES; BEZERRA, 2003; BAKKE, 2012; CASTRO, 2015). This hypothetical one-dimensional structure was proposed in the 1970s (KIBBLE, 1976), and the interest in its study is due to the fact that this object may have contributed to the agglutination of matter that would give rise to super clusters of galaxies, justifying the filamentary structure found in some observations (SATO, 1986). The presence of the cosmic string has significant effects on the structure of the spacetime in which it is embedded – for instance, we can expect gravitationally lensed objects, and indeed a potential candidate is currently under scrutiny (SAFONOVA *et al.*, 2023). The space with the cosmic string is globally conical, although its geometry is locally flat/pseudo-Euclidean, with an azimuthal deficit angle related to the string tension (VILENKIN; SHELLARD, 2000; VILENKIN, 1985).

The general form for the metric associated with a static cosmic string spacetime with signature $diag(-1,+1,+1,+1)$ in cylindrical coordinates (t,ρ,ϕ,z) is

$$ds^2 = -dt^2 + d\rho^2 + \alpha^2\rho^2 d\phi^2 + dz^2, \tag{1.1}$$

where $\rho \in (0,\infty)$, $\phi \in [0,2\pi]$, $z \in (-\infty,\infty)$, and the parameter α is the deficit angle that is determined by the the string linear mass density μ, by means of the relations $\alpha = 1 - 4\mu \in (0,1]$. The presence of α in the metric implies that that the propagation of fields in this spacetime will be modified in comparison with the propagation in Minkowsky space.

An interesting approach is the interaction between such spacetime and quantum fields. This approach is appealing once it opens the perspective of detection of cosmic strings by means other than the gravitational effects. Two of the most studied fields in spacetimes with topological defects are the Dirac (CUZINATTO *et al.*, 2019) and Klein-Gordon fields (BOUMALI; MESSAI, 2014; AHMED, 2020; AHMED, 2021).

The latter is the field of our interest in the present work, particularly, the Klein–Gordon Oscillator (KGO), which was recently examined in the cosmic string spacetime (CUZINATTO *et al.*, 2022).

The KGO is represented by a complex field presenting U(1) symmetry, which means that it describes a spin-zero charged particle. Hence, the KGO interacts electromagnetically and its dynamics is modified in the presence of electric and magnetic fields, which are known to be generated by astrophysics objects. In this context, we are interested in the dynamics of the Klein-Gordon oscillator in a static cosmic string spacetime in the presence of an electric and a magnetic field. The latter can be introduced through the minimal coupling prescription, as usual in gauge theories (UTIYAMA, 1956; ACEVEDO R.R. CUZINATTO; POMPEIA, 2018), but can also be interpreted as a parameter in non-commutative geometry (CUZINATTO *et al.*, 2022).

The implementation of potentials in the Klein Gordon field under the presence of cosmic strings has recently become something of great interest, especially in the presence of scalar potentials (MEDEIROS; MELLO, 2012). In this context, a simple potential that incorporates both the Coulomb potential and the confinement potential is the Cornell potential

$$S\left(\rho\right) = \frac{\eta_C}{\rho} + \eta_L \rho, \tag{1.2}$$

where η_C and η_L are constants. This potential is composed of a confinement linear term plus a Coulombian-type term. The work (DOSCH *et al.*,) shows us that the minimal coupling is not the only way to couple a potential, in particular to the Dirac equation. It was suggested that the non-electromagnetic potential can be introduced though a modification in the mass term as $M \to M + S\left(\rho\right)$.

Recently, a magnetic flux has been taken into account in the context of topological defects (AHMED, 2021). The main reason is the interpretation of the effect generated by the flux as an Aharonov-Bohm effect, a quantum phenomenon that describes phase changes in the wave function of a quantum particle due to the presence of a quantum flux, in this case produced by topological defects in space-time.

In this work, we analyze the Klein-Gordon Oscillator in a spacetime produced by an idealized cosmic string in the presence of a constant electric field and a uniform magnetic field. In order to develop this analysis, we separated in four distinct configurations for the interactions fields: *i*) free particle, *ii*) a constant radial electric field, *iii*) a constant axial electric field *iv*) a static angular magnetic field, and *v*) a static axial magnetic field in the presence of an electrostatic and a scalar potential, and a magnetic flux.

We begin with a description of the spacetime created by the presence of a cosmic string, finding the metric tensor associated, section 2. Then in the section 3, we developed the Klein-Gordon Oscillator via a non-minimal coupling in the usually Klein-Gordon Lagrangian density. We show that the KGO is invariant under global phase transformation and how to preserve it under a local phase transformation by imposing an interaction field, A, via minimal coupling. This invariance allow us to introduce the interaction field, electric and magnetic field, via gauge theory. The two configurations of the electric field are analyzed in the section 4, the fields are introduced by minimal coupling and the resulting equations of motions are solved and energy spectra are explicitly found for the axial field scenario. The section 5 is related to the KGO in magnetic fields, two scenarios are explored here, the magnetic field in the angular direction and axial direction. For angular magnetic field, the equation of motion is presented and we use a numerical method to find a general solution for probability density of the particle's position. We also show that the magnetic field configuration in the angular direction can be interpreted as a non-commutative geometry. For the axial magnetic field, we introduce the Cornell potential by making a modification in the mass term, we also introduce an electrostatic potential and a magnetic flux via minimal coupling. The scenarios are analyzed with each potential individually, the associated equation of motion is solved and the energy is quantized. A mapping is also made for the case of the magnetic field with the magnetic flux with non-commutative geometry.

2 cosmic string

Phase transitions in gauge theories can lead to the spontaneous symmetries breaking in the evolution of the Universe (KIBBLE, 2015). As we go back in time the Universe becomes hotter and for a sufficiently large temperature, $T > T_c$[1], these symmetries can be restored (LINDE, 1979). The phase transition at $T = T_c$ can give rise to some cosmological consequences, some of them are a vacuum domain structures, as in a ferromagnet cooled through its Curie point, which separates domains with different magnetization formation of domain walls. As the Universe's temperature goes lower than T_c, the Higgs field ϕ obtains non-zero vacuum expectation value $< \phi >$. In each region of space, the direction of the $< \phi >$ in the manifold of degenerate vacuum states can be different, the topology of the resulting vacuum structure is related to the topology of the manifold. The possible topological configurations that arise in the Universe's evolution are domain walls, strings and monopoles (KIBBLE, 1976). Our goal in this chapter is to describe gravitation properties of a cosmic string in the framework of general relativity.

2.1 Energy-momentum tensor

In this work, we are interested in the macroscopic behavior in spacetime generated by a stretched cosmic string in the z-direction. In order to construct it, we shall use the energy-momentum tensor as a starting point. To do this, we need to find the tensor for a thin string approximation. The stress energy tensor in field theory is defined as (LANDAU *et al.*, 1975)

$$T_\mu^\nu = \sum_i \frac{\partial L}{\partial \left(\partial_\nu \phi^{(i)}\right)} \left(\partial_\mu \phi^{(i)}\right) - \delta_\mu^\nu L, \tag{2.1}$$

where $L\left(\phi^{(i)}; \partial_\mu \phi^{(i)}\right)$ is the Lagrangian of the theory and the summation is taken over all fields $\phi^{(i)}$.

Our interest is in a macroscopic effect of the string. So it is reasonable to approximate it by an infinitely thin curve. Let us consider a static string parallel to the z axis in a flat

[1] T_c is the critical temperature.

spacetime, the energy-momentum tensor can be written as

$$\tilde{T}_{\mu}^{\nu}(x,y) = \delta(x-a)\,\delta(y-b)\int T_{\mu}^{\nu}(x',y')\,dx'dy', \tag{2.2}$$

where, $x = a$ and $y = b$ is the position of the string, and δ is Dirac delta distribution.

Assuming that all the fields $\phi^{(i)}$ are functions only of x and y, from the definition 2.1 we see

$$\tilde{T}_{0}^{0} = \tilde{T}_{3}^{3}, \tag{2.3}$$

since the stress-energy tensor is invariant under spacetime translation (SCHWARTZ, 2013), it means that the divergence of the tensor is zero, in our case we can express it in terms of the partial derivatives as

$$\partial_{\nu}\tilde{T}_{\mu}^{\nu} = 0. \tag{2.4}$$

Considering a perfect fluid, in this case, $\tilde{T}_{\mu}^{\nu}$ has only diagonal components. Therefore, for the spacetime index $\mu = 1$ and $\mu = 2$:

$$\partial_1 T_1^1 = 0 \rightarrow T_1^1 = \text{const}; \tag{2.5a}$$

$$\partial_2 T_2^2 = 0 \rightarrow T_2^2 = \text{const}. \tag{2.5b}$$

We desire that the stress-energy tensor vanish at the infinity, $x \rightarrow \pm\infty$ and $y \rightarrow \pm\infty$, we conclude that $T_1^1 = T_2^2 = 0$. Thus, the general form of the energy-momentum tensor of a homogeneous cosmic string is as follows:

$$\tilde{T}_{\mu}^{\nu}(x,y) = \delta(x-a)\,\delta(y-b)\begin{pmatrix}\mu & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -p\end{pmatrix}, \tag{2.6}$$

where μ is the linear energy density and p is the pressure in the z direction. The specific case where $p = -\mu$, string vacuum (VILENKIN, 1981), $\tilde{T}_{\mu}^{\nu}$ has the form

$$\tilde{T}_{\mu}^{\nu}(x,y) = \mu\delta(x-a)\,\delta(y-b)\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{2.7}$$

We can also calculate the trace of energy-momentum tensor,

$$\tilde{T}(x,y) = \delta(x-a)\,\delta(y-b)\,(\mu - p)\,. \tag{2.8}$$

2.2 Weak-Field Approximation

Just as Maxwell's equations govern how the electric and magnetic fields respond to charges and currents, Einstein's field equation govern how the metric responds to energy and momentum (CARROLL *et al.*, 2004). Therefore, to find the effect due to the string, we must solve the field equations proposed in 1915 by Albert Einstein (EINSTEIN, 1915), that can be written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \chi T_{\mu\nu}, \tag{2.9}$$

where $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ is the Ricci curvature, $R = g^{\mu\nu}R_{\mu\nu}$ is the trace of $R_{\mu\nu}$ and it is called scalar of curvature or the Ricci scalar, χ is the Einstein gravitational constant and $T_{\mu\nu}$ is the stress-energy tensor.

As our aim is to find the metric generated by the presence of a static cosmic string in the z-direction, we need to solve the Einstein field equation for the energy-momentum tensor found in the previous section, Eq. 2.6.

For weak fields, an interesting approach is to rewrite the metric tensor, $g_{\mu\nu}$, as being a Minkowski metric $\eta_{\mu\nu}$ plus a small perturbation term $h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$. Thus, we can represent the metric tensor as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.10}$$

and Eq. 2.9 can be written as (SABBATA; GASPERINI, 1986)

$$\frac{1}{2}\left(\nabla^2 - \partial_t^2\right) h_{\mu\nu} = 8\pi \left(\eta_{\sigma\nu}T^{\sigma}_{\mu} - \frac{1}{2}\eta_{\mu\nu}T\right), \tag{2.11}$$

with the harmonic coordinate conditions (also called harmonic gauge and Lorenz gauge)

$$\partial_\nu \left(h^{\nu}_{\mu} - \frac{1}{2}\delta^{\nu}_{\mu}h\right) = 0, \tag{2.12}$$

where T and h are the trace of $T_{\mu\nu}$ and $h_{\mu\nu}$ respectively.

In this work, we use the natural units (Planck units), that means

$$c = \hbar = G = 1, \tag{2.13}$$

where c is the speed of light, $\hbar$ is the reduced Planck constant and G is the gravitational constant.

Since $\eta_{\mu\nu}$ is a diagonal matrix, we only have four non-zero fields equations. Substituting the energy-momentum tensor of a homogeneous static massive cosmic string, Eq. 2.6, into

Eq. 2.11, we obtain

$$\frac{1}{2}\nabla^2 h_{00} = -4\pi\delta\left(x-a\right)\delta\left(y-b\right)\left(\mu+p\right); \tag{2.14a}$$
$$\frac{1}{2}\nabla^2 h_{11} = -4\pi\delta\left(x-a\right)\delta\left(y-b\right)\left(\mu-p\right); \tag{2.14b}$$
$$\frac{1}{2}\nabla^2 h_{22} = -4\pi\delta\left(x-a\right)\delta\left(y-b\right)\left(\mu-p\right); \tag{2.14c}$$
$$\frac{1}{2}\nabla^2 h_{33} = -4\pi\delta\left(x-a\right)\delta\left(y-b\right)\left(\mu+p\right). \tag{2.14d}$$

As we have a static cosmic string, the perturbation does not depend on time and therefore the time derivative in the Eq. 2.11 vanish. However, instead of working directly with these expressions, we shall consider summarizing the equations as

$$\nabla^2 h_{ii} = a_i\delta\left(x-a\right)\delta\left(y-b\right), \quad i = 0, 1, 2 \text{ and } 3, \tag{2.15}$$

with

$$a_0 = a_3 = -8\pi\left(\mu+p\right), \tag{2.16a}$$
$$a_1 = a_2 = -8\pi\left(\mu-p\right). \tag{2.16b}$$

In order to solve the Eq. 2.12, we can rewrite it in polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h_{ii}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 h_{ii}}{\partial \theta^2} = a_i\delta\left(r\right), \tag{2.17}$$

where $r = \sqrt{\left(x-a\right)^2 + \left(y-b\right)^2}$. Considering that the perturbations is a function only of the radial coordinate, we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h_{ii}}{\partial r}\right) = a_i\delta\left(r\right). \tag{2.18}$$

For $r \neq 0$, this is a Euler–Cauchy differential equation (KREYSZIG, 2010), the solution for this problem are well known but for the sake of completeness we are going to develop it. We seek to solve this problem using a Green's function G that satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) = 0, \quad h_{ii}\left(r\right) = \int G\left(r, r_0\right)\delta\left(r_0\right)dr_0, \tag{2.19}$$

for $r > 0$.

The approach to find G is to multiply by r and then integrate both sides of the equation in relation to r. So, we get

$$\frac{\partial G}{\partial r} = \frac{A}{r}, \tag{2.20}$$

where A is a constant.

This is a separable ODE and the solution is as follows:

$$G(r) = A \ln r + B, \tag{2.21}$$

where B is a constant.

Now, our goal is to find the constant A. Therefore, we go back to the Eq. 2.19 and calculate an integral over a disk D_ε of radius ε,

$$\int\int_{D_\varepsilon} \nabla^2 G dA = \int\int_{D_\varepsilon} a_i \delta(r)\, dA, \tag{2.22}$$

on the left side we use the Divergence Theorem to rewrite the divergence integral over the disc in term of the integral on the disc boundary,

$$\int\int_{D_\varepsilon} \nabla(\nabla G)\, dA = \int_{C_\varepsilon} \vec{\nabla} G \cdot \vec{n} dS, \tag{2.23}$$

where C_ε is the boundary of the disc D_ε, that means that C_ε is a circle of circumference $2\pi\varepsilon$. So we can develop Eq. 2.23 as

$$\int_{C_\varepsilon} \vec{\nabla} G \cdot \vec{n} dS = \int_{C_\varepsilon} \left.\frac{dG}{dr}\right|_{r=\varepsilon} dS = \int_{C_\varepsilon} \left.\frac{d}{dr}(A \ln r + B)\right|_{r=\varepsilon} dS = \int_{C_\varepsilon} \frac{A}{\varepsilon} dS = 2\pi A. \tag{2.24}$$

The right side of Eq. 2.22 is

$$\int\int_{D_\varepsilon} a_i \delta(r)\, dA = a_i. \tag{2.25}$$

Using the results 2.24 and 2.25 into Eq. 2.22, we can find the constant A,

$$A = \frac{a_i}{2\pi}, \tag{2.26}$$

and defining the constant B as $B = -A \ln r_0$, where r_0 is a constant, which we can set to be equal to the radius of the string (VILENKIN, 1981). Thus, the complete solution can be express as

$$G(r) = \frac{a_i}{2\pi} \ln \frac{r}{r_0}. \tag{2.27}$$

Using the Eq. 2.19, we can write the components of the perturbation term as

$$h_{00} = h_{33} = -4(\mu + p) \ln \frac{r}{r_0}, \tag{2.28a}$$

$$h_{11} = h_{22} = -4(\mu - p) \ln \frac{r}{r_0}. \tag{2.28b}$$

For a vacuum string, $\mu = -p$ (VILENKIN, 1981) and

$$h_{00} = h_{33} = 0, \tag{2.29a}$$

$$h_{11} = h_{22} = -8\mu \ln \frac{r}{r_0}. \tag{2.29b}$$

Writing the Minkowski metric $\eta_{\mu\nu}$ and the pertubation $h_{\mu\nu}$ in the cylindrical coordinates, we can use the weak-field approximation, Eq. 2.10, to find the metric

$$(g_{\mu\nu}) = (\eta_{\mu\nu}) + (h_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 - 8\mu \ln \frac{r}{r_0} & 0 & 0 \\ 0 & 0 & r^2 - 8r^2\mu \ln \frac{r}{r_0} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.30}$$

and the line element associated with the presence of the cosmic string is

$$ds^2 = -dt^2 + (1 - \Lambda)\left(dr^2 + r^2 d\phi^2\right) + dz^2, \tag{2.31}$$

where $\Lambda = 8\mu \ln (r/r_0)$. Introducing a new radial coordinate ρ as

$$(1 - \Lambda)\, r^2 = (1 - 8\mu)\, \rho^2. \tag{2.32}$$

In order to take its differential, start by rewriting Eq. 2.32 as

$$r^2 \frac{\left(1 - 8\mu \ln \frac{r}{r_0}\right)}{(1 - 8\mu)} = \rho^2. \tag{2.33}$$

Now we use the approximation $\frac{1}{1-x} \approx 1 + x$, $x \ll 1$. Then

$$r^2 \left(1 - 4\mu \ln \frac{r}{r_0} + 4\mu\right)^2 \approx \rho^2. \tag{2.34}$$

Note that the term in parenthesis can be expressed as below if we consider the linear order of approximation:

$$r^2 \left(1 - 8\mu \ln \frac{r}{r_0} + 8\mu\right) \approx \rho^2. \tag{2.35}$$

Considering that terms proportional to μ^2 are ignored since they are of second order compared to those linear in μ, the following equation is valid

$$r^2 \left(1 - 4\mu \ln \frac{r}{r_0} + 4\mu\right)^2 \approx \rho^2. \tag{2.36}$$

Rewriting it as

$$r_0\left(\frac{r}{r_0}-4\mu\left(\frac{r}{r_0}\ln\frac{r}{r_0}-\frac{r}{r_0}\right)\right)\approx\rho. \tag{2.37}$$

and we use the following result

$$x\ln x-x=\int dx\ln x, \tag{2.38}$$

we obtain

$$\int\left(1-4\mu\ln\frac{r}{r_0}\right)dr\approx\rho. \tag{2.39}$$

In particular, when $x \ll 1$ we have $\sqrt{1-x}\approx 1-\frac{x}{2}$ (recall that we are using the approximation $h_{\mu\nu}\ll 1$). For the present case:

$$\int\sqrt{\left(1-8\mu\ln\frac{r}{r_0}\right)}dr=\rho \tag{2.40}$$

and the following result is obtained

$$(1-\Lambda)\,dr^2=d\rho^2. \tag{2.41}$$

Applying this result in Eq. 2.31, we obtain

$$ds^2=-dt^2+d\rho^2+\alpha^2\rho^2d\phi^2+dz^2, \tag{2.42}$$

where $\alpha^2=(1-4\mu)^2$. This is the spacetime metric of the static cosmic string with signature diag$(-1,1,1,1)$ in cylindrical coordinates (t,ρ,ϕ,z), where $\rho\in(0,\infty)$, $\phi\in[0,2\pi]$, $z\in(-\infty,\infty)$, and parameter α is related to the deficit angle and it obeys $\alpha=1-4\mu\in(0,1]$, μ is the string linear mass density. The matrix form and the inverse of the metric tensor are

$$(g_{\mu\nu})=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&\alpha^2\rho^2&0\\0&0&0&1\end{pmatrix}\rightarrow(g^{\mu\nu})=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&\frac{1}{\alpha^2\rho^2}&0\\0&0&0&1\end{pmatrix}, \tag{2.43}$$

and its determinant is

$$g=\det\left(g_{\mu\nu}\right)=-\alpha^2\rho^2. \tag{2.44}$$

The presence of the cosmic string does not describe an Euclidean space, since the angular coordinate, ϕ, used to make a complete turn from 0 to 2π, however, the presence of the string causes the angular coordinates changes from 0 to $2\pi\,(1-4\mu)$. The angle is changed by a deficit angle, $\theta=2\pi\,(1-\alpha)$, which is related to the string tension (ZWIEBACH,

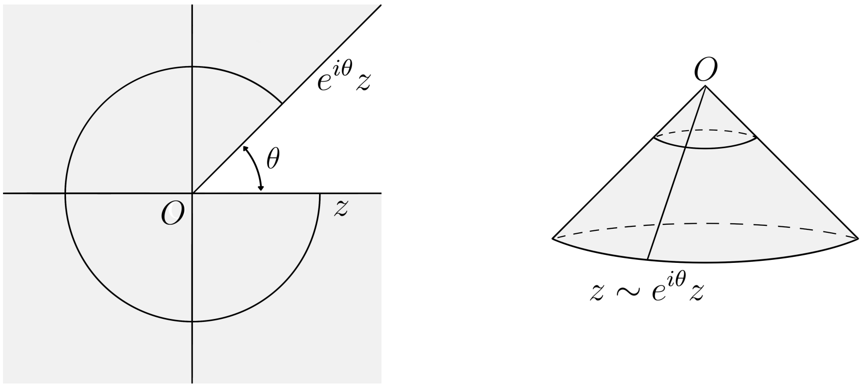


FIGURE 2.1 – (a) The complex plane, $z = x + iy$, cut by the region $0 \leq \arg(z) \leq \theta$. (b) The cone create by the identification $z \sim e^{i\theta}z$.

2004). Such space can be called conical since the string runs along the apexes of the cones. This can be seen in the construction of a cone from the plane. If we use the complex variable $z = x + iy$ to represent the plane, the cone appears cutting through the region $0 \leq \arg(z) \leq \theta$, see figure 2.1(a), and identifying the resulting boundaries through $z \sim e^{i\theta}z$, as showing in the figure 2.1 (b). For $\alpha = 1$, that means $\mu = 0$, we do not have a string in our background and the metric becomes the Minkowski metric in cylindrical coordinates.

An interesting aspect about topological defects is that the presence of them do not play a role in the spacetime curvature. They do not create a gravitation field, since they are defects in the topology of spacetime. In particular the cosmic string create an azimuthal deficit angle related to its tension. Now we are going to explore the curvature tensor generated by the cosmic string metric and show that geometry is locally flat/pseudo-Euclidean.

In General Relativity (GR), the object that tell us about curvature is the Riemann curvature tensor, ${R_{\alpha\nu\mu}}^{\rho}$, also known as the Riemann-Christoffel curvature tensor (WEINBERG, 1972),

$$ {R_{\alpha\nu\mu}}^{\rho} = \partial_\alpha \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\alpha\mu} + \Gamma^{\beta}_{\nu\mu}\Gamma^{\rho}_{\alpha\beta} - \Gamma^{\beta}_{\alpha\mu}\Gamma^{\rho}_{\nu\beta}, \tag{2.45} $$

where $g_{\mu\nu}$ is the metric tensor and ${\Gamma_{\mu\nu}}^{\rho}$ are the Christoffel symbols defined as

$$ {\Gamma_{\mu\nu}}^{\rho} = \frac{1}{2} g^{\rho\beta} \left(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu} \right). \tag{2.46} $$

The curvature tensor has some symmetry properties over its indices, as shown below

$$R_{\mu\nu\alpha\beta} = - R_{\mu\nu\beta\alpha}, \tag{2.47a}$$

$$R_{\mu\nu\alpha\beta} = - R_{\nu\mu\alpha\beta}, \tag{2.47b}$$

$$R_{\mu\nu\alpha\beta} = R_{\mu\alpha\nu\beta}. \tag{2.47c}$$

These symmetry properties help us to calculate all the components of the tensor.

The contraction of two indices of the Riemann tensor define a new tensor called the Ricci curvature tensor, $R_{\mu\nu} = g^{\rho\alpha} R_{\rho\alpha\mu\nu}$, of which we can also contract the indices and obtain the scalar curvature (Ricci scalar), $R = g^{\mu\nu} R_{\mu\nu}$.

Here, we will calculate the Riemann tensor using $g_{\mu\nu}$ generated by the presence of the cosmic string, Eq. 2.42. First, we calculate the Christoffel symbols using 2.46 and the only non-zero symbols are

$${\Gamma_{22}}^{1} = -\alpha^2 \rho, \tag{2.48a}$$

$${\Gamma_{12}}^{2} = {\Gamma_{21}}^{2} = \frac{1}{\rho}. \tag{2.48b}$$

Now we can calculate the curvature tensor applying 2.48 into 2.45, the results is

$${R_{\alpha\nu\mu}}^{\rho} = 0, \tag{2.49}$$

for all values of the space time indices α, ν, μ and ρ.

The Riemann tensor vanishes, hence the presence of the string does not bend the spacetime, it still has a flat geometry. As a consequence of 2.49, the associated Ricci tensor, $R_{\mu\nu}$, and the Ricci scalar, R, are also equal to zero.

3 Klein-Gordon Oscillator

This chapter is dedicated to study of the Klein-Gordon Oscillator (KGO) in a cosmic string background and we briefly introduce gauge theory. This chapter is organized as follows. We start by developing the Klein-Gordon Oscillator by applying a coupling and then the KGO equation on a cosmic string spacetime is solved. An introduction to the gauge theory developed by Utiyma explains how to preserve the invariance of a system of fields $\phi^a(x)$ under a local transformation, and we show that it is necessary to introduce a new field $A_\mu(x)$ to maintain the invariance. This auxiliary field interacts with ϕ as manifested by the covariant derivative $\nabla_\mu\phi$. We then show the invariance of the KGO under a global phase transformation and how we preserve it for a local phase transformation by introducing the gauge field.

3.1 Klein-Gordon Oscillator

In this section, we are going to introduce the Klein-Gordon equation, a relativistic wave equation that describes spin-0 particles and antiparticles with charge, and then we shall introduce the oscillator term in the momentum by applying the non-minimal coupling in KG equation.

The Lagrangian for the complex scalar field $\psi(x)$ on a curved spacetime is

$$\mathcal{L} = \sqrt{-g}\left(-g^{\mu\nu}\partial_\mu\psi^*\partial_\nu\psi - m^2\psi^*\psi\right), \tag{3.1}$$

where $g^{\mu\nu}$ is the inverse of the metric tensor, g is the determinant of the metric tensor and m is the field mass. Using the Euler-Lagrange equation for the ψ^* field, given by

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}\right) - \frac{\partial\mathcal{L}}{\partial\psi^*} = 0, \tag{3.2}$$

we can find that the equation governing the massive scalar field has the form

$$\left(\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\nu\mu}\partial_\nu\right) - m^2\right)\psi = 0, \tag{3.3}$$

this is known as the Klein-Gordon equation. It is relativistically covariant and it has solution that we expect for a free relativistic particle in Minkowski spacetime. We expect the time dependence to be like $\exp(-i\mathcal{E}t)$, where $\mathcal{E}$ is an eigenvalue of the Hamiltonian, and the spatial dependence to be a plane wave, $\exp(i\vec{p}\cdot\vec{x})$ for momentum $\vec{p}$ (SAKURAI; NAPOLITANO, 2020).

The equation for the Klein-Gordon Oscillator is obtained from the KG Lagrangian density, Eq. 3.1, by applying the non-minimal coupling (MOSHINSKY; SZCZEPANIAK, 1989; BAKKE, 2013; NEDJADI; BARRETT, 1994; CUZINATTO *et al.*, 2022)

$$\vec{p}\psi \to (\vec{p}-im\omega_0\vec{r})\psi, \quad \vec{p}\psi^* \to (\vec{p}+im\omega_0\vec{r})\psi^*, \tag{3.4}$$

the KG field has a mass m and the oscillator's frequency is denoted by ω_0. Hereafter, we use cylindrical coordinates and set

$$\vec{r} = \rho\hat{e}_\rho + z\hat{e}_z. \tag{3.5}$$

Since the momentum operator in quantum mechanics is $p_j = -i\partial_j$ ($j = 1,2,3$, respectively for ρ, ϕ and z), we can rewrite the expression 3.4 as $-i(\nabla \pm m\omega_0\vec{r})$, where $+$ applies to ψ and $-$ for its conjugate. Therefore, we make the following prescription:

$$\partial_\rho \to \partial_\rho \pm m\omega_0\rho, \tag{3.6a}$$

$$\partial_z \to \partial_z \pm m\omega_0 z. \tag{3.6b}$$

By making this transformation in the Lagrangian density 3.1, we obtain the Lagrangian density of the KGO in a curved space-time,

$$\begin{aligned}\mathcal{L} = &-\sqrt{-g}\left[g^{00}\partial_0\psi^*\partial_0\psi + g^{11}(\partial_1 - m\omega_0\rho)\psi^*(\partial_1 + m\omega_0\rho)\psi + g^{22}\partial_2\psi^*\partial_2\psi\right.\\ &\left.+g^{33}(\partial_3 - m\omega_0 z)\psi^*(\partial_3 + m\omega_0 z)\psi + m^2\psi^*\psi\right].\end{aligned} \tag{3.7}$$

Applying the metric 2.42 and its trace, Eq. 2.44, into 3.7, we find the Lagrangian density of the Klein-Gordon Oscillator in a cosmic string spacetime:

$$\begin{aligned}\mathcal{L} =&\alpha\rho(\partial_t\psi^*\partial_t\psi) - \alpha\rho(\partial_\rho\psi^*\partial_\rho\psi + m\omega_0\rho\psi\partial_\rho\psi^* - m\omega_0\rho\psi^*\partial_\rho\psi - m^2\omega_0^2\rho^2\psi^*\psi)\\ &-\frac{1}{\alpha\rho}(\partial_\phi\psi^*\partial_\phi\psi) - \alpha\rho(\partial_z\psi^*\partial_z\psi + m\omega_0 z\psi\partial_z\psi^* - m^2\omega_0^2 z^2\psi^*\psi - m\omega_0 z\psi^*\partial_z\psi)\\ &-\alpha\rho m^2\psi^*\psi.\end{aligned} \tag{3.8}$$

The associated equation of motion for the complex field is

$$-\partial_t^2\psi + \frac{1}{\rho}\partial_\rho\psi + \partial_\rho^2\psi + 3m\omega_0\psi + 2m\omega_0\rho\partial_\rho\psi + \frac{1}{\alpha^2\rho^2}\partial_\phi^2\psi$$
$$+\partial_z^2\psi + 2m\omega_0 z\partial_z\psi - m^2\psi + m^2\omega_0^2\rho^2\psi + m^2\omega_0^2 z^2\psi = 0. \tag{3.9}$$

Our goal is to solve this equation, we start by proposing a separate solution as $\psi(t,\rho,\phi,z) = T(t)R(\rho)\Phi(\phi)Z(z)$. Applying in the equation and dividing by $TR\Phi Z$, we get

$$\frac{1}{T}\partial_t^2 T - \frac{1}{R}\partial_\rho^2 R - \frac{1}{\rho R}\partial_\rho R - 2\rho m\omega_0\frac{1}{R}\partial_\rho R - \frac{1}{\alpha^2\rho^2}\frac{1}{\Phi}\partial_\phi^2\Phi$$
$$-\frac{1}{Z}\partial_z^2 Z - 2m\omega_0 z\frac{1}{Z}\partial_z Z - \rho^2 m^2\omega_0^2 - m^2\omega_0^2 z^2 - 3m\omega_0 + m^2 = 0. \tag{3.10}$$

In order to obtain an temporal independet equation, we propose the following ansatz

$$T(t) = e^{-i\mathcal{E}t}, \tag{3.11}$$

where $\mathcal{E}$ is interpreted as the energy. Applying this ansatz in the Eq. 3.10 we perform separation of variables in the spatial part

$$-\mathcal{E}^2 - \frac{1}{R}\partial_\rho^2 R - \frac{1}{\rho R}\partial_\rho R - 2\rho m\omega_0\frac{1}{R}\partial_\rho R - \frac{1}{\alpha^2\rho^2}\frac{1}{\Phi}\partial_\phi^2\Phi - \frac{1}{Z}\partial_z^2 Z$$
$$-2m\omega_0 z\frac{1}{Z}\partial_z Z - \rho^2 m^2\omega_0^2 - m^2\omega_0^2 z^2 - 3m\omega_0 + m^2 = 0. \tag{3.12}$$

The solution to the equation is constructed by imposing the following ansatz for the angular part,

$$\Phi(\phi) = e^{iL\phi}, \qquad L = 0, \pm1, \pm2, \pm3, \dots, \tag{3.13}$$

where L is the angular momentum. Its quantization comes from the boundary conditions upon the angular part. By substituting 3.13 into Eq. 3.12, we obtain

$$-\mathcal{E}^2 - \frac{1}{R}\partial_\rho^2 R - \frac{1}{\rho R}\partial_\rho R - 2\rho m\omega_0\frac{1}{R}\partial_\rho R + \frac{L^2}{\alpha^2\rho^2} - \frac{1}{Z}\partial_z^2 Z$$
$$-2m\omega_0 z\frac{1}{Z}\partial_z Z - \rho^2 m^2\omega_0^2 - m^2\omega_0^2 z^2 - 3m\omega_0 + m^2 = 0. \tag{3.14}$$

By imposing a separation constant k, we see that this equation decouples into a part dependent on z,

$$-\frac{1}{Z}\partial_z^2 Z - 2m\omega_0 z\frac{1}{Z}\partial_z Z - m^2\omega_0^2 z^2 = 2mk, \tag{3.15}$$

and a ρ-dependent branch

$$-\mathcal{E}^2-\frac{1}{R}\partial_\rho^2 R-\frac{1}{\rho R}\partial_\rho R-3m\omega_0-2\rho m\omega_0\frac{1}{R}\partial_\rho R+\frac{L^2}{\alpha^2\rho^2}-\rho^2m^2\omega_0^2+m^2+2mk=0, \quad (3.16)$$

where k is a constant resulting from the separation.

Lets start analyzing the z-equation. It is convenient to define the dimensionless variables $\zeta=\sqrt{m\omega_0}z$ and the Eq. 3.15 becomes

$$\partial_\zeta^2 Z+2\zeta\partial_\zeta Z+\left(\zeta^2+\frac{2k}{\omega_0}\right)Z=0, \quad (3.17)$$

which the solution is

$$Z\left(\zeta\right)=A_1e^{-\frac{\zeta^2}{2}-\zeta\sqrt{1-\frac{2k}{\omega_0}}}+A_2e^{-\frac{\zeta^2}{2}+\zeta\sqrt{1-\frac{2k}{\omega_0}}}, \quad (3.18)$$

where A_1 and A_2 are constants to be determined by the boundary conditions.

Now our focus is on the radial Eq. 3.16. The idea is to propose an ansartz that leads us to a known equation. But first we start as was done for the axial part, introducing a new dimensionless variable $\xi=\sqrt{m\omega_0}\rho$, so the equation becomes

$$-\xi^2\partial_\xi^2 R-\left(\xi+2\xi^3\right)\partial_\xi R+\left(\frac{L^2}{\alpha^2}-\xi^4+\xi^2\frac{\beta}{m\omega_0}\right)R=0, \quad (3.19)$$

where $\beta=\left(-3m\omega_0-\mathcal{E}^2+m^2+2mk\right)$. We then propose the following form for the radial function R

$$R\left(\xi\right)=e^{-\frac{\xi^2}{2}}F\left(\xi\right). \quad (3.20)$$

Now our search is for the unknown function $F\left(\xi\right)$. Calculating the derivatives we get

$$\partial_\xi R=e^{-\frac{\xi^2}{2}}\left(\partial_\xi F-\xi F\right), \quad (3.21a)$$

$$\partial_\xi^2 R=e^{-\frac{\xi^2}{2}}\left(\partial_\xi^2 F-2\xi\partial_\xi F-F+\xi^2F\right). \quad (3.21b)$$

By substituting these expressions into Eq. 3.19, it becomes

$$\partial_\xi^2 F+\frac{1}{\xi}\partial_\xi F+\left[\left(\frac{\beta}{m\omega_0}+2\right)+\frac{L^2}{\alpha^2}\frac{1}{\xi^2}\right]F=0. \quad (3.22)$$

This is a Modified Bessel's Equation (BUTKOV, 1978), which is related to the Bessel equation via the change of variable

$$\xi=-i\frac{1}{\left(\frac{\beta}{m\omega_0}+2\right)}x. \quad (3.23)$$

The first and second derivatives becomes

$$\partial_\xi = i\left(\frac{\beta}{m\omega_0}+2\right)\partial_x, \tag{3.24a}$$

$$\partial_\xi^2 = -\left(\frac{\beta}{m\omega_0}+2\right)^2\partial_x^2. \tag{3.24b}$$

Applying these modifications into 3.22, resulting equation is

$$\partial_x^2 F + \frac{1}{x}\partial_x F + \left(1-\frac{L^2}{\alpha^2}\frac{1}{x^2}\right)F = 0. \tag{3.25}$$

This is the Bessel equation which appears in various physics problems, such as the solution of the Laplace and Helmholtz equations in cylindrical coordinates, solving the radial part of Schrödinger equation in spherical and cylindrical coordinates for the free particle, etc. It has the solution given by Bessel functions of the first kind $J_a(x)$, and the second kind $Y_a(x)$, usually called Neumann function, as

$$F(x) = B_1 J_{\frac{L}{\alpha}}(x) + B_2 Y_{\frac{L}{\alpha}}(x), \tag{3.26}$$

where B_1 and B_2 are constants.

Now using 3.20, 3.23 and 3.26, we can write the solution for $R(\xi)$ as

$$R(\xi) = B_1 e^{-\frac{\xi^2}{2}} J_{\frac{L}{\alpha}}(\alpha_c\xi) + B_2 e^{-\frac{\xi^2}{2}} Y_{\frac{L}{\alpha}}(\alpha_c\xi), \tag{3.27}$$

where B_1, B_2 are constants and the new parameter α_c is

$$\alpha_c = \sqrt{1+\frac{\mathcal{E}^2}{m\omega_0}-\frac{m}{\omega_0}-\frac{2k}{\omega_0}}. \tag{3.28}$$

Our solution is a composition of a Gaussian with a Bessel function, J_a, and a Gaussian with Neumann function, Y_a. It is known that $Y_a(x)$ is singular at the origin, $\xi = 0$, so we discard this solution imposing $B_2 = 0$. Our final result is

$$R(\xi) = B_1 e^{-\frac{\xi^2}{2}} J_{\frac{L}{\alpha}}(\alpha_c\xi). \tag{3.29}$$

In quantum mechanics, for a particle confined in a infinite spherical well, we can find the energy quantization imposing the wave function to be zero at the boundary and finding the roots of the Bessel Function. Here we can do the same. We shall impose that the wave function is zero for $\xi \to \infty$, the strategy is to define a finite radius ρ_0, where

$\psi(t, \rho_0, \phi, \zeta) = 0$. Then we will take the limit where this radius goes to infinity. Therefore

$$R(\rho_0) = B_1 e^{-\frac{\xi^2}{2}} J_{\frac{L}{\alpha}}(\alpha_c \rho_0). \tag{3.30}$$

Let $j_{\frac{L}{\alpha}p}$ be the p root of Bessel function and our boundary is at the radius ρ_0, so

$$B_1 e^{-\frac{\xi^2}{2}} J_{\frac{L}{\alpha}}\left(j_{\frac{L}{\alpha}p}\right) = 0, \tag{3.31}$$

Comparing 3.30 with 3.31, we must have $\alpha_c \rho_0 = j_{\frac{L}{\alpha}p}$. Since the energy $\mathcal{E}$ is in α_c, we can find it explicitly as

$$\mathcal{E}_{Lkp}^2 = m^2 - m\omega_0 + 2mk \pm m\omega_0 \frac{j_{\frac{L}{\alpha}p}^2}{\rho_0^2}. \tag{3.32}$$

For the boundary in the infinity we have $\rho_0 \to \pm\infty$, so the energy becomes:

$$\mathcal{E}_k^2 = m^2 - m\omega_0 + 2mk. \tag{3.33}$$

3.2 Gauge Theory

In this section, we briefly review the gauge theory for semi-simple Lie groups as elaborated by Utiyama in (UTIYAMA, 1956). This is a importance section, since the electric and magnetic field in the following sections will exploit results described here.

Lets start with a first order Lagrangian density $\mathcal{L}_M(x) = \mathcal{L}_M\left[\phi^A; \partial_\mu \phi^A\right](x)$ of N free matter fields $\phi^A(x)$, $(A = 1, 2, \cdots, N)$, defined in a region Ω. The field equation is

$$\frac{\partial \mathcal{L}_M}{\partial \phi^A} - \partial_\mu \frac{\partial \mathcal{L}_M}{\partial(\partial \phi^A)} = 0. \tag{3.34}$$

Let us postulate the action

$$S[\phi] = \int_\Omega d^4x \mathcal{L}_M(x), \tag{3.35}$$

which is invariant under the following global infinitesimal transformation

$$\phi^A \to \phi^A + \delta\phi^A, \tag{3.36}$$

$$\delta\phi^A = \epsilon^a I_{(a)\ B}^{\ A} \phi^B, \tag{3.37}$$

where ϵ^a is an infinitesimal parameter independent of x, $(a = 1, 2, \cdots, N)$, and $I_{(a)\ B}^{\ A}$ a constant coefficient. We say global transformation because the parameter ϵ^a does not depend on the spacetime point.

The symmetry Lie group is described by a set of constants $f_{a\ c}^{\ b}$ called structure constants that are defined by the commutation relation between the group generators $I_{(a)}$ as below

$$\left[I_{(a)}, I_{(b)}\right]^{A}_{\ B} = I_{(a)\ C}^{\ A} I_{(b)\ B}^{\ C} - I_{(b)\ C}^{\ A} I_{(a)\ B}^{\ C} = f_{a\ b}^{\ c} I_{(c)\ B}^{\ A}. \tag{3.38}$$

The constants satisfies

$$f_{a\ b}^{\ m} f_{m\ c}^{\ l} + f_{b\ c}^{\ m} f_{m\ a}^{\ l} + f_{c\ a}^{\ m} f_{m\ b}^{\ l} = 0, \tag{3.39}$$

which follows from the Jacobi identity

$$\left[\left[I_{(a)}, I_{(b)}\right], I_{(c)}\right] + \left[\left[I_{(b)}, I_{(c)}\right], I_{(a)}\right] + \left[\left[I_{(c)}, I_{(a)}\right], I_{(b)}\right] = 0, \tag{3.40}$$

and they are antisymmetric in the lower indices

$$f_{a\ b}^{\ c} = -f_{b\ a}^{\ c}. \tag{3.41}$$

The group G_n is an abelian group if its structures constants are zero, $f_{a\ b}^{\ c} = 0$, that means the commutator 3.38 is equal to zero. On the other hand if $f_{a\ b}^{\ c} \neq 0$, the group is called non-abelian.

Supposed that the action S is invariant under the infinitesimal transformation described above in any domain of Ω, the variation of the Lagrangian density must be equal to zero,

$$\begin{aligned}
\delta\mathcal{L}_M &= \frac{\partial\mathcal{L}_M}{\partial\phi^A}\delta\phi^A + \frac{\partial\mathcal{L}_M}{\partial\left(\partial_\mu\phi^A\right)}\delta\left(\partial_\mu\phi^A\right) \\
&= \frac{\partial\mathcal{L}_M}{\partial\phi^A}\delta\phi^A + \frac{\partial\mathcal{L}_M}{\partial\left(\partial_\mu\phi^A\right)}\partial_\mu\left(\delta\phi^A\right) \\
&= \frac{\partial\mathcal{L}_M}{\partial\phi^A}\epsilon^a I_{(a)\ B}^{\ A}\phi^B + \frac{\partial\mathcal{L}_M}{\partial\left(\partial_\mu\phi^A\right)}\epsilon^a I_{(a)\ B}^{\ A}\partial_\mu\phi^B = 0,
\end{aligned} \tag{3.42}$$

using the Leibniz rule in the first step of the equation above, we get

$$\left[\frac{\partial\mathcal{L}_M}{\partial\phi^A} - \partial_\mu\frac{\partial\mathcal{L}_M}{\partial\left(\partial_\mu\phi^A\right)}\right]\delta\phi^A + \partial_\mu\left[\frac{\partial\mathcal{L}_M}{\partial\left(\partial_\mu\phi^A\right)}\delta\phi^A\right] = 0. \tag{3.43}$$

The term on the left must vanish, so we get the Euler-Lagrange equation and, in the term on the right, we have the conserved current,

$$\partial_\mu J^\mu_{\ a} = 0, \qquad J^\mu_{\ a} = \frac{\partial\mathcal{L}_M}{\partial\left(\partial_\mu\phi^A\right)}\delta\phi^A. \tag{3.44}$$

Now lets extend our study to the group $G_{\infty n}$, a more general group in the sense that the parameters ϵ^a are exchanged for a set of arbitrary functions $\epsilon^a(x)$, $G_{\infty n}$ is called a local group. The transformation is as follows:

$$\phi^A \to \phi^A + \delta\phi^A, \tag{3.45}$$

$$\delta\phi^A = \epsilon^a(x)\, I_{(a)\ B}^{\ A}\phi^B, \tag{3.46}$$

where ϵ^a is an infinitesimal function of x, $(a = 1, 2, \cdots, N)$, and $I_{(a)\ B}^{\ A}$ a constant coefficient. The associated Lagrangian density variation is

$$\begin{aligned}\delta\mathcal{L}_M &= \frac{\partial\mathcal{L}_M}{\partial\phi^A}\delta\phi^A + \frac{\partial\mathcal{L}_M}{\partial\left(\partial_\mu\phi^A\right)}\delta\left(\partial_\mu\phi^A\right)\\ &= \frac{\partial\mathcal{L}_M}{\partial\phi^A}\delta\phi^A + \frac{\partial\mathcal{L}_M}{\partial\left(\partial_\mu\phi^A\right)}\partial_\mu\left(\delta\phi^A\right)\\ &= \frac{\partial\mathcal{L}_M}{\partial\phi^A}\epsilon^a(x)\, I_{(a)}{}^A{}_B\phi^B + \frac{\partial\mathcal{L}_M}{\partial\left(\partial_\mu\phi^A\right)}\left(\partial_\mu\epsilon^a I_{(a)}{}^A{}_B\phi^B + \epsilon^a I_{(a)}{}^A{}_B\partial_\mu\phi^B\right)\\ &= \left[\frac{\partial\mathcal{L}_M}{\partial\phi^A}I_{(a)}{}^A{}_B\phi^B + \frac{\partial\mathcal{L}_M}{\partial\left(\partial_\mu\phi^A\right)}I_{(a)}{}^A{}_B\partial_\mu\phi^B\right]\epsilon^a + \left[\frac{\partial\mathcal{L}_M}{\partial\left(\partial_\mu\phi^A\right)}I_{(a)}{}^A{}_B\phi^B\right]\partial_\mu\epsilon^a.\end{aligned} \tag{3.47}$$

Note that the term with ϵ^a is the same as showing in the Eq. 3.42, so from the hypothesis that the Eq. 3.42 is valid, we have

$$\delta\mathcal{L}_M = \left[\frac{\partial\mathcal{L}_M}{\partial\left(\partial_\mu\phi^A\right)}I_{(a)}{}^A{}_B\phi^B\right]\partial_\mu\epsilon^a. \tag{3.48}$$

Then, the variation $\delta\mathcal{L}_M$ is nonzero for the local group $G_{\infty n}$, since the Lagrangian density always has a kinetic term, the derivatives of $\epsilon^a(x)$ are non-zero and the generator $I_{(a)}{}^A{}_B$ are also non-zero. One way to force $\delta\mathcal{L}_M = 0$ is introducing a new set of fields

$$A^J(x), \qquad J = 1, 2, \cdots, M, \tag{3.49}$$

which transform as

$$\delta A^J(x) = \epsilon^a(x)\, U_{(a)}{}^J{}_K A^K + \frac{1}{g}C^J{}_a{}^\mu\partial_\mu\epsilon^a(x), \tag{3.50}$$

in such a way that the term in 3.48 is canceled. In Eq. 3.50, g is a parameter that characterizes the intensity of the dependence of the fields A^J on the derivatives of the parameters and the coefficients U and C are constants to be determined.

Inserting the fields A^J in the Lagrangian density, it becomes

$$\mathcal{L}'_M(x) = \mathcal{L}'_M\left[\phi^A; \partial_\mu\phi^A; A^J\right](x), \tag{3.51}$$

and the action

$$S'[\phi, A] = \int_{\Omega} d^4x \mathcal{L}'_M(x). \tag{3.52}$$

Suppose that $\mathcal{L}'_M$ is invariant under the $G_{\infty n}$, that means

$$\begin{aligned}
\delta\mathcal{L}_M &= \frac{\partial\mathcal{L}_M}{\partial\phi^A}\delta\phi^A + \frac{\partial\mathcal{L}_M}{\partial(\partial_\mu\phi^A)}\delta\left(\partial_\mu\phi^A\right) + \frac{\partial\mathcal{L}_M}{\partial A^J}\delta A^J \\
&= \frac{\partial\mathcal{L}_M}{\partial\phi^A}\epsilon^a I_{(a)}{}^A{}_B\phi^B + \frac{\partial\mathcal{L}_M}{\partial(\partial_\mu\phi^A)}\left(\partial_\mu\epsilon^a I_{(a)}{}^A{}_B\phi^B + \epsilon^a I_{(a)}{}^A{}_B\partial_\mu\phi^B\right) \\
&\quad + \frac{\partial\mathcal{L}_M}{\partial A^J}\left(\epsilon^a U_{(a)}{}^J{}_K A^K + \frac{1}{g}C^J{}_a^\mu\partial_\mu\epsilon^a\right) \\
&= \left[\frac{\partial\mathcal{L}_M}{\partial\phi^A}I_{(a)}{}^A{}_B\phi^B + \frac{\partial\mathcal{L}_M}{\partial(\partial_\mu\phi^A)}I_{(a)}{}^A{}_B\partial_\mu\phi^B + \frac{\partial\mathcal{L}_M}{\partial A^J}U_{(a)}{}^J{}_K A^K\right]\epsilon^a \\
&\quad + \left[\frac{\partial\mathcal{L}_M}{\partial(\partial_\mu\phi^A)}I_{(a)}{}^A{}_B\phi^B + \frac{\partial\mathcal{L}_M}{\partial A^J}\frac{1}{g}C^J{}_a^\mu\right]\partial_\mu\epsilon^a = 0.
\end{aligned} \tag{3.53}$$

This results in a system of equations, since there is no relationship between ϵ^a and its derivatives, we can write

$$\frac{\partial\mathcal{L}_M}{\partial\phi^A}I_{(a)}{}^A{}_B\phi^B + \frac{\partial\mathcal{L}_M}{\partial(\partial_\mu\phi^A)}I_{(a)}{}^A{}_B\partial_\mu\phi^B + \frac{\partial\mathcal{L}_M}{\partial A^J}U_{(a)}{}^J{}_K A^K = 0, \tag{3.54}$$

$$\frac{\partial\mathcal{L}_M}{\partial(\partial_\mu\phi^A)}I_{(a)}{}^A{}_B\phi^B + \frac{\partial\mathcal{L}_M}{\partial A^J}\frac{1}{g}C^J{}_a^\mu = 0. \tag{3.55}$$

We remember that the indices in the system of equations assume the values: $J = 1, \cdots, M$ (number of components of the gauge potential A); $\mu = 0, 1, 2, 3$ (the four coordinates of spacetime); $a = 1, \cdots, n$ (number of parameters of the gauge transformation ϵ). We can write the Eq. 3.55 in the matrix form as

$$\begin{pmatrix} \frac{\partial\mathcal{L}_M}{\partial(\partial_0\phi^A)}I_{(1)}{}^A{}_B\phi^B \\ \vdots \\ \frac{\partial\mathcal{L}_M}{\partial(\partial_3\phi^A)}I_{(n)}{}^A{}_B\phi^B \end{pmatrix}_{4n\times 1} + \frac{1}{g}\begin{pmatrix} C^1 & \cdots & C^M \end{pmatrix}_{4n\times M}\begin{pmatrix} \frac{\partial\mathcal{L}_M}{\partial A^1} \\ \vdots \\ \frac{\partial\mathcal{L}_M}{\partial A^J} \end{pmatrix}_{M\times 1} = 0, \tag{3.56}$$

where each C^J is the column with $4n$ elements $C^J{}_a^\mu$. In order to the system can be determined univocally, the matrix $[C]_{4n\times M}$ must be square and invertible, so we find that $4n = M$.

From the inversibitlity condition of $[C]_{4n\times M}$, we infer the existence of the inverse matrix $[C^{-1}]_{M\times 4n}$ with elements $(C^{-1})_J{}^a_\mu$.

Now we define the gauge potential $A^a{}_\mu(x)$ to the parameter $\epsilon^a(x)$ as

$$A^a{}_\mu(x) \equiv \left(C^{-1}\right)_\mu{}^a{}_J A^J(x), \tag{3.57}$$

which is a vector. We can write the auxiliary field in term of the gauge potential as

$$A^J(x) \equiv C^{J\ \mu}_{\ a} A^a{}_\mu(x). \tag{3.58}$$

Now let us find the derivative of the Lagrangian density with respect to the gauge potential. Therefore, using the chain rule and Eq. 3.58 we obtain

$$\frac{\partial \mathcal{L}_M}{\partial A^a{}_\mu} = \frac{\partial \mathcal{L}_M}{\partial A^J}\frac{\partial A^J}{\partial A^a{}_\mu} = \frac{\partial \mathcal{L}_M}{\partial A^J}\frac{\partial}{\partial A^a{}_\mu}\left(C^{J\ \nu}_{\ b} A^b{}_\nu(x)\right) = \frac{\partial \mathcal{L}_M}{\partial A^J} C^{J\ \nu}_{\ b}\delta^\mu_\nu \delta^b_a = \frac{\partial \mathcal{L}_M}{\partial A^J} C^{J\ \mu}_{\ a}. \tag{3.59}$$

We can rewrite the Eq. 3.55 with the above result as

$$\frac{\partial \mathcal{L}_M}{\partial\left(\partial_\mu \phi^A\right)} I_{(a)}{}^A{}_B \phi^B + \frac{1}{g}\frac{\partial \mathcal{L}_M}{\partial A^a{}_\mu} = 0. \tag{3.60}$$

This equation is satisfied if the dependence of $\mathcal{L}'_M$ with $\partial_\mu \phi^A$ and $A^a{}_\mu$ occurs through the relation

$$\nabla_\mu \phi^A = \partial_\mu \phi^A + g A^a{}_\mu I_{(a)}{}^A{}_B \phi^B, \tag{3.61}$$

which is called the covariant derivative.

We can define a new Lagrangian density that depends on the covariant derivative

$$\mathcal{L}'_M\left[\phi^A; \partial_\mu \phi^A; A^a{}_\mu\right] = \mathcal{L}''_M\left[\phi^A; \nabla_\mu \phi^A\right]. \tag{3.62}$$

Let us explore this new Lagrangian density by calculating its derivative in terms of the fields ϕ^A and the gauge potential $A^a{}_\mu$ in order to check whether 3.60 is satisfied.

The derivative of $\mathcal{L}''_M$ with respect to the ϕ^A fields can be developed as

$$\begin{aligned}\frac{\partial \mathcal{L}''_M}{\partial\left(\partial_\mu \phi^A\right)} &= \frac{\partial \mathcal{L}''_M}{\partial\left(\nabla_\nu \phi^B\right)}\frac{\partial\left(\nabla_\nu \phi^B\right)}{\partial\left(\partial_\mu \phi^A\right)} = \frac{\partial \mathcal{L}''_M}{\partial\left(\nabla_\nu \phi^B\right)}\frac{\partial}{\partial\left(\partial_\mu \phi^A\right)}\left(\partial_\nu \phi^B - g A^b{}_\nu I_{(b)}{}^B{}_C \phi^C\right) \\ &= \frac{\partial \mathcal{L}''_M}{\partial\left(\nabla_\nu \phi^B\right)}\delta^\mu_\nu \delta^A_B = \frac{\partial \mathcal{L}''_M}{\partial\left(\nabla_\mu \phi^A\right)},\end{aligned} \tag{3.63}$$

and its derivative with respect to the gauge potential $A^a{}_\mu$ can be expressed as

$$\begin{aligned}\frac{\partial \mathcal{L}''_M}{\partial A^a{}_\mu} &= \frac{\partial \mathcal{L}''_M}{\partial\left(\nabla_\nu \phi^B\right)}\frac{\partial\left(\nabla_\nu \phi^B\right)}{\partial A^a{}_\mu} = \frac{\partial \mathcal{L}''_M}{\partial\left(\nabla_\nu \phi^B\right)}\frac{\partial}{\partial A^a{}_\mu}\left(\partial_\nu \phi^B - g A^b{}_\nu I_{(b)}{}^B{}_C \phi^C\right) \\ &= \frac{\partial \mathcal{L}''_M}{\partial\left(\nabla_\nu \phi^B\right)}\delta^\mu_\nu \delta^b_a g I_{(b)}{}^B{}_C \phi^C = \frac{\partial \mathcal{L}''_M}{\partial\left(\nabla_\mu \phi^B\right)} g I_{(a)}{}^B{}_C \phi^C.\end{aligned} \tag{3.64}$$

Applying 3.63 and 3.64 into the relation 3.60 and changing the dummy indices we get

$$\frac{\partial \mathcal{L}_M}{\partial\left(\partial_\mu \phi^A\right)} I_{(a)}{}^A{}_B \phi^B+\frac{1}{g} \frac{\partial \mathcal{L}_M}{\partial A^a{}_\mu}=\frac{\partial \mathcal{L}_M^{\prime\prime}}{\partial\left(\nabla_\mu \phi^A\right)} I_{(a)}{}^A{}_B \phi^B-\frac{\partial \mathcal{L}_M^{\prime\prime}}{\partial\left(\nabla_\mu \phi^A\right)} I_{(a)}{}^A{}_B \phi^B=0. \tag{3.65}$$

Thus, the equation is satisfied.

Now let us express δA^J, Eq. 3.50, in terms of $A^a{}_\mu$. To do this, we substitute the definition of A^J, Eq. 3.58, into Eq. 3.50 and we obtain

$$\begin{aligned}
\delta A^J &= \epsilon^a U_{(a)}{}^J{}_K A^K+\frac{1}{g} C^J{}_a^\mu \partial_\mu \epsilon^a, \\
\delta\left[C^J{}_a^\mu A^a{}_\mu\right] &= \epsilon^a U_{(a)}{}^J{}_K\left(C^K{}_b{}^\mu A^b{}_\mu\right)+\frac{1}{g} C^J{}_a^\mu \partial_\mu \epsilon^a, \\
\left(C^{-1}\right)_\nu^c{}_J \delta\left[C^J{}_a^\mu A^a{}_\mu\right] &= \left(C^{-1}\right)_\nu^c{}_J \epsilon^a U_{(a)}{}^J{}_K\left(C^K{}_b{}^\mu A^b{}_\mu\right)+\left(C^{-1}\right)_\nu^c{}_J \frac{1}{g} C^J{}_a^\mu \partial_\mu \epsilon^a, \\
\delta\left(\delta_\nu^\mu \delta_a^c A^a{}_\mu\right) &= \epsilon^a\left(C^{-1}\right)_\nu^c{}_J U_{(a)}{}^J{}_K C^K{}_b{}^\mu A^b{}_\mu+\frac{1}{g} \delta_\nu^\mu \delta_a^c \partial_\mu \epsilon^a, \\
\delta A^c{}_\nu &= \epsilon^a S_{(a)}{}^c{}_\nu^\mu{}_b A^b{}_\mu+\frac{1}{g} \partial_\nu \epsilon^c.
\end{aligned} \tag{3.66}$$

where

$$S_{(a)}{}^c{}_\nu^\mu{}_b=\left(C^{-1}\right)_\nu^c{}_J U_{(a)}{}^J{}_K C^K{}_b{}^\mu. \tag{3.67}$$

This definition switch the problem to finding $U_{(a)}{}^J{}_K$ and $C^K{}_b{}^\mu$ to seek for $S_{(a)}{}^c{}_\nu^\mu{}_b$.

To confirm the invariance of $\mathcal{L}_M^{\prime\prime}\left[\phi^A ; \nabla \phi^A\right]$ under the local transformation 3.45, let us calculate its variation. Applying the variation operator δ to $\mathcal{L}_M^{\prime\prime}$ we get

$$\delta \mathcal{L}_M^{\prime\prime}=\frac{\partial \mathcal{L}_M}{\partial \phi^A} \delta \phi^A+\frac{\partial \mathcal{L}_M}{\partial\left(\nabla_\mu \phi^A\right)} \delta\left(\nabla_\mu \phi^A\right), \tag{3.68}$$

using the equations 3.46 and 3.66 we can express $\delta\left(\nabla_\mu \phi^A\right)$ as

$$\begin{aligned}
\delta\left(\nabla_\mu \phi^A\right) &= \delta\left(\partial_\mu \phi^A-g A^a{}_\mu I_{(a)}{}^A{}_B \phi^B\right) \\
&= \delta\left(\partial_\mu \phi^A\right)-g \delta A^a{}_\mu I_{(a)}{}^A{}_B \phi^B-g A^a{}_\mu I_{(a)}{}^A{}_B \delta \phi^B \\
&= \partial_\mu \epsilon^a I_{(a)}{}^A{}_B \phi^B+\epsilon^a I_{(a)}{}^A{}_B \partial_\mu \phi^B-g I_{(a)}{}^A{}_B \phi^B\left(\epsilon^c S_{(c)}{}^a{}_\mu^\nu{}_b A^b{}_\nu+\frac{1}{g} \partial_\mu \epsilon^a\right) \\
&\quad -g A^a{}_\mu I_{(a)}{}^A{}_B\left(\epsilon^b I_{(b)}{}^B{}_C \phi^C\right) \\
&= \left(I_{(a)}{}^A{}_B \partial_\mu \phi^B-g A^b{}_\mu I_{(b)}{}^A{}_C I_{(a)}{}^C{}_B \phi^B-g I_{(c)}{}^A{}_B \phi^B S_{(a)}{}^c{}_\mu^\nu{}_b A^b{}_\nu\right) \epsilon^a \\
&\quad +\left(I_{(a)}{}^A{}_B \phi^B-I_{(a)}{}^A{}_B \phi^B\right) \partial_\mu \epsilon^a.
\end{aligned} \tag{3.69}$$

Note that by introducing the auxiliary field A^J, the term of the derivative transformation parameter disappears as desired. From Eq. 3.38 we can write

$$I_{(b)}{}^{A}{}_{C}I_{(a)}{}^{C}{}_{B} = I_{(a)}{}^{A}{}_{C}I_{(b)}{}^{C}{}_{B} - \left[I_{(a)}, I_{(b)}\right]^{A}{}_{B} = I_{(a)}{}^{A}{}_{C}I_{(b)}{}^{C}{}_{B} - f_{a}{}^{c}{}_{b}I_{(c)}{}^{A}{}_{B}. \quad (3.70)$$

Therefore, the non-zero term in 3.69 can be expressed as

$$\begin{aligned}\delta\left(\nabla_{\mu}\phi^{A}\right) &= \left(I_{(a)}{}^{A}{}_{B}\partial_{\mu}\phi^{B} - gA^{b}{}_{\mu}\left(I_{(b)}{}^{A}{}_{C}I_{(a)}{}^{C}{}_{B}\right)\phi^{B} - gI_{(c)}{}^{A}{}_{B}\phi^{B}S_{(a)}{}^{c}{}_{\mu}{}^{\nu}{}_{b}A^{b}{}_{\nu}\right)\epsilon^{a} \\ &= \left(I_{(a)}{}^{A}{}_{B}\partial_{\mu}\phi^{B} - gA^{b}{}_{\mu}\left(I_{(a)}{}^{A}{}_{C}I_{(b)}{}^{C}{}_{B} - f_{a}{}^{c}{}_{b}I_{(c)}{}^{A}{}_{B}\right)\phi^{B} - gI_{(c)}{}^{A}{}_{B}\phi^{B}S_{(a)}{}^{c}{}_{\mu}{}^{\nu}{}_{b}A^{b}{}_{\nu}\right)\epsilon^{a} \\ &= \left[\left(\partial_{\mu}\phi^{B} - gA^{b}{}_{\mu}I_{(b)}{}^{B}{}_{C}\phi^{C}\right)I_{(a)}{}^{A}{}_{B} + \left(f_{a}{}^{c}{}_{b}\delta^{\nu}_{\mu} - S_{(a)}{}^{c}{}_{\mu}{}^{\nu}{}_{b}\right)gA^{b}{}_{\nu}I_{(c)}{}^{A}{}_{B}\phi^{B}\right]\epsilon^{a}. \quad (3.71)\end{aligned}$$

We can identify the covariant derivative and write the Eq. 3.71 in the following compact form

$$\delta\left(\nabla_{\mu}\phi^{A}\right) = I_{(a)}{}^{A}{}_{B}\left(\nabla_{\mu}\phi^{B}\right)\epsilon^{a} + \left(f_{a}{}^{c}{}_{b}\delta^{\nu}_{\mu} - S_{(a)}{}^{c}{}_{\mu}{}^{\nu}{}_{b}\right)gA^{b}{}_{\nu}I_{(c)}{}^{A}{}_{B}\phi^{B}\epsilon^{a}. \quad (3.72)$$

Here, the unknown $S_{(a)}{}^{c}{}_{\mu}{}^{\nu}{}_{b}$ coefficients are determined in such a way as to nullfy the last term, making explicit the covariant character of ∇_{μ}, since it has a transformation law with the same form as that presented by ϕ^{A}, Eq. 3.37.

$$\delta\left(\nabla_{\mu}\phi^{A}\right) = I_{(a)}{}^{A}{}_{B}\left(\nabla_{\mu}\phi^{B}\right)\epsilon^{a} \quad (3.73)$$

The last important definition is the field strenght tensor $F^{a}_{\mu\nu}$, which is defined as

$$\begin{aligned}F^{a}_{\mu\nu} &= \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} - gf_{b}{}^{a}{}_{c}A^{b}_{\mu}A^{c}_{\nu}. \\ &= A^{a}_{[\mu,\nu]} - gf_{b}{}^{a}{}_{c}A^{b}_{\mu}A^{c}_{\nu}, \quad (3.74)\end{aligned}$$

where $A^{a}_{[\mu,\nu]} = \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu}$

This complete our section about the Gauge Theory. More could be presented about the theory, but this is enough for the scope of this work.

3.3 Klein-Gordon Oscillator under local phase transformation

In this section, we shall explore the invariance of the KGO in a cosmic string spacetime under phase transformation. We will see that the KGO Lagrangian density is invariant under the global transformation but not under the local one. The local invariance is restored by making a minimal coupling of the interaction field, the gauge field.

We start by making the global phase transformation in the field as

$$\psi \to \psi e^{i\Lambda}, \tag{3.75a}$$
$$\psi^* \to \psi^* e^{-i\Lambda}. \tag{3.75b}$$

It is called global because the change of the phase parameter of the field is the same for all points of the spacetime, Λ does not depend of the spacetime points. This transformation corresponds to the group of the unitary matrices of dimension 1, U(1), which means complex numbers of norm 1 or as we parameterized $e^{i\Lambda}$, Λ is a parameter to describe the elements of the group and $\Lambda \in [0, 2\pi)$. U(1) corresponds geometrically to a circle, and multiplication is equivalent to adding the rotation of an angle Λ around the circle.

Applying the transformation 3.75 in the Eq. 3.8 we get a new Lagrangian density denoted by $\mathcal{L}'$,

$$\begin{aligned}\mathcal{L}' =&\alpha\rho(\partial_t(\psi^* e^{-i\Lambda})\partial_t(\psi e^{i\Lambda})) - \alpha\rho(\partial_\rho(\psi^* e^{-i\Lambda})\partial_\rho(\psi e^{i\Lambda}) + m\omega_0\rho(\psi e^{i\Lambda})\partial_\rho(\psi^* e^{-i\Lambda})\\ &- m\omega_0\rho(\psi^* e^{-i\Lambda})\partial_\rho(\psi e^{i\Lambda}) - m^2\omega_0^2\rho^2(\psi^* e^{-i\Lambda})(\psi e^{i\Lambda})) - \frac{1}{\alpha\rho}\left(\partial_\phi(\psi^* e^{-i\Lambda})\partial_\phi(\psi e^{i\Lambda})\right)\\ &- \alpha\rho(\partial_z(\psi^* e^{-i\Lambda})\partial_z(\psi e^{i\Lambda}) + m\omega_0 z(\psi e^{i\Lambda})\partial_z(\psi^* e^{-i\Lambda}) - m^2\omega_0^2 z^2(\psi^* e^{-i\Lambda})(\psi e^{i\Lambda})\\ &- m\omega_0 z(\psi^* e^{-i\Lambda})\partial_z(\psi e^{i\Lambda}) - \alpha\rho m^2(\psi^* e^{-i\Lambda})(\psi e^{i\Lambda}).\end{aligned} \tag{3.76}$$

Since the parameter Λ is independent of the coordinates (it is a global transformation), the derivative does not act on it and we obtain

$$\begin{aligned}\mathcal{L}' =&\alpha\rho(\partial_t\psi^*\partial_t\psi) - \alpha\rho(\partial_\rho\psi^*\partial_\rho\psi + m\omega_0\rho\psi\partial_\rho\psi^* - m\omega_0\rho\psi^*\partial_\rho\psi - m^2\omega_0^2\rho^2\psi^*\psi)\\ &- \frac{1}{\alpha\rho}(\partial_\phi\psi^*\partial_\phi\psi) - \alpha\rho(\partial_z\psi^*\partial_z\psi + m\omega_0 z\psi\partial_z\psi^* - m^2\omega_0^2 z^2\psi^*\psi - m\omega_0 z\psi^*\partial_z\psi)\\ &- \alpha\rho m^2\psi^*\psi = \mathcal{L}.\end{aligned} \tag{3.77}$$

Thus, the global phase transformation does not alter the Lagrangian density, which proves the invariance of the KGO under this transformation. Now, our goal is to extend this procedure to the local transformation

$$\psi \to \psi e^{i\Lambda(t,\rho,\phi,z)}, \tag{3.78a}$$
$$\psi^* \to \psi^* e^{-i\Lambda(t,\rho,\phi,z)}. \tag{3.78b}$$

The dependence of Λ on the spacetime points is explicit and the parameter changes for each point. Making the transformation in the Eq. 3.8 we find a new Lagrangian density

$\mathcal{L}''$,

$$\begin{aligned}\mathcal{L}'' =&\alpha\rho\left(\partial_t\left(\psi^*e^{-i\Lambda}\right)\partial_t\left(\psi e^{i\Lambda}\right)\right)-\alpha\rho\left[\partial_\rho\left(\psi^*e^{-i\Lambda}\right)\partial_\rho\left(\psi e^{i\Lambda}\right)+m\omega_0\rho\left(\psi e^{i\Lambda}\right)\partial_\rho\left(\psi^*e^{-i\Lambda}\right)\right.\\&\left.-m\omega_0\rho\left(\psi^*e^{-i\Lambda}\right)\partial_\rho\left(\psi e^{i\Lambda}\right)-m^2\omega_0^2\rho^2\left(\psi^*e^{-i\Lambda}\right)\left(\psi e^{i\Lambda}\right)\right]-\frac{1}{\alpha\rho}\left(\partial_\phi\left(\psi^*e^{-i\Lambda}\right)\partial_\phi\left(\psi e^{i\Lambda}\right)\right)\\&-\alpha\rho\left[\partial_z\left(\psi^*e^{-i\Lambda}\right)\partial_z\left(\psi e^{i\Lambda}\right)+m\omega_0 z\left(\psi e^{i\Lambda}\right)\partial_z\left(\psi^*e^{-i\Lambda}\right)-m^2\omega_0^2z^2\left(\psi^*e^{-i\Lambda}\right)\left(\psi e^{i\Lambda}\right)\right.\\&\left.-m\omega_0 z\left(\psi^*e^{-i\Lambda}\right)\partial_z\left(\psi e^{i\Lambda}\right)\right]-\alpha\rho m^2\left(\psi^*e^{-i\Lambda}\right)\left(\psi e^{i\Lambda}\right).\end{aligned} \tag{3.79}$$

With a few manipulation we can rewrite $\mathcal{L}''$ in terms of the original $\mathcal{L}$ plus extra terms related to the derivative of the transformation parameter Λ,

$$\begin{aligned}\mathcal{L}'' =&\mathcal{L}+\alpha\rho\partial_t\psi^*i\partial_t\Lambda-\alpha\rho i\psi^*\partial_t\Lambda\partial_t\psi-\alpha\rho i\psi^*\partial_t\Lambda i\partial_t\Lambda-\alpha\rho\partial_\rho\psi^*i\psi\partial_\rho\Lambda\\&+\alpha\rho i\psi^*\partial_\rho\Lambda\partial_\rho\psi+i\psi^*\partial_\rho\Lambda i\psi\partial_\rho\Lambda-\frac{1}{\alpha\rho}\partial_\phi\psi^*i\psi\partial_\phi\Lambda+\frac{1}{\alpha\rho}i\psi^*\partial_\phi\Lambda\partial_\phi\psi\\&+\frac{1}{\alpha\rho}i\psi^*\partial_\phi\Lambda i\psi\partial_\phi\Lambda-\alpha\rho\partial_z\psi^*i\psi\partial_z\Lambda+\alpha\rho i\psi^*\partial_z\Lambda\partial_z\psi+\alpha\rho i\psi^*\partial_z\Lambda i\psi\partial_z\Lambda\\&-\alpha\rho m\omega_0 z\psi i\psi^*\partial_z\Lambda+\alpha\rho m\omega_0 z\psi^*i\psi\partial_z\Lambda.\end{aligned} \tag{3.80}$$

Therefore, the KGO is not invariant under local phase transformation, since $\mathcal{L}''\neq\mathcal{L}$. To make $\mathcal{L}$ locally symmetric under 3.78, we will introduce a field A_μ, called the gauge field as described in the section 3.2.

Expanding the local phase transformation 3.78 into a Taylor series, we write the infinitesimal transformation as

$$e^{i\Lambda}\psi=\psi+i\Lambda\psi, \tag{3.81a}$$
$$e^{-i\Lambda}\psi^*=\psi^*-i\Lambda\psi^*. \tag{3.81b}$$

Comparing them with 3.45, we can identify $\epsilon=\Lambda$ and $I=\pm i$, where + (resp. $-$) applies to ψ (resp. ψ^*). Calculating the commutator of the group generators $I_{(a)}$, we obtain

$$[I_{(a)},I_{(b)}]=ii-ii=0. \tag{3.82}$$

That confirm the fact that U(1) is an abelian group and also mean $f_{(c)}{}^{a}{}_{b}=0$, see Eq. 3.38.

The variation of the fields ψ and ψ^* can be found using 3.46 and they are

$$\delta\psi=i\Lambda\psi, \tag{3.83a}$$
$$\delta\psi^*=-i\Lambda\psi^*. \tag{3.83b}$$

The gauge potential variation δA_μ can be written using 3.66,

$$\delta A_\mu = \frac{1}{g}\partial_\mu \Lambda, \tag{3.84}$$

as well as the covariant derivative using the expression 3.61,

$$\nabla_\mu \psi = \partial_\mu \psi - igA_\mu \psi, \tag{3.85a}$$

$$\nabla_\mu \psi^* = \partial_\mu \psi^* + igA_\mu \psi^*, \tag{3.85b}$$

and its variation is, by the Eq. 3.73,

$$\delta(\nabla_\mu \psi) = i\Lambda(\nabla_\mu \psi), \tag{3.86a}$$

$$\delta(\nabla_\mu \psi^*) = -i\Lambda(\nabla_\mu \psi^*). \tag{3.86b}$$

Here we have made the complete map between the phase group and the generic group described in the previous section.

To make the KGO Lagrangian density invariant under local phase transformation, we introduce the gauge field A_μ by making a minimal coupling, replacing the ordinary derivative by the covariant derivative ($\partial_\mu \to \nabla_\mu$). The covariant derivative for the ψ field can be explicit written as

$$\nabla_\mu \psi = \partial_\mu \psi - igA_\mu \psi \begin{cases} \nabla_t \psi = \partial_t \psi - igA_t \psi \\ \nabla_\rho \psi = \partial_\rho \psi - igA_\rho \psi \\ \nabla_\phi \psi = \partial_\phi \psi - igA_\phi \psi \\ \nabla_z \psi = \partial_z \psi - igA_z \psi \end{cases}, \tag{3.87}$$

and for ψ^*,

$$\nabla_\mu \psi^* = \partial_\mu \psi^* + igA_\mu \psi^* \begin{cases} \nabla_t \psi^* = \partial_t \psi^* + igA_t \psi^* \\ \nabla_\rho \psi^* = \partial_\rho \psi^* + igA_\rho \psi^* \\ \nabla_\phi \psi^* = \partial_\phi \psi^* + igA_\phi \psi^* \\ \nabla_z \psi^* = \partial_z \psi^* + igA_z \psi^* \end{cases}. \tag{3.88}$$

Since we are working with an abelian group, the field strength tensor $F_{\mu\nu}$ defined in 3.74 is just

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.89}$$

Making the minimal coupling in the Lagrangian density 3.8, we get

$$\begin{aligned}\mathcal{L} =&\alpha\rho(\nabla_t\psi^*\nabla_t\psi) - \alpha\rho(\nabla_\rho\psi^*\nabla_\rho\psi + m\omega_0\rho\psi\nabla_\rho\psi^* - m\omega_0\rho\psi^*\nabla_\rho\psi - m^2\omega_0^2\rho^2\psi^*\psi)\\ &- \frac{1}{\alpha\rho}(\nabla_\phi\psi^*\nabla_\phi\psi) - \alpha\rho(\nabla_z\psi^*\nabla_z\psi + m\omega_0 z\psi\nabla_z\psi^* - m^2\omega_0^2 z^2\psi^*\psi - m\omega_0 z\psi^*\nabla_z\psi)\\ &- \alpha\rho m^2\psi^*\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},\end{aligned} \tag{3.90}$$

where the last term is the kinetic term associated with the interaction field A_μ. Analyzing its variation under the local transformation U(1), we can write the variation of the Lagrangian density explicitly as

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\psi^*}\delta\psi^* + \frac{\partial\mathcal{L}}{\partial(\nabla_\lambda\psi)}\delta(\nabla_\lambda\psi) + \frac{\partial\mathcal{L}}{\partial(\nabla_\lambda\psi^*)}\delta(\nabla_\lambda\psi^*) + \frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\delta F_{\mu\nu}. \tag{3.91}$$

First, calculating the kinetic term $\delta F_{\mu\nu}$ by using the commutation between the ordinary derivative with the variation operator and Eq. 3.84, we obtain

$$\delta F_{\mu\nu} = \delta(\partial_\mu A_\nu) - \delta(\partial_\nu A_\mu) = \partial_\mu\frac{1}{g}\partial_\nu\Lambda - \partial_\nu\frac{1}{g}\partial_\mu\Lambda = 0. \tag{3.92}$$

As the $\delta F_{\mu\nu}$ vanishes, our analysis becomes the remaining terms of 3.91, expanding them to all coordinates explicitly,

$$\begin{aligned}\delta\mathcal{L} =&\frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\psi^*}\delta\psi^* + \frac{\partial\mathcal{L}}{\partial(\nabla_t\psi)}\delta(\nabla_t\psi) + \frac{\partial\mathcal{L}}{\partial(\nabla_t\psi^*)}\delta(\nabla_t\psi^*) + \frac{\partial\mathcal{L}}{\partial(\nabla_\rho\psi)}\delta(\nabla_\rho\psi)\\ &+ \frac{\partial\mathcal{L}}{\partial(\nabla_\rho\psi^*)}\delta(\nabla_\rho\psi^*) + \frac{\partial\mathcal{L}}{\partial(\nabla_\phi\psi)}\delta(\nabla_\phi\psi) + \frac{\partial\mathcal{L}}{\partial(\nabla_\phi\psi^*)}\delta(\nabla_\phi\psi^*) + \frac{\partial\mathcal{L}}{\partial(\nabla_z\psi)}\delta(\nabla_z\psi)\\ &+ \frac{\partial\mathcal{L}}{\partial(\nabla_z\psi^*)}\delta(\nabla_z\psi^*).\end{aligned} \tag{3.93}$$

Using Eq. 3.90 and 3.86 into the above equation we get

$$\begin{aligned}\delta\mathcal{L} =&\frac{\partial}{\partial\psi}\left(-\alpha\rho m\omega_0\rho\psi\nabla_\rho\psi^* + \alpha\rho m^2\omega_0^2\rho^2\psi^*\psi - \alpha\rho m\omega_0 z\psi\nabla_z\psi^* + \alpha\rho m^2\omega_0^2 z^2\psi^*\psi\right.\\ &\left.-\alpha\rho m^2\psi^*\psi\right) i\Lambda\psi - \frac{\partial}{\partial\psi^*}\left(\alpha\rho m\omega_0\rho\psi^*\nabla_\rho\psi - \alpha\rho m^2\omega_0^2\rho^2\psi^*\psi + \alpha\rho m\omega_0 z\psi^*\nabla_z\psi\right.\\ &\left.+\alpha\rho m^2\omega_0^2 z^2\psi^*\psi - \alpha\rho m^2\psi^*\psi\right) i\Lambda\psi^* + \alpha\rho\left(\nabla_t\psi^*\right) i\Lambda\left(\nabla_t\psi\right) - \alpha\rho\left(\nabla_t\psi\right) i\Lambda\left(\nabla_t\psi^*\right)\\ &- \alpha\rho\left(\nabla_\rho\psi^* - m\omega_0\rho\psi^*\right) i\Lambda\left(\nabla_\rho\psi\right) + \alpha\rho\left(\nabla_\rho\psi + m\omega_0\rho\psi\right) i\Lambda\left(\nabla_\rho\psi^*\right)\\ &- \frac{1}{\alpha\rho}\left(\nabla_\phi\psi^*\right) i\Lambda\left(\nabla_\phi\psi\right) + \frac{1}{\alpha\rho}\left(\nabla_\phi\psi\right) i\Lambda\left(\nabla_\phi\psi^*\right) - \alpha\rho\left(\nabla_z\psi^* - m\omega_0 z\psi^*\right) i\Lambda\left(\nabla_z\psi\right)\\ &+ \alpha\rho\left(\nabla_z\psi + m\omega_0 z\psi\right) i\Lambda\left(\nabla_z\psi^*\right).\end{aligned} \tag{3.94}$$

With a few manipulations we see that all terms vanish,

$$\delta\mathcal{L} = 0. \tag{3.95}$$

In this example, we see that by introducing the auxiliary field A via minimum coupling, the invariance of the Lagrangian density holds for the local phase transformation. This will be useful in the next section where we introduce the interaction fields, electric and magnetic fields.

4 Electric Field

As little has been discussed about electric fields in the Klein-Gordon context, in this chapter we decided to extend the study of this interaction field to the Klein-Gordon Oscillator in a cosmic string background, exploring two scenarios with different types of electric fields. The first scenario is a constant radial electric field and the second is parallel to the string. For both systems, we obtain the equation of motion and their solutions, we also quantize the related physical observables (energy, linear momentum, etc.).

4.1 Radial electric field

In this section we present the first physical environment under which we shall consider a Klein-Gordon Oscillator. We introduce a vector potential associated with a static electric field in the radial direction. We obtain analytical solutions of the corresponding KGO equations.

A static radial electric field can be written as

$$\vec{E} = E_0 \hat{e}_\rho, \tag{4.1}$$

where E_0 is a constant that define the field intensity.

Using the definition of Maxwell tensor, $F_{\mu\nu}$, we can determine the vector potential associated to the desired electric field. The Maxwell tensor is defined as following (GRIFFITHS, 2017)

$$[F_{\mu\nu}] = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}. \tag{4.2}$$

Since the system under consideration has cylindrical symmetry, we have

$$E_x = -F_{01} = F_{10} = \left(\partial_x A_0 - \partial_t A_1\right), \tag{4.3}$$

$$E_y = -F_{02} = F_{20} = (\partial_y A_0 - \partial_t A_2)\,. \tag{4.4}$$

Assuming that the gauge field has no time dependence, we get

$$\partial_x A_0 = E_x, \tag{4.5}$$
$$\partial_y A_0 = E_y. \tag{4.6}$$

The component A_0 that satisfy both equation is

$$A_0 = xE_x + yE_y. \tag{4.7}$$

Using the Eq. 4.2 and the definition of $F_{\mu\nu}$, Eq. 3.74, we can easily verify that A_0 only create the E_x and E_y components. Analyzing each components we obtain

$$B_x = -F_{32} = F_{23} = \partial_y A_3 - \partial_z A_2 = 0, \tag{4.8a}$$
$$B_y = -F_{13} = F_{31} = \partial_z A_1 - \partial_x A_3 = 0, \tag{4.8b}$$
$$B_z = -F_{21} = F_{12} = \partial_x A_2 - \partial_y A_1 = 0, \tag{4.8c}$$
$$E_x = -F_{01} = F_{10} = \partial_x A_0 - \partial_t A_1 = E_x, \tag{4.8d}$$
$$E_y = -F_{02} = F_{20} = \partial_y A_0 - \partial_t A_2 = E_y, \tag{4.8e}$$
$$E_z = -F_{03} = F_{30} = \partial_z A_0 - \partial_t A_3 = 0. \tag{4.8f}$$

As we are working in a scenario with cylindrical symmetry, it is convenient to rewrite the vector potential in this coordinate system,

$$x = \rho\cos\phi, \tag{4.9}$$

$$y = \rho\sin\phi, \tag{4.10}$$

and the field components are

$$E_x = E_0\cos\phi, \tag{4.11}$$

$$E_y = E_0\sin\phi, \tag{4.12}$$

where $\rho = \sqrt{x^2+y^2}$ is the radial Euclidean distance from the z-axis to a point $\vec{r}$ and ϕ is the azimuth angle. Therefore, the vector potential associated with a static angular direction electric field, can be written as

$$\vec{A} = E_0\rho\hat{e}_t. \tag{4.13}$$

Note that we choose the Coulomb gauge, since $\vec{A}$ satisfies $\nabla\cdot\vec{A} = 0$.

We consider the minimal coupling between the Klein-Gordon fields and the electromagnetic fields by changing the ordinary derivatives to covariant derivatives as follows:

$$\nabla_t \psi = \partial_t \psi - igE_0 \rho \psi, \tag{4.14}$$

$$\nabla_t \psi^* = \partial_t \psi^* + igE_0 \rho \psi^*. \tag{4.15}$$

This minimal coupling is described in detail in the previous section 3.2. The Klein-Gordon Oscillator's Lagrangian density 3.8 becomes:

$$\begin{aligned}\mathcal{L} =&\alpha\rho\left(\partial_t\psi^* + igE_0\rho\psi^*\right)\left(\partial_t\psi - igE_0\rho\psi\right) - \alpha\rho\partial_\rho\psi^*\partial_\rho\psi - \alpha\rho^2 m\omega_0\psi\partial_\rho\psi^* \\ &+ \alpha\rho^2 m\omega_0\psi^*\partial_\rho\psi + \alpha\rho^3 m^2\omega_0^2\psi^*\psi - \alpha\rho\partial_z\psi^*\partial_z\psi - \alpha\rho m\omega_0 z\psi\partial_z\psi^* \\ &+ \alpha\rho m^2\omega_0^2 z^2\psi^*\psi + \alpha\rho m\omega_0 z\psi^*\partial_z\psi - \frac{1}{\alpha\rho}\left(\partial_\phi\psi^*\partial_\phi\psi\right) - \alpha\rho m^2\psi^*\psi,\end{aligned} \tag{4.16}$$

and the corresponding Euler-Lagrange equation is

$$\begin{aligned}&\partial_t^2\psi - 2igE_0\rho\partial_t\psi - \frac{1}{\rho}\partial_\rho\psi - \partial_\rho^2\psi - 2\rho m\omega_0\partial_\rho\psi - 3m\omega_0\psi - m^2\omega_0^2\rho^2\psi \\ &-g^2E_0^2\rho^2\psi - \frac{1}{\alpha^2\rho^2}\partial_\phi^2\psi - \partial_z^2\psi - 2m\omega_0 z\partial_z\psi - m^2\omega_0^2 z^2\psi + m^2\psi = 0.\end{aligned} \tag{4.17}$$

Our goal is to solve this second-order partial differential equation. We use the separation of variables by proposing a solution as $\psi(t,\rho,\phi,z) = G(t,\phi)R(\rho)Z(z)$. The equation of motion develops into

$$\begin{aligned}&\frac{1}{G}\partial_t^2 G - 2igE_0\rho\frac{1}{G}\partial_t G - \frac{1}{\rho}\frac{1}{R}\partial_\rho R - \frac{1}{R}\partial_\rho^2 R - 2\rho m\omega_0\frac{1}{R}\partial_\rho R - m^2\omega_0^2\rho^2 - \\ &g^2E_0^2\rho^2 - \frac{1}{\alpha^2\rho^2}\frac{1}{G}\partial_\phi^2 G - \frac{1}{Z}\partial_z^2 Z - 2m\omega_0 z\frac{1}{Z}\partial_z Z - m^2\omega_0^2 z^2 + m^2 - 3m\omega_0 = 0.\end{aligned} \tag{4.18}$$

For the temporal and angular part, we propose the following ansatz,

$$G\left(t,\phi\right) = e^{-i(\mathcal{E}t - L\phi)}, \tag{4.19}$$

where $L = 0, \pm1, \pm2, \pm3, \ldots$ is the angular momentum. Its quantization comes from the periodic boundary condition upon the angular coordinate, ϕ, and $\mathcal{E}$ is interpreted as the energy. Applying the ansatz above we find a time and angular independent equation,

$$\begin{aligned}&-\mathcal{E}^2 - \frac{1}{\rho}\frac{1}{R}\partial_\rho R - \frac{1}{R}\partial_\rho^2 R - 2\rho m\omega_0\frac{1}{R}\partial_\rho R - m^2\omega_0^2\rho^2 - g^2E_0^2\rho^2 \\ &+\frac{L^2}{\alpha^2\rho^2} - \frac{1}{Z}\partial_z^2 Z - 2m\omega_0 z\frac{1}{Z}\partial_z Z - m^2\omega_0^2 z^2 + m^2 - 3m\omega_0 = 0.\end{aligned} \tag{4.20}$$

Introducing a separation constant k, this differential equation decouples into a ρ and z

independent parts. The ρ-dependent sector is

$$\left[-\partial_\rho^2 - \left(\frac{1}{\rho} + 2\rho m\omega_0\right)\partial_\rho - \left(m^2\omega_0^2 + g^2E_0^2\right)\rho^2 - 2gE_0\mathcal{E}\rho + \frac{L^2}{\alpha^2\rho^2} + \gamma\right]R(\rho) = 0, \quad (4.21)$$

where $\gamma = -3m\omega_0 - \mathcal{E}^2 + m^2 + 2mk$, and the z-dependent equation is

$$-\frac{1}{Z}\partial_z^2 Z - 2m\omega_0 z\frac{1}{Z}\partial_z Z - m^2\omega_0^2 z^2 = 2mk. \quad (4.22)$$

First, let us focus on the axial equation. It is convenient to make the change of variables that gives us a dimensionless coordinate, so we define $\zeta = \sqrt{m\omega_0}z$ and obtain

$$\partial_\zeta^2 Z + 2\zeta\partial_\zeta Z + \left(\zeta^2 + \frac{2k}{\omega_0}\right)Z = 0, \quad (4.23)$$

which solution is

$$Z(\zeta) = A_1 e^{-\frac{\zeta^2}{2} - \zeta\sqrt{1-\frac{2k}{\omega_0}}} + A_2 e^{-\frac{\zeta^2}{2} + \zeta\sqrt{1-\frac{2k}{\omega_0}}}, \quad (4.24)$$

where A_1 and A_2 are constants to be determined by the boundary conditions.

Now, we concentrate our analysis on the radial equation. We can make a change of variables similar to the previous one, $\xi = \sqrt{m\omega_0}\rho$. The radial Eq. 4.21 becomes

$$\left[-\partial_\xi^2 - \left(\frac{1}{\xi} + 2\xi\right)\partial_\xi - \left(1 + \frac{g^2E_0^2}{m^2\omega_0^2}\right)\xi^2 - 2gE_0\mathcal{E}\frac{\xi}{(m\omega_0)^{\frac{3}{2}}} + \frac{L^2}{\alpha^2\xi^2} + \frac{\gamma}{m\omega_0}\right]R(\xi) = 0. \quad (4.25)$$

We propose the ansatz $R(\xi) = e^{\xi^2 b_1 + \xi b_2}\xi^{b_3}F(\xi)$, where the constants b_1, b_2 and b_3 are

$$b_1 = \frac{1}{2}\left(-1 \pm i\frac{gE_0}{m\omega_0}\right), \quad (4.26a)$$

$$b_2 = \pm\left(\frac{\gamma}{m\omega_0} - \frac{2gE_0\mathcal{E}}{m\omega_0\sqrt{m\omega_0}}\right)\left(2i\frac{gE_0}{m\omega_0}\right)^{-1}, \quad (4.26b)$$

$$b_3 = \frac{L}{\alpha}. \quad (4.26c)$$

Then, Eq. 4.25 can be written as

$$F''(\xi) + \left(\frac{\bar{\gamma}}{\xi} + \delta + \xi\varepsilon\right)F'(\xi) + \left(\frac{\xi\overline{\alpha} - q}{\xi}\right)F(\xi) = 0, \quad (4.27)$$

where

$$\bar{\gamma} = 2\frac{L}{\alpha} + 1, \tag{4.28a}$$

$$\varepsilon = \pm 2i\frac{gE_0}{m\omega_0}, \tag{4.28b}$$

$$q = \mp\left(\left(\frac{1}{2} + \frac{L}{\alpha}\right)\left(\frac{\gamma}{m\omega_0} - \frac{2gE_0\mathcal{E}}{m\omega_0\sqrt{m\omega_0}}\right)\left(i\frac{gE_0}{m\omega_0}\right)^{-1}\right), \tag{4.28c}$$

$$\delta = \pm\left(\frac{\gamma}{m\omega_0} - \frac{2gE_0\mathcal{E}}{m\omega_0\sqrt{m\omega_0}}\right)\left(i\frac{gE_0}{m\omega_0}\right)^{-1}, \tag{4.28d}$$

$$\bar{\alpha} = -\frac{\gamma}{m\omega_0} + \left(\frac{\gamma}{m\omega_0} - \frac{2gE_0\mathcal{E}}{m\omega_0\sqrt{m\omega_0}}\right)^2\left(-4\frac{g^2E_0^2}{m^2\omega_0^2}\right)^{-1} + 2\frac{L}{\alpha} + 2\left(-1 \pm i\frac{g\left|E_0\right|}{m\omega_0}\right)\left(1 + \frac{L}{\alpha}\right). \tag{4.28e}$$

Note that Eq. 4.27 is the biconfluent Heun equation, BCH, (RONVEAUX; ARSCOTT, 1995). It has a regular singularity at $\xi = 0$ and an irregular singularity at ∞ of rank 2 (ARRIOLA *et al.*, 1991). The solution for this equation is described by a biconfluent Heun solution, $\mathrm{H_B}$:

$$F\left(\xi\right) = B_1H_B\left(q, \bar{\alpha}, \bar{\gamma}, \delta, \epsilon, \xi\right) + B_2\xi^{1-\bar{\gamma}}H_B\left(q - \left(1 - \bar{\gamma}\right)\delta, \bar{\alpha} + \left(1 - \gamma\right)\epsilon, 2 - \bar{\gamma}, \delta, \epsilon, \xi\right), \tag{4.29}$$

where B_1 and B_2 are constants.

We propose a power series solutions of Eq. 4.27 by means of the Frobenius method.

$$F\left(\xi\right) = \sum_{n=0} c_n\xi^{n+r}, \tag{4.30}$$

where c_n are constant coefficients and r is natural number.

By substituting the series expansions into Eq. 4.27, we get

$$\begin{aligned} & r\left[r - 1 + \bar{\gamma}\right]c_0\xi^{r-1} \\ & + \left[c_1\left(r + r^2 + \bar{\gamma} + \bar{\gamma}r\right) + c_0\left(\delta r - q\right)\right]\xi^r \\ & + \sum_{n=0}\left[\left(n + 2 + r\right)\left(n + 1 + r\right) + \bar{\gamma}\left(n + 2 + r\right)\right]c_{n+2}\xi^{n+1+r} \\ & + \sum_{n=0}\left[\delta\left(n + 1 + r\right) - q\right]c_{n+1}\xi^{n+r+1} \\ & + \sum_{n=0}\left[\varepsilon\left(n + r\right) + \overline{\alpha}\right]c_n\xi^{n+r+1} = 0. \end{aligned} \tag{4.31}$$

From the coefficients of ξ^{r-1} , we find two solutions for r,

$$r = 0 \text{ or } r = 1 - \bar{\gamma}. \tag{4.32}$$

For ξ^{r}, we obtain

$$c_1 = \frac{\delta r - q}{r\left(1+r\right) + \bar{\gamma}\left(1+r\right)} c_0, \tag{4.33}$$

and the coefficients of ξ^{n+r+1} lead us to the recurrent relation

$$\begin{aligned} c_{n+2} =& \frac{\left[-\delta\left(n+1+r\right)+q\right]}{\left(n+2+r\right)\left(n+1+r\right)+\bar{\gamma}\left(n+2+r\right)} c_{n+1} \\ &- \frac{\varepsilon\left(n+r\right)+\overline{\alpha}}{\left(n+2+r\right)\left(n+1+r\right)+\bar{\gamma}\left(n+2+r\right)} c_n. \end{aligned} \tag{4.34}$$

The power series becomes a polynomial if there is a value of n, $n = n_0$, such that the numerator of the last term vanish,

$$\varepsilon\left(n_0 + r\right) + \overline{\alpha} = 0, \tag{4.35}$$

and

$$c_{n_0+1} = 0. \tag{4.36}$$

So, the quantization conditions are

$$\begin{cases} \overline{\alpha}/\varepsilon = -n_0, & \text{for } r = 0, \\ \bar{\gamma} - \overline{\alpha}/\varepsilon - 1 = n_0, & \text{for } r = 1 - \bar{\gamma}. \end{cases} \tag{4.37}$$

We will use this result to obtain the Heun polynomials and, consequently, the quantization of certain physical observables. It is also common to see the biconfluent Heun equation in its canonical form,

$$\xi F''\left(\xi\right) + \left[1 + a - b\xi - 2\xi^2\right] F'\left(\xi\right) + \left[-\frac{1}{2}\left[d + \left(1+a\right)\beta\right] + \left(g - a - 2\right)\xi\right] F\left(\xi\right) = 0. \tag{4.38}$$

We can easily make a map between the form of Eq.4.27 with its canonical form defining

$$\bar{\gamma} = 1 + a, \tag{4.39a}$$

$$\delta = -b, \tag{4.39b}$$

$$\varepsilon = -2, \tag{4.39c}$$

$$\overline{\alpha} = (g - a - 2)\,, \tag{4.39d}$$

$$q = \frac{1}{2}\left[d + (1 + a)\,\beta\right]. \tag{4.39e}$$

So the condition 4.37 becomes, as presented in (ARRIOLA *et al.*, 1991),

$$\begin{cases} g - 2 - a = 2n_0, & \text{for } r = 0, \\ g - 2 + a = 2n_0, & \text{for } r = -a. \end{cases} \tag{4.40}$$

The condition 4.35 leads us to the quantization of the constant k,

$$K = \pm i\frac{gE_0}{m\omega_0}\left[2\left(n_0 + \frac{1}{2}\right) + \left(\frac{L}{\alpha}\right)^2\right] - \frac{m}{\omega_0}, \tag{4.41}$$

where K is a dimensionless constant defined by

$$K \equiv \frac{2k}{\omega_0}, \tag{4.42}$$

but it is only possible if the Eq. 4.36 is also satisfied. Using the recurrence relation 4.34 to express c_{n_0+1}, we have

$$\begin{aligned} c_{n_0+1} =& \frac{\left[-\delta\,(n_0 + r) + q\right]}{(n_0 + 1 + r)\,(n_0 + r) + \bar{\gamma}\,(n_0 + 1 + r)} c_{n_0} \\ & - \frac{\varepsilon\,(n_0 - 1 + r) + \overline{\alpha}}{(n_0 + 1 + r)\,(n_0 + r) + \bar{\gamma}\,(n_0 + 1 + r)} c_{n_0-1}. \end{aligned} \tag{4.43}$$

Imposing $c_{n_0+1} = 0$, thus

$$\left[-\delta\,(n_0 + r) + q\right] c_{n_0} - \left[\varepsilon\,(n_0 - 1 + r) + \overline{\alpha}\right] c_{n_0-1} = 0. \tag{4.44}$$

From 4.35 we can find $\overline{\alpha}$,

$$\overline{\alpha} = -\varepsilon\,(n_0 + r)\,, \tag{4.45}$$

and substituting in the Eq. 4.43,

$$\left[-\delta\,(n_0 + r) + q\right] c_{n_0} + \varepsilon c_{n_0-1} = 0. \tag{4.46}$$

In our situation, there is not a general expression for the biconfluent Heun function for a generic n_0. We have to solve it for a given degree of the polynomial considered. We shall show that the condition for $n_0 = 1$, which corresponds to a linear polynomial, can be satisfied by adjusting the parameter α, either the energy or the electric field.

For $n_0 = 1$, we have

$$\left[-\delta\left(1+r\right)+q\right]c_1+\varepsilon c_0=0. \tag{4.47}$$

Isolating C_1 in Eq. 4.33 and substituting in Eq. 4.47 we get

$$\left[-\delta\left(1+r\right)+q\right]\frac{\delta r-q}{r\left(1+r\right)+\bar{\gamma}\left(1+r\right)}+\varepsilon=0. \tag{4.48}$$

For $r = 0$, the above equation is

$$-\left[\left(\frac{1}{2}+\frac{L}{\alpha}\right)+\left(\frac{1}{2}+\frac{L}{\alpha}\right)^2\right]\left[\frac{1}{g^2E_0^2}\left(\gamma-\frac{2gE_0\mathcal{E}}{\sqrt{m\omega_0}}\right)^2\right]\pm 4i\frac{|gE_0|}{m\omega_0}\left(\frac{L}{\alpha}+\frac{1}{2}\right)=0. \tag{4.49}$$

Here we have two possible approaches, the first and the most intuitive is to make

$$\frac{1}{2}+\frac{L}{\alpha}=0. \tag{4.50}$$

This choice is satisfies if $L = -\alpha/2$. Since the quantization of the angular momentum give us $L \in \mathbb{Z}$, the possible solution of this equation is for $\alpha > 1$. On the other hand α is defined as $\alpha = 1 - 4\mu \in (0, 1]$, where μ is the string linear mass density. The Eq. 4.50 is only satisfied for a negative linear mass density. Therefore we discards this possibility.

A second approach is to find the energy, $\mathcal{E}$, that satisfies this equation. This provides a condition on the energy spectrum of the KGO:

$$\mathcal{E}_{L,k}=-\frac{E_0g}{\sqrt{m\omega_0}}\pm\sqrt{\frac{E_0^2g^2}{m\omega_0}+\frac{\left(2+2i\right)E_0g}{\sqrt{\frac{2L}{\alpha}+3}}\sqrt{\frac{E_0g}{m\omega_0}}+2km+m^2-3m\omega_0}, \tag{4.51}$$

$$\mathcal{E}_{L,k}=-\frac{E_0g}{\sqrt{m\omega_0}}\pm\sqrt{\frac{E_0^2g^2}{m\omega_0}-\frac{\left(2+2i\right)E_0g}{\sqrt{\frac{2L}{\alpha}+3}}\sqrt{\frac{E_0g}{m\omega_0}}+2km+m^2-3m\omega_0}. \tag{4.52}$$

In the system without electric field, $E_0 = 0$, the energy is

$$\mathcal{E}_k^2=2km+m^2-3m\omega_0. \tag{4.53}$$

4.2 Axial electric Field

This section is dedicated to the study of the KGO under a static electric field parallel to the cosmic string, an electric field in the z direction. We modify the Lagrangian density of the KGO by introducing a vector potential associated with the axial field and analyze its dynamics in a cosmic string spacetime. We introduce an electric field in the z-direction as

$$\vec{E} = E_0 \hat{e}_z, \tag{4.54}$$

where E_0 is a constant. We can use the definition of the Maxwell tensor $F_{\mu\nu}$, eq 4.2, to find the components of the gauge field. The simplest vector potential associated with a static electric field in the axial direction can be written as

$$\vec{A} = E_0 z \hat{e}_t. \tag{4.55}$$

We consider the minimal coupling between the KGO fields and the electromagnetic field by changing the ordinary derivatives to covariant derivatives (ACEVEDO R.R. CUZINATTO; POMPEIA, 2018; UTIYAMA, 1956) as follows:

$$\nabla_\mu \psi = \partial_\mu \psi - igA_\mu \psi \begin{cases} \nabla_t \psi = \partial_t \psi - igE_0 z\psi, \\ \nabla_\rho \psi = \partial_\rho \psi, \\ \nabla_\phi \psi = \partial_\phi \psi, \\ \nabla_z \psi = \partial_z \psi, \end{cases} \tag{4.56}$$

$$\nabla_\mu \psi^* = \partial_\mu \psi^* + igA_\mu \psi^* \begin{cases} \nabla_t \psi^* = \partial_t \psi^* + igE_0 z\psi^*, \\ \nabla_\rho \psi^* = \partial_\rho \psi^*, \\ \nabla_\phi \psi^* = \partial_\phi \psi^*, \\ \nabla_z \psi^* = \partial_z \psi^*. \end{cases} \tag{4.57}$$

The Lagrangian density 3.8 becomes:

$$\begin{aligned} \mathcal{L} =& \alpha\rho \left(\partial_t \psi^* + igE_0 z\psi^*\right)\left(\partial_t \psi - igE_0 z\psi\right) - \alpha\rho\partial_\rho\psi^*\partial_\rho\psi - \alpha\rho m\omega_0\rho\psi\partial_\rho\psi^* \\ &+ \alpha\rho m\omega_0\rho\psi^*\partial_\rho\psi + \alpha\rho m^2\omega_0^2\rho^2\psi^*\psi - \frac{1}{\alpha\rho}\left(\partial_\phi\psi^*\partial_\phi\psi\right) - \alpha\rho\partial_z\psi^*\partial_z\psi \\ &- \alpha\rho m\omega_0 z\psi\partial_z\psi^* + \alpha\rho m^2\omega_0^2 z^2\psi^*\psi + \alpha\rho m\omega_0 z\psi^*\partial_z\psi - \alpha\rho m^2\psi^*\psi, \end{aligned} \tag{4.58}$$

and the corresponding Euler-Lagrange equation is

$$\partial_t^2\psi - 2igE_0 z\partial_t\psi - \frac{1}{\rho}\partial_\rho\psi - \partial_\rho^2\psi - 3m\omega_0\psi - 2\rho m\omega_0\partial_\rho\psi - \frac{1}{\alpha^2\rho^2}\left(\partial_\phi^2\psi\right)$$
$$-\partial_z^2\psi - 2m\omega_0 z\partial_z\psi - g^2E_0^2z^2\psi - m^2\omega_0^2\rho^2\psi - m^2\omega_0^2z^2\psi + m^2\psi = 0. \qquad (4.59)$$

In order to analyze the motion of the KGO, let us admit that the function, solution of 4.59, has the general form $\psi(t,\rho,\phi,z) = G(t,\phi)R(\rho)Z(z)$. The equation of motion develop into

$$\frac{1}{G}\partial_t^2G - 2igE_0z\frac{1}{G}\partial_tG - \frac{1}{\rho}\frac{1}{R}\partial_\rho R - \frac{1}{R}\partial_\rho^2R - 3m\omega_0 - 2\rho m\omega_0\frac{1}{R}\partial_\rho R - \frac{1}{\alpha^2\rho^2}\frac{1}{G}\left(\partial_\phi^2G\right)$$
$$-\frac{1}{Z}\partial_z^2Z - 2m\omega_0z\frac{1}{Z}\partial_zZ - g^2E_0^2z^2 - m^2\omega_0^2\rho^2 - m^2\omega_0^2z^2 + m^2 = 0.$$
$$(4.60)$$

As we have done in the previous section, we propose an ansatz for the temporal and angular part in the form

$$G\left(t,\phi\right) = e^{-i(\mathcal{E}t - L\phi)}, \qquad (4.61)$$

where $L = 0, \pm1, \pm2, \pm3, \ldots$ is the angular momentum, its quantization also comes from the boundary condition upon angular coordinate and $\mathcal{E}$ is the energy. Using this solution in equation of motion 4.60 we obtain

$$-\frac{1}{Z}\partial_z^2Z - 2m\omega_0z\frac{1}{Z}\partial_zZ - g^2E_0^2z^2 - 2gE_0z\mathcal{E} - \frac{1}{\rho}\frac{1}{R}\partial_\rho R - \frac{1}{R}\partial_\rho^2R$$
$$-2\rho m\omega_0\frac{1}{R}\partial_\rho R - m^2\omega_0^2\rho^2 + \frac{L^2}{\alpha^2\rho^2} - 3m\omega_0 - m^2\omega_0^2z^2 + m^2 - \mathcal{E}^2 = 0. \qquad (4.62)$$

Separating the radial and axial parts by introducing a separation constant k with units of energy, the ρ-part is

$$-\partial_\rho^2R - \left(2\rho m\omega_0 + \frac{1}{\rho}\right)\partial_\rho R + \left(\frac{L^2}{\alpha^2\rho^2} - m^2\omega_0^2\rho^2 - \mathcal{E}^2 - 3m\omega_0 + m^2 - k\right)R = 0, \quad (4.63)$$

and the z-dependent sector reads

$$-\partial_z^2Z - 2m\omega_0z\partial_zZ - \left[\left(g^2E_0^2 + m^2\omega_0^2\right)z^2 + 2gE_0z\mathcal{E} - k\right]Z = 0. \qquad (4.64)$$

Solving the radial equation, proposing $R\left(\rho\right) = e^{-\frac{1}{2}m\omega_0\rho^2}F\left(\rho\right)$, we obtain

$$-\partial_\rho^2R - \frac{1}{\rho}\partial_\rho R + \left(\frac{L^2}{\alpha^2\rho^2} + \beta\right)R = 0, \qquad (4.65)$$

where $\beta \equiv -k - \mathcal{E}^2 - m\omega_0 + m^2$.

Note that Eq. 4.65 is the Bessel differential equation (BUTKOV, 1978). In order to obtain its canonical form, we define

$$x = -i\sqrt{\beta}\rho. \tag{4.66}$$

In term of the new variable we get the ordinary Bessel differential equation

$$x^2\partial_x^2 F + x\partial_x F + \left(x^2 - \frac{L^2}{\alpha^2}\right) F = 0, \tag{4.67}$$

which solution is

$$F(x) = B_1 J_{\frac{L}{\alpha}}(x) + B_2 Y_{\frac{L}{\alpha}}(x), \tag{4.68}$$

where B_1 and B_2 are constants.

As mentioned in the section 3.1, J_a is the Bessel function which is non-singular at the origin and Y_a is the Neumann function, which diverges in the limit $\rho \to 0$ and therefore we choose $B_2 = 0$. Finally, the general solution of ρ-dependent part is

$$R(\rho) = B_1 e^{-\frac{1}{2}m\omega_0\rho^2} J_{\frac{L}{\alpha}}\left(-i\rho\sqrt{\beta}\right). \tag{4.69}$$

The series expansion of $J_a(x)$ around $x = 0$ is found by the Frobenius method and gives

$$J_a(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a}, \tag{4.70}$$

where $\Gamma(z)$ is the gamma function (ARFKEN *et al.*, 2011).

The quantization of the energy is obtained imposing the wave function is zero for $\rho \to \infty$. The strategy is to define a finite radius ρ_0, where $\psi(t, \rho_0, \phi, z) = 0$. Then we will take the limit $\rho_0 \to \infty$. From $R(\rho_0) = 0$, we have

$$c_1 e^{-\frac{1}{2}mw_0\rho_0^2} J_{\frac{L}{\alpha}}\left(-i\sqrt{\beta}\rho_0\right) = 0. \tag{4.71}$$

Let $j_{\frac{L}{\alpha}p}$ be the p-root of Bessel function $J_{\frac{L}{\alpha}}(x)$, that means

$$c_1 e^{-\frac{1}{2}mw_0\rho_0^2} J_{\frac{L}{\alpha}}\left(j_{\frac{L}{\alpha}p}\right) = 0. \tag{4.72}$$

Comparing 4.71 and 4.72, we must have $-i\sqrt{\beta}\rho_0 = j_{\frac{L}{\alpha}p}$. Therefore, the energy $\mathcal{E}$ can be calculated in terms of the roots of the Bessel function, as

$$\mathcal{E}_{Lkp}^2 = m^2 - m\omega_0 + \frac{j_{\frac{L}{\alpha}p}^2}{\rho_0^2} - k. \tag{4.73}$$

In the limit where the boundary goes to infinity, $\rho_0 \to \pm\infty$, we obtain the energy levels as

$$\mathcal{E}_k^2 = m^2 - m\omega_0 - k. \tag{4.74}$$

Once we have established the radial solution, we return to the z-dependent differential Eq. 4.64 and proposing the following solution form $Z(z) = e^{-\frac{1}{2}(iE_0 g + m\omega_0)z^2 - i\mathcal{E}z} F(z)$ we get

$$\partial_z^2 F - 2i\left(E_0 g z + \mathcal{E}\right)\partial_z F - \left(k + iE_0 g + \mathcal{E}^2 + m\omega_0\right) F = 0. \tag{4.75}$$

Making the change of variables below in search of a known equation

$$x = -\frac{i}{\sqrt{iE_0 g}}\left(E_0 g z + \mathcal{E}\right), \tag{4.76}$$

we obtain

$$\partial_x^2 F - 2x\partial_x F - \lambda F = 0, \tag{4.77}$$

where $\lambda = -\left(-ik + E_0 g - i\mathcal{E}^2 - im\omega_0\right)/E_0 g$. This is the Hermite differential equation (OLVER *et al.*, 2020), which solution is

$$F(x) = A_1 M\left(\frac{1}{4}\lambda, \frac{1}{2}, x^2\right) + A_2 H_{-\frac{\lambda}{2}}(x), \tag{4.78}$$

where A_1 and A_2 are constants. $H_n(x)$ is the Hermite polynomial of order n, and $M(a, b, x)$ is the Kummer's confluent hypergeometric function.

The Hermite function $H_n(x)$ is well defined for all values of x and defines a sequence of orthogonal polynomials where the number n is the polynomial degree, for $n \in \mathbb{N}$. The function is defined by the Rodrigues formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right). \tag{4.79}$$

The Kummer's confluent hypergeometric function is defined by

$$M(a, b, x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}, \tag{4.80}$$

with the Pochhammer's symbol defined by

$$(a)_0 = 1, \tag{4.81}$$

$$(a)_n = a(a+1)(a+2)\dots(a+n-1). \tag{4.82}$$

When $a = -n_0$, $n_0 = 1, 2, 3, \dots$, the Kummer's confluent hypergeometric function is a polynomial of degree not exceeding n_0. If b is also a non-positive integer, $M(a, b, x)$ is not defined (ABRAMOWITZ; STEGUN, 1965).

As Hermite's solution covers all polynomial solutions including those obtained by Kummer's confluent hypergeometric function, we chose to keep the function H and discard the function M by imposing $c_1 = 0$.

The equation has the solution

$$Z\left(z\right) = A_2 e^{-\frac{1}{2}(iE_0 g + m\omega_0)z^2 - i\mathcal{E}z} H_{-\frac{\lambda}{2}}\left(\chi z + \gamma\right), \tag{4.83}$$

where

$$\gamma = \frac{(-1)^{\frac{1}{4}}\,\mathcal{E}}{\sqrt{E_0 g}} \quad \text{and} \quad \chi = (-1)^{\frac{1}{4}}\sqrt{E_0 g}. \tag{4.84}$$

Here, it is interesting to note that if the electric field is given by

$$E_0 = \frac{im^2}{g\left(1 - 2n_0\right)}, \tag{4.85}$$

the Hermite function becomes a polynomial of degree n_0.

5 Magnetic Field

In this chapter, we explore the Klein-Gordon Oscillator on a cosmic string background under the presence of two types of magnetic fields. The first system, not yet explored in this context in the literature, is the KGO in the presence of a static angular magnetic field, in which the associated equation of motion is solved numerically and the effects of the interaction field and the presence of the string is studied. A description of the magnetic field as a non-commutative geometry is also presented. The second scenario is the magnetic field parallel to the string in the presence of an external electrostatic potential, a scalar potential and a magnetic flux. We obtain the equation of motion and its solutions, finding the quantization of the related physical observables.

5.1 Angular magnetic field

The first scenario is an constant magnetic field in the angular direction, therefore we introduce an magnetic field as

$$\vec{B} = B_0 \hat{e}_\phi, \tag{5.1}$$

where B_0 is a constant. In the cosmic string spacetime, the vector potential associated with an static angular magnetic field in the Coulomb gauge can be expressed by

$$\vec{A} = \frac{1}{2} B_0 z \hat{e}_\rho - \frac{1}{2} B_0 \rho \hat{e}_z. \tag{5.2}$$

Using field strength tensor 4.2, it is straightforward to verify that this choice for the vector potential create the desired interaction field 5.1,

$$\vec{B} = \vec{\nabla} \times \vec{A} = B_0 \hat{e}_\phi. \tag{5.3}$$

Thus, making the minimal coupling in Eq. 3.8, the Lagrangian associated with the Klein–Gordon Oscillator in cosmic string spacetime in the presence of an angular magnetic field is

$$\begin{aligned}\mathcal{L} =&\alpha\rho\partial_t\psi^*\partial_t\psi + \left(m^2\omega_0^2 z^2 - m^2 - g^2\frac{1}{2}B_0^2 z^2 + m^2\omega_0^2\rho^2\right)\alpha\rho\psi^*\psi \\ &- \frac{1}{\alpha\rho}\partial_\phi\psi^*\partial_\phi\psi - \alpha\rho\partial_\rho\psi\partial_\rho\psi^* + \alpha\rho i g\frac{1}{2}B_0 z\left(\psi\partial_\rho\psi^* - \psi^*\partial_\rho\psi\right) \\ &+ \alpha\rho m\omega_0\rho\left(\psi^*\partial_\rho\psi - \psi\partial_\rho\psi^*\right) - \alpha\rho\partial_z\psi^*\partial_z\psi \\ &+ \alpha\rho i g\frac{1}{2}B_0\rho\left(\psi^*\partial_z\psi - \psi\partial_z\psi^*\right) + \alpha\rho m\omega_0 z\left(\psi^*\partial_z\psi - \psi\partial_z\psi^*\right). \end{aligned} \tag{5.4}$$

This Lagrangian density leads us to the following equation of motion for the field ψ:

$$\begin{aligned}\partial_t^2\psi - \frac{1}{\rho}\partial_\rho\psi - \partial_\rho^2\psi + \frac{1}{\rho}ig\frac{1}{2}B_0 z\psi + igB_0 z\partial_\rho\psi - 3m\omega_0\psi - 2m\omega_0\rho\partial_\rho\psi \\ -\frac{1}{\alpha^2\rho^2}\partial_\phi^2\psi - \partial_z^2\psi - igB_0\rho\partial_z\psi - 2m\omega_0 z\partial_z\psi + g^2\frac{1}{4}B_0^2 z^2\psi \\ -m^2\omega_0^2\rho^2\psi + g^2\frac{1}{4}B_0^2\rho^2\psi - m^2\omega_0^2 z^2\psi + m^2\psi = 0.\end{aligned} \tag{5.5}$$

For the temporal and angular part, we propose the following ansatz,

$$\psi(t,\rho,\phi,z) = e^{-i(\mathcal{E}t - L\phi)}F(\rho,z), \tag{5.6}$$

where $\mathcal{E}$ is the energy and L is the angular momentum, L is quantized by the periodic boundary condition upon ϕ and $L \in \mathbb{Z}$. By substituting this ansatz into Eq. 5.5, it becomes

$$\begin{aligned}\Bigg[-\partial_\rho^2 - \frac{1}{\rho}\left(\partial_\rho - ig\frac{1}{2}B_0 z\right) + igB_0\left(z\partial_\rho - \rho\partial_z\right) - 2m\omega_0\left(\rho\partial_\rho + z\partial_z\right) \\ +\frac{L^2}{\alpha^2\rho^2} - \partial_z^2 + \left(g^2\frac{1}{4}B_0^2 - m^2\omega_0^2\right)\left(z^2+\rho^2\right) + \gamma\Bigg]F(\rho,z) = 0,\end{aligned} \tag{5.7}$$

where $\gamma = -\mathcal{E}^2 - 3m\omega_0 + m^2$. It is convenient to make a change of variables that makes the coordinates dimensionless, therefore performing the following changes, $\xi = \sqrt{m\omega_0}\rho$ and $\zeta = \sqrt{m\omega_0}z$, we obtain

$$\begin{aligned}\Bigg[-\partial_\xi^2 - \frac{1}{\xi}\left(\partial_\xi - \frac{1}{2}i\bar{B}_0\zeta\right) + i\bar{B}_0\left(\zeta\partial_\xi - \xi\partial_\zeta\right) - 2\left(\xi\partial_\xi + \zeta\partial_\zeta\right) \\ +\frac{\bar{L}^2}{\xi^2} - \partial_\zeta^2 + \left(\frac{1}{4}\bar{B}_0^2 - 1\right)\left(\zeta^2+\xi^2\right) + \bar{\gamma}\Bigg]F(\xi,\zeta) = 0,\end{aligned} \tag{5.8}$$

where $\bar{B}_0 = gB_0/m\omega_0$, $\bar{L} = L/\alpha$ and $\bar{\gamma} = \gamma/m\omega_0$. These parameters are dimensionless and they are directly associated with the magnetic field, angular momentum, string parameter α and energy, respectively. In this coordinate system (ξ,ζ) the equation is not separable by the usual method of separation of variables, because the magnetic field couples the dependence over ξ and ζ. We take a different approach from the previous chapters and calculate a numerical solution to the equation of motion 5.8 and use numerical analysis to understand the behavior of the probability density of the complex scalar field.

5.1.1 Numerical Solution

Our approach in this section is to compute a numerical solution for the Eq. 5.8 and find the probability density of the complex scalar field, $|\psi(t,\xi,\phi,\zeta)|^2 = |F(\xi,\zeta)|^2$.

Although there are various methods for solving differential equations, such as finite elements, Monte Carlo, spectral and variational methods, we will use the finite difference method here. From a computational point of view, the finite difference method is, in general, relatively easy to implement, efficient and accurate for most physical problems. In this work, we show two finite difference methods for solving the PDE: the first and most intuitive is the explicit method, described in detail in section A.1 of the appendix A, which is the method used to find the results presented here, and the second is the Successive Overrelaxation (SOR) method with Chebyshev acceleration, section A.2.

Since we want a solution that vanishes far from the center of the string and smooth, without any singularities at the origin, we impose the boundary conditions below:

$$\lim_{\zeta\to\pm\infty} \psi(t,\xi,\phi,\zeta) = 0, \tag{5.9a}$$

$$\lim_{\xi\to\infty} \psi(t,\xi,\phi,\zeta) = 0, \tag{5.9b}$$

$$\partial_\xi \psi(t,\xi,\phi,\zeta)\big|_{\xi=0} = 0. \tag{5.9c}$$

As we can see from the Eq. 5.8, the system has three free parameters: $\bar{B}_0$, $\bar{\gamma}_0$ and $\bar{L}$. In this work, we set $\bar{B}_0$ to two values, 0.1 and 10, $\bar{L}$ to four values, 0, 1.1, 2 and 10, and $\bar{\gamma}_0$ to three values, -100, 0 and 100.

We note that as we increase the parameter $\bar{L}$, we induce our probability density to move away from the string, see Fig. 5.1, this can be interpreted as a change in angular momentum or in the parameter α of the string. For the case $\bar{L} = 1.1$, the only possibility is $L = 1$ and $\alpha = 0.9$, for $\bar{L} = 2$, we have $L = 2$ and $\alpha = 1$ or $L = 1$ and $\alpha = 0.5$, for the case $\bar{L} = 10$, we have ten possibilities. The case $\bar{L} = 0$ the cosmic string does not play a role in the solution.

We can also see that, for integer values of $\bar{L}$, it is always possible to find a system in which $\bar{L} = L$ and $\alpha = 1$. This scenario is described by Minkowski spacetime and there is no cosmic string present. A more general representation for this case where $\bar{L} \in \mathbb{Z}$ is to write $\bar{L} = L - n_0$, where n_0 is an integer smaller than $\bar{L}$. In this situation the string parameter is $\alpha = L/(L - n_0)$. Therefore, the system can always be interpreted as a space with a cosmic string, $n_0 \neq 0$, or a space without it $n_0 = 0$.

The magnetic field plays a significant role in the solution. For low values of the field, $\bar{B}_0 < 1$, the probability has a single peak, as shown in Fig. 5.2. For high values, $\bar{B}_0 > 1$, the probability splits into two peaks, see Fig. 5.3.

The parameter associated with the energy is $\bar{\gamma}$ and changing its values means a shift in the probability density peak in radial and axial directions, see Fig. 5.4 and 5.5, and a dispersion of the Gaussian behavior, since as we increase the value of $\bar{\gamma}$ the peak of the solution is higher and more concentrated, see Fig. 5.6.

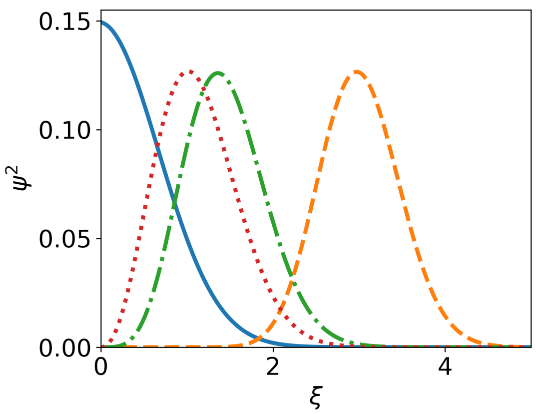


FIGURE 5.1 – $|\psi(t, \xi, \phi, \zeta)|^2$ with $\bar{B}_0 = 0.1$ and $\bar{\gamma}_0 = 0$ at $\zeta = 0$. The solid line represents the case $\bar{L} = 0$, dotted line $\bar{L} = 1.1$, the dashed dotted line $\bar{L} = 2$ and dashed line $\bar{L} = 10$.

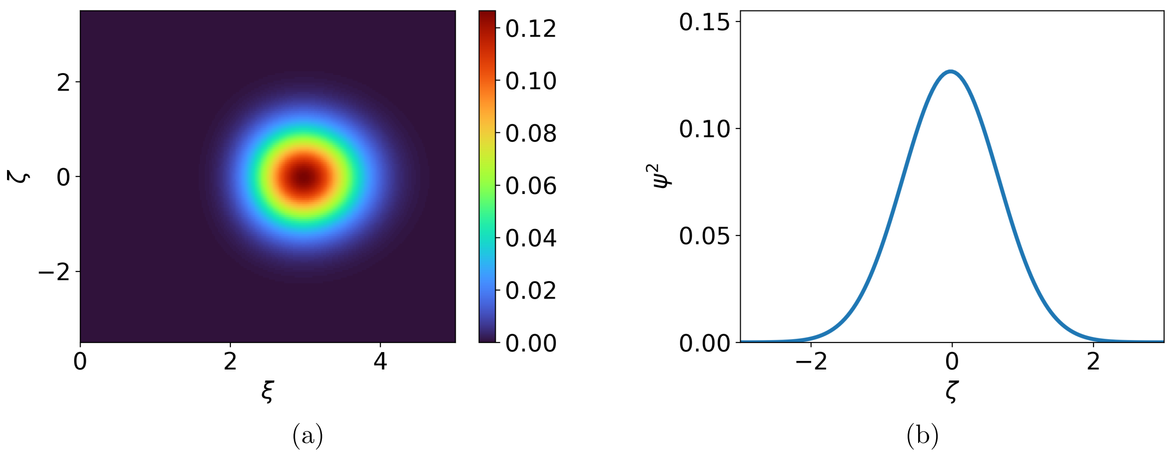
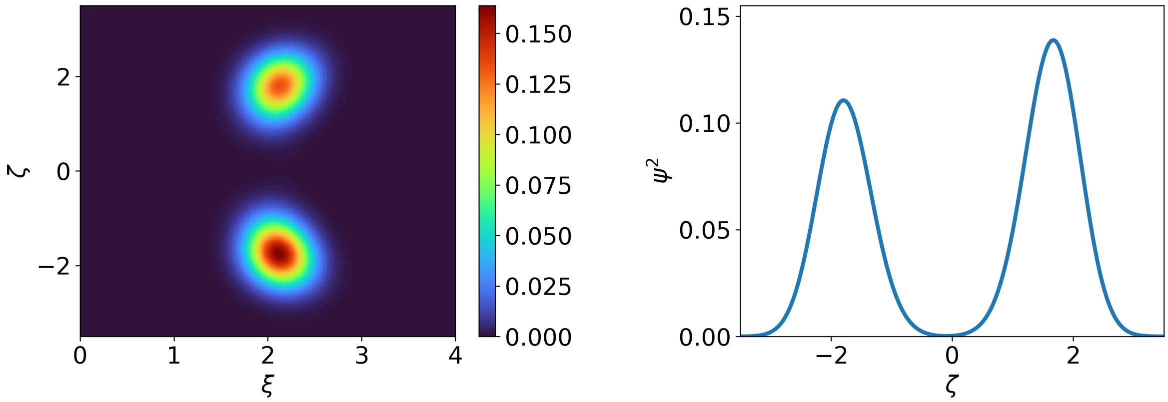
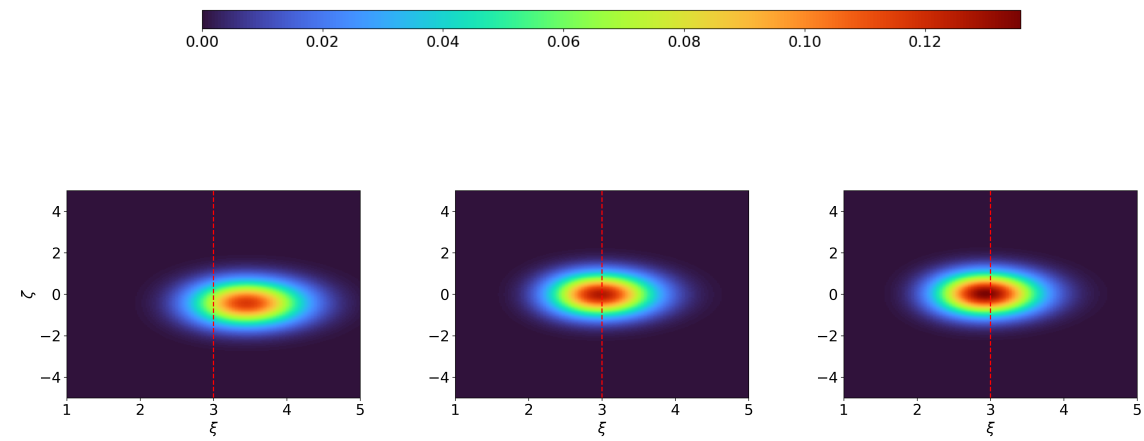


FIGURE 5.2 – $|\psi(t, \xi, \phi, \zeta)|^2$ with $\bar{B}_0 = 0.1$, $\bar{\gamma} = 0$ and $\bar{L} = 10$, (a) is a density plot of the solution $|\psi(t, \xi, \phi, \zeta)|^2$ and (b) is a slice at $\xi = 3$.



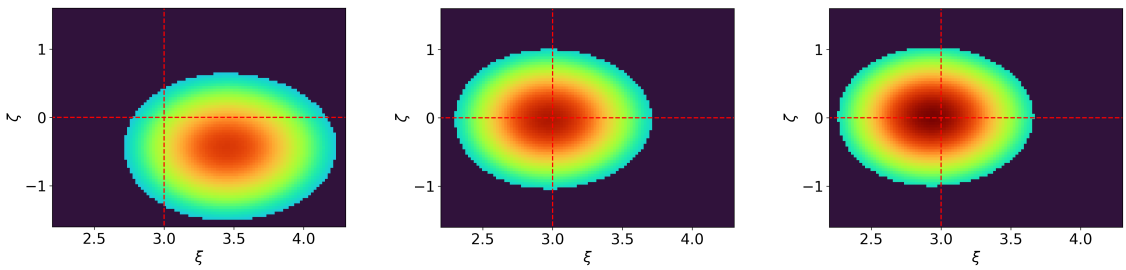
(a) (b)

FIGURE 5.3 – $|\psi(t,\xi,\phi,\zeta)|^2$ with $\bar{B}_0 = 10$, $\bar{\gamma} = 0$ and $\bar{L} = 10$, (a) is a density plot of the solution $|\psi(t,\xi,\phi,\zeta)|^2$ and (b) is a slice at $\xi = 2$.



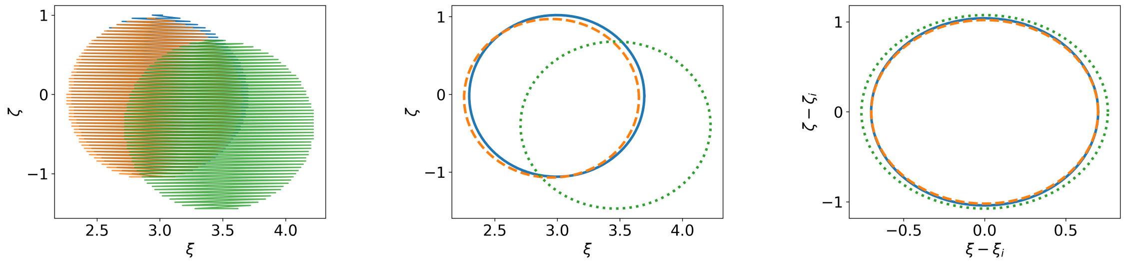
(a) (b) (c)

FIGURE 5.4 – $|\psi(t,\xi,\phi,\zeta)|^2$ with $\bar{B}_0 = 0.1$ and $\bar{L} = 10$. (a) $\bar{\gamma} = -100$, (b) $\bar{\gamma} = 0$ and (c) $\bar{\gamma} = 100$. The red dashed line is a vertical marker at $\xi = 3$.



(a) (b) (c)

FIGURE 5.5 – $|\psi(t,\xi,\phi,\zeta)|^2$ with $\bar{B}_0 = 0.1$ and $\bar{L} = 10$. (a) $\bar{\gamma} = -100$, (b) $\bar{\gamma} = 0$ and (c) $\bar{\gamma} = 100$. The red dashed line is a vertical marker at $\xi = 3$ and a horizontal marker at $\zeta = 0$. These figures show a cut of a $\xi\zeta$-plane in Fig. 5.4 at a height that keeps the volume for the standard deviation equal to σ.



(a) (b) (c)

FIGURE 5.6 – $|\psi(t,\xi,\phi,\zeta)|^2$ with $\bar{B}_0 = 0.1$ and $\bar{L} = 10$. The solid line and the blue disk is the case $\bar{\gamma} = -100$, the dash line and the orange disk is $\bar{\gamma} = 0$ and the dot line and the green disk $\bar{\gamma} = 100$. (a) The surfaces of Fig. 5.5 overlapping, (b) the boundary of the surface and (c) The surfaces translated to the same center, ξ_i and ζ_i represent the translation factor where $i = -100, 0$ and 100 respectively for the case $\gamma = -100, 0$ and 100. Their values are $\xi_{-100} = 2.96$, $\xi_0 = 3.005$, $\xi_{100} = 3.47$, $\zeta_{-100} = 0.02$, $\zeta_0 = 0$ and $\zeta_{100} = -0.43$.

5.1.2 The non-commutative geometry

In this section we shall make a map between the Klein-Gordon Oscillator with a static magnetic field in angular direction in a cosmic string background scenario with a particular case of the non-commutative phase space, the non-commutative momentum space.

The non-commutative momentum space is described by the operators $\hat{r}_i$ and $\hat{p}_i$, defined via the following generalized Bopp shift (CARVALHO *et al.*, 2011; CUZINATTO *et al.*, 2022):

$$\hat{r}_i = r_i - \frac{\Theta_{ij}}{2\hbar}p_j = r_i + \frac{\left(\vec{\Theta}\times\vec{p}\right)_i}{2\hbar}, \tag{5.10}$$

$$\hat{p}_i = p_i + \frac{\Omega_{ij}}{2\hbar}r_j = p_i - \frac{\left(\vec{\Omega}\times\vec{r}\right)_i}{2\hbar}. \tag{5.11}$$

Which satisfy the following commutation relations:

$$[\hat{r}_i,\hat{r}_j] = i\Theta_{ij}, \quad [\hat{p}_i,\hat{p}_j] = i\Omega_{ij}, \quad [\hat{r}_i,\hat{p}_j] = i\hbar\Delta_{ij}. \tag{5.12}$$

where

$$\Delta_{ij} = \left(1 + \frac{\vec{\Theta}\cdot\vec{\Omega}}{4\hbar^2}\right)\delta_{ij} - \frac{\Omega_i\Theta_j}{4\hbar^2}. \tag{5.13}$$

Since we are interested in the non-commutativity in momentum space, we take $\Theta_j = 0$ for $j = 1, 2, 3$, which allow us to maintain the spatial configuration for the cosmic string commutative. The relations 5.10 and 5.11 become

$$\hat{r}_i = r_i, \tag{5.14}$$

$$\hat{p}_i = p_i + \frac{\Omega_{ij}}{2\hbar} r_j = r_i - \frac{\left(\vec{\Omega} \times \vec{p}\right)_i}{2\hbar}. \tag{5.15}$$

The density Lagrangian for the non-commutative Klein-Gordon oscillator can be written as

$$\begin{aligned}\mathcal{L} = &- \alpha\rho \left[-\partial_t \psi^* \partial_t \psi + \left(\partial_\rho \psi^* - m\omega_0 \rho \psi^* - \frac{1}{2} i\Omega_2 z \psi^*\right)\left(\partial_\rho \psi + m\omega_0 \rho\psi - \frac{1}{2} i\Omega_2 z\psi\right)\right.\\ &+ \alpha^2\rho^2 \left(\partial_\phi \psi^* - \frac{1}{2} i\Omega_3 \rho\psi^* + \frac{1}{2} i\Omega_1 z\psi^*\right) \times \left(\partial_\phi \psi + \frac{1}{2} i\Omega_3\rho\psi - \frac{1}{2} i\Omega_1 z\psi\right)\\ &\left.+ \left(\partial_z \psi^* - m\omega_0 z\psi^* + \frac{1}{2} i\Omega_2 \rho\psi^*\right)\left(\partial_z \psi + m\omega_0 z\psi - \frac{1}{2} i\Omega_2 \rho\psi\right) + m^2 \psi^* \psi\right],\end{aligned} \tag{5.16}$$

and the related equation of motion is

$$\begin{aligned}&-m^2\psi + 3m\omega_0\psi + m^2\omega_0^2\rho^2\psi - \frac{1}{4}\Omega_2^2 z^2\psi + m^2\omega_0^2 z^2\psi + \frac{1}{4}\Omega_2^2\rho^2\psi\\ &+ \left(\frac{1}{\alpha^2\rho^2}\right)\left(-i\frac{1}{2}\Omega_3\rho + i\frac{1}{2}\Omega_1 z\right)^2 \psi + \left(i\frac{1}{2}\Omega_2 z\right)\frac{1}{\rho}\psi - \partial_t^2\psi + \partial_z^2\psi\\ &+2\left(m\omega_0\rho + i\frac{1}{2}\Omega_2 z\right)\partial_\rho\psi - 2\left(\frac{1}{\alpha^2\rho^2}\right)\left(-i\frac{1}{2}\Omega_3\rho + i\frac{1}{2}\Omega_1 z\right)^2 \partial_\phi\psi\\ &+2\left(m\omega_0 z - i\frac{1}{2}\Omega_2\rho\right)\partial_z\psi + \partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi + \left(\frac{1}{\alpha^2\rho^2}\right)\partial_\phi^2\psi = 0.\end{aligned} \tag{5.17}$$

Comparing the above equation with Eq. 5.5, we can map the non-commutative case with a magnetic field in angular direction as $\Omega_2 = -B_0 g$ and $\Omega_1 = \Omega_3 = 0$.

5.2 Axial magnetic field in the presence of a Coulomb and Cornell potentials and magnetic flux

In this section we shall study the KGO in cosmic string spacetime in the presence of static magnetic field parallel to the string with a electrostatic and Cornell Potentials, and a magnetic flux produced by the topological defect. We modify the Lagrangian density of the KGO by introducing a vector potential associated with the interaction field and coupling the non-electromagnetic potential by making a modification in the mass term as $m \to m + S(\rho)$, being $S(\rho)$ the scalar potential. The resulting equation of motion are solved for each scenario individually.

We introduce an constant magnetic field parallel to the string as

$$\vec{B} = B_0 \hat{e}_z. \tag{5.18}$$

Using the definition of the field strength tensor, Eq. 4.2, we can express vector potential associated with the axial magnetic field as the following:

$$\vec{A} = \frac{1}{2}\alpha B_0 \rho \hat{e}_\phi, \tag{5.19}$$

and we can easily verify that this potential create a constant magnetic field 5.18,

$$\vec{B} = \vec{\nabla} \times \vec{A} = B_0 \hat{e}_z. \tag{5.20}$$

Lets assume that the cosmic string has an internal magnetic flux. Thus, we have the angular component of the four-vector electromagnetic potential given by the interaction field plus the magnetic flux,

$$A_\phi = \frac{1}{2}\alpha B_0 \rho + \frac{\Phi_B}{2\pi}, \tag{5.21}$$

where Φ_B is a constant quantum flux through the core of the topological defects (AHARONOV; BOHM, 1959; AHMED, 2021; MARQUES *et al.*, 2001; AHMED, 2020; FURTADO; MORAES, 2000). We can introduce the electrostatic potential $V(\rho)$ through the zero component of Lorentz vector (BONDARCHUK *et al.*, 2007) as

$$A_0 = V(\rho) = \frac{\xi_C}{\rho}, \tag{5.22}$$

where $\xi_C > 0$ is the Coulomb potential parameter. Therefore, the electromagnetic vector potential of the magnetic field plus the Coulomb potential and the magnetic flux is

$$\vec{A} = \frac{\xi_C}{\rho}\hat{e}_0 + A_\phi \hat{e}_\phi. \tag{5.23}$$

This choice of vector potential has never been explored in the context of the presence of an external electrostatic potential, a scalar potential and a magnetic flux.

We will also consider in this analysis the presence of cylindrical symmetric scalar potentials, this scalar potential is a Cornell-type potential (LEITE *et al.*, 2020) and it reads

$$S(\rho) = \frac{\eta_C}{\rho} + \eta_L \rho, \tag{5.24}$$

here, η_C and η_L are regarded as free parameters. This is a Cornell-type potential developed in the 1970s to create a model that incorporates the confinement of quarks and was used to determine the quantum numbers and estimate the masses and decay widths of quark-antiquark bound states (EICHTEN *et al.*, 1975). The potential consists of two parts. The first one, which has a Coulomb-like form, dominates at short distances and the second term of the potential is the linear confinement term. A plot of the potential can be seen in Fig. 5.7.

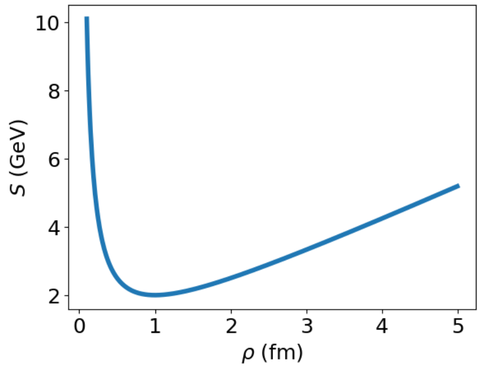


FIGURE 5.7 – Cornell-type potential with $\eta_L = 1$ GeV/fm and $\eta_C = 1$GeVfm.

Non-electromagnetic potential can be introduced in the system by making a modification in the mass term as $m \to m + S(\rho)$, being $S(\rho)$ the scalar potential (MEDEIROS; MELLO, 2012; BONDARCHUK *et al.*, 2007) and we consider the minimal coupling between the KGO and electromagnetic fields. So the density Lagrangian 3.1 becomes

$$\begin{aligned}
\mathcal{L} = \alpha\rho\left(\partial_t\psi^* + ig\frac{\xi_C}{\rho}\psi^*\right)\left(\partial_t\psi - ig\frac{\xi_C}{\rho}\psi\right) - \alpha\rho\left(m + S(\rho)\right)^2\psi^*\psi \\
-\alpha\rho\left(\partial_\rho\psi^*\partial_\rho\psi + m\omega_0\rho\psi\partial_\rho\psi^* - m\omega_0\rho\psi^*\partial_\rho\psi - m^2\omega_0^2\rho^2\psi^*\psi\right) \\
-\frac{1}{\alpha\rho}\left(\partial_\phi\psi^*\partial_\phi\psi - ig\frac{1}{2}\alpha B_0\rho\psi\partial_\phi\psi^* - \frac{ig\Phi_B}{2\pi}\psi\partial_\phi\psi^* + ig\frac{1}{2}\alpha B_0\rho\psi^*\partial_\phi\psi\right. \\
\left.+g^2\frac{1}{4}\alpha^2B_0^2\rho^2\psi^*\psi + \frac{g\Phi_B}{2\pi}gB_0\alpha\rho\psi^*\psi + \frac{ig\Phi_B}{2\pi}\psi^*\partial_\phi\psi + \frac{g^2\Phi_B^2}{4\pi^2}\psi^*\psi\right) \\
-\alpha\rho\left(\partial_z\psi^*\partial_z\psi + m\omega_0z\psi\partial_z\psi^* - m^2\omega_0^2z^2\psi^*\psi - m\omega_0z\psi^*\partial_z\psi\right)
\end{aligned} \tag{5.25}$$

and the related equation of motion is

$$\begin{aligned}
\partial_t^2\psi - 2ig\frac{\xi_C}{\rho}\partial_t\psi - \partial_\rho^2\psi - \left(\frac{1}{\rho} + 2m\omega_0\rho\right)\partial_\rho\psi - \frac{1}{\alpha^2\rho^2}\partial_\phi^2\psi + \left(\frac{1}{\alpha\rho}igB_0 + \frac{ig\Phi_B}{\pi\alpha^2\rho^2}\right)\partial_\phi\psi \\
-\partial_z^2\psi - 2m\omega_0z\partial_z\psi - m^2\omega_0^2z^2\psi + \left(\eta_L^2 - m^2\omega_0^2\right)\rho^2\psi + 2m\eta_L\rho\psi + \left(2m\eta_C + \frac{g\Phi_B}{2\pi\alpha}gB_0\right)\frac{1}{\rho}\psi \\
+\left(\eta_C^2 - g^2\xi_C^2 + \frac{g^2\Phi_B^2}{4\pi^2\alpha^2}\right)\frac{1}{\rho^2}\psi + m^2\psi + 2\eta_C\eta_L\psi - 3m\omega_0\psi + \frac{1}{4}g^2B_0^2\psi = 0.
\end{aligned} \tag{5.26}$$

In similar way of we have done in the previous sections, for the angular and temporal part we propose the ansatz with a separable solution as

$$\psi(t,\rho,\phi,z) = e^{i(L\phi-\mathcal{E}t)}R(\rho)\,Z(z)\,. \tag{5.27}$$

$\mathcal{E}$ is interpreted as the energy and L is the angular momentum, $L = 0, \pm1, \pm2, \pm3\ldots$, which quantization comes from the periodic boundary condition upon the angular coordinate. Then introducing a separable constant k, the equation decouples in a radial dependent equation

$$-\partial_\rho^2 R - \left(\frac{1}{\rho} + 2m\omega_0\rho\right)\partial_\rho R + \left[-m^2\omega_0^2\rho^2 + \frac{1}{\rho^2}\left(\frac{1}{\alpha^2}\left(L - \frac{g\Phi_B}{2\pi}\right)^2 - g^2\xi_C^2\right)\right.$$
$$\left. -\frac{1}{\rho}\left(\frac{gB_0}{\alpha}\left(L - \frac{g\Phi_B}{2\pi}\right) - 2g\xi_C\mathcal{E}\right) + (m + S(\rho))^2 + \beta\right]R = 0 \tag{5.28}$$

where $\beta = -\mathcal{E}^2 - 3m\omega_0 + \frac{1}{4}g^2B_0^2 + 2mk$, and z-dependent equation

$$-\partial_z^2 Z - 2m\omega_0 z\partial_z Z + \left(2mk - m^2\omega_0^2 z^2\right)Z = 0. \tag{5.29}$$

Looking at the z-dependent equation, we make the following change of variables:

$$z = \frac{\zeta}{\sqrt{m\omega_0}}, \tag{5.30}$$

where ζ is a dimensionless variable. The result equation is

$$-\partial_\zeta^2 Z - 2\zeta\partial_\zeta Z - \left(\frac{2k}{\omega_0} + \zeta^2\right)Z = 0, \tag{5.31}$$

which solution is

$$Z(\zeta) = A_1 e^{-\frac{\zeta^2}{2} - \zeta\sqrt{1-\frac{2k}{\omega_0}}} + A_2 e^{-\frac{\zeta^2}{2} + \zeta\sqrt{1-\frac{2k}{\omega_0}}} \tag{5.32}$$

where A_1 and A_2 are constants to be determined from the boundary conditions.

Now, returning to the radial Eq. 5.28, we first introduce a new dimensionless variable ξ as

$$\rho = \frac{\xi}{\sqrt{m\omega_0}} \tag{5.33}$$

and we obtain the following radial equation:

$$-\partial_\xi^2 - \left(\frac{1}{\xi} + 2\xi\right)\partial_\xi R + \left[-\xi^2 - \frac{1}{\xi\sqrt{m\omega_0}}\left(\frac{gB_0}{\alpha}\left(L - \frac{g\Phi_B}{2\pi}\right) + 2g\xi_C\mathcal{E}\right)\right.$$
$$\left. + \frac{1}{\xi^2}\left(-g^2\xi_C^2 + \frac{1}{\alpha^2}\left(L - \frac{g\Phi_B}{2\pi}\right)^2\right) + \frac{1}{m\omega_0}(m+S)^2 + \frac{\beta}{m\omega_0}\right] R = 0. \tag{5.34}$$

It is worth noting that the presence of Coulomb potential makes energy appear in different place of the ODE, which impacts the energy spectrum as we are going to see further.

We can also observe that the angular momentum L of the system is shifted as $L \to \bar{L}$, where $\bar{L} = \frac{1}{\alpha}\left(L - \frac{g\Phi_B}{2\pi}\right)$. This change is related to the presence of a quantum flux produced by the topological defect and it is analogous to the Aharonov–Bohm effect (AHARONOV; BOHM, 1959; PESHKIN, 2005), a quantum mechanical phenomena that describe phase shifts of the wave-function of a quantum particle. We shall see how this magnetic flux affect the physical observables.

In the following subsections, we shall analyze the radial differential equation obtained earlier, considering different physical situations: I) only the presence of the magnetic field, II) linear confinement potential, III) Coulomb-type potential, IV) Cornell potential V) Electrostatic potential VI) Magnetic flux and VII) Complete scenario. In all situations, the magnetic field is present.

5.2.1 Landau Levels

The first scenario is characterized by the presence of the magnetic field and the absence of potentials and magnetic flux, which means that $\eta_C = \eta_L = \xi_C = \Phi_B = 0$. This system is a particular case the one discussed in (CUZINATTO *et al.*, 2022). The Eq. 5.34 becomes

$$-\partial_\xi^2 R - \left(\frac{1}{\xi} + 2\xi\right)\partial_\xi R + \left(-\xi^2 + \frac{L^2}{\alpha^2\xi^2} - \frac{gB_0L}{\alpha\sqrt{m\omega_0}\xi} + \frac{m}{\omega_0} + \frac{\beta}{m\omega_0}\right) R = 0. \tag{5.35}$$

We propose the following ansatz:

$$R(\xi) = \xi^{L/\alpha}\exp\left(-\frac{\xi\sqrt{\beta + m^2 + 2m\omega_0}}{\sqrt{m\omega_0}} - \frac{\xi^2}{2}\right) F(\xi), \tag{5.36}$$

where the $F(\xi)$ is a function to be determined. Thus, the differential Eq. 5.35 becomes

$$\xi\partial_\xi^2 F + (1 + a - b\xi)\partial_\xi F + nF = 0, \tag{5.37}$$

with the following constants:

$$a = 2\xi\sqrt{\frac{\beta}{m\omega_0} + \frac{m}{\omega_0} + 2}, \tag{5.38a}$$

$$b = 2\frac{L}{\alpha}, \tag{5.38b}$$

$$n = \frac{B_0 g L}{\alpha\sqrt{m\omega_0}} - \frac{\left(1 + 2\frac{L}{\alpha}\right)}{\sqrt{m\omega_0}}\sqrt{\beta + m^2 + 2m\omega_0}. \tag{5.38c}$$

This is the associated Laguerre differential equation (ZWILLINGER; DOBRUSHKIN, 2021) with the solution

$$F(\xi) = B_1 U\left(-\frac{n}{b}, a+1, b\xi\right) + B_2 L^{a}_{\frac{n}{b}}(b\xi), \tag{5.39}$$

where B_1 and B_2 are constants.

$U(a, b, z)$ is the Tricomi confluent hypergeometric function introduced by Francesco Tricomi (TRICOMI, 1947) and $L^{k}_{n_0}(z)$ is the associated Laguerre polynomial.

The function U has a singularity at $z = 0$ and it can be express in terms of Kummer's function of the first kind $M(a, b, z)$ as (ABRAMOWITZ; STEGUN, 1965)

$$U(a, b, z) = \frac{\pi}{\sin(\pi b)}\left[\frac{M(a, b, z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(a+1-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)}\right], \tag{5.40}$$

where the function M was explored in the section 4.2. Since U is divergent at the origin, we discard this solution imposing $c_1 = 0$.

The associated Laguerre polynomial can be express as

$$L^{k}_{n_0}(z) = \sum_{a}^{n_0} \frac{(k+a+1)_{n_0-a}}{(n_0-a)!a!}(-z)^a, \tag{5.41}$$

where the lower index is used to represent the Pochhammer's symbol, which definition is in Eq. 4.82.

We have a polynomial solution for n_0 being a non-negative integer. Imposing this condition in our solution 5.39, that means $n/b = n_0$, we find the energy quantization expressed

$$\mathcal{E}_{k,L,n_0} = \pm\sqrt{-\frac{B_0^2 g^2 L^2}{4\alpha^2\left(\frac{L}{\alpha} + n_0 + \frac{1}{2}\right)^2} + \frac{B_0^2 g^2}{4} + 2mk + m^2 - m\omega_0}. \tag{5.42}$$

In the first term, we can see that the presence of the string factor α and the magnetic field B_0 modify the degenerate spectrum of the KGO. Each set of wave functions with the same value of n_0 is called a Landau level.

Eq. 5.42 also reveals that the quantization is lost when $B_0 \to 0$ or $L = 0$ and complex values for the energy are possible for

$$\frac{B_0^2 g^2 L^2}{4\alpha^2 \left(\frac{L}{\alpha} + n_0 + \frac{1}{2}\right)^2} + m\omega_0 > \frac{B_0^2 g^2}{4} + 2km + m^2. \tag{5.43}$$

In fact, we would have a pure complex energy for the above situation and it can be interpreted as a dissipative term for the minus sign or a gain term for the plus sign, which means that $\psi = e^{-i\mathcal{E}t}\varphi = e^{\pm \mathrm{Im}(\mathcal{E})t}\varphi$. The positive sign for $\mathrm{Im}\left(\mathcal{E}\right)$ lead us to the divergence of the wave function, therefore we discard this possibility since it is unphysical.

Note that the condition 5.43 gives a restriction on the constant k; for real values of the energy, the condition becomes:

$$k > \frac{B_0^2 g^2 L^2}{8m\alpha^2 \left(\frac{L}{\alpha} + n_0 + \frac{1}{2}\right)^2} + \frac{\omega_0}{2} - \frac{B_0^2 g^2}{8m} - \frac{m}{2}. \tag{5.44}$$

This constraint is essential when we do not admit energy dissipation.

The same energy spectrum has been developed in (CUZINATTO *et al.*, 2022) considering a cosmic string space-time equipped with a rotating frame. If the frame is static their energy eigenvalues match with Eq. 5.42.

In order to observe the effect of the presence of the magnetic field and angular momentum we define

$$\bar{B}_0 = \frac{B_0 g}{\sqrt{m\omega_0}}, \tag{5.45a}$$

$$\bar{L} = \frac{L}{\alpha}, \tag{5.45b}$$

$$\bar{m} = \frac{m}{\omega_0}. \tag{5.45c}$$

These parameters are dimensionless and are directly associated with the physical quantities of the system. $\bar{B}_0$, $\bar{L}$ and $\bar{m}$ are free parameters. In this section, we have selected four values for the magnetic field, 0.1, 1, 5 and 10, and for the angular momentum we have chosen five values, 2, 4, 6, 8 and 10.

The magnetic field has a big effect on the probability density. For small values of the field, $\bar{B}_0 \leq 1$, the probability has a single peak dispersed in the radial direction, while for high values, $\bar{B}_0 > 1$, the probability density begins to concentrate near the origin and a second peak begins to appear, see Fig. 5.8.

The change in angular momentum also has an effect on $|\psi\left(t, \xi, \phi, \zeta\right)|^2$. For larger values of $\bar{L}$, the peak of the probability density moves away from the cosmic string. This behavior can be seen in the Fig. 5.9. Similar to what happened in the previous section, the presence of the cosmic string can be seen as a change in angular momentum.

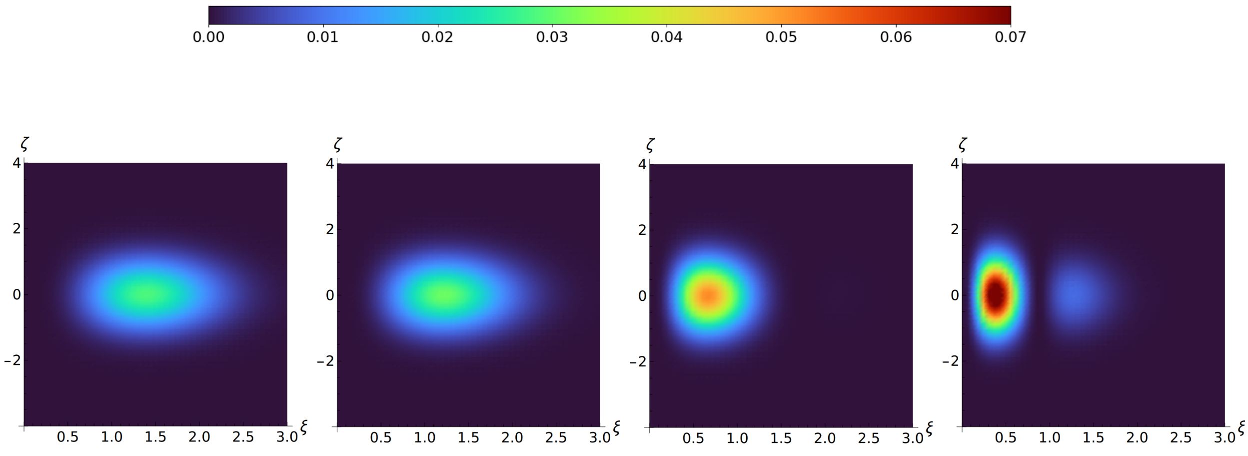


FIGURE 5.8 – $|\psi(t,\xi,\phi,\zeta)|^2$ with $\bar{L}=2$ and $\bar{m}=1$ (a) is the case $\bar{B}_0=0.1$, (b) $\bar{B}_0=1$, (c) $\bar{B}_0=5$ and (d) $\bar{B}_0=10$.

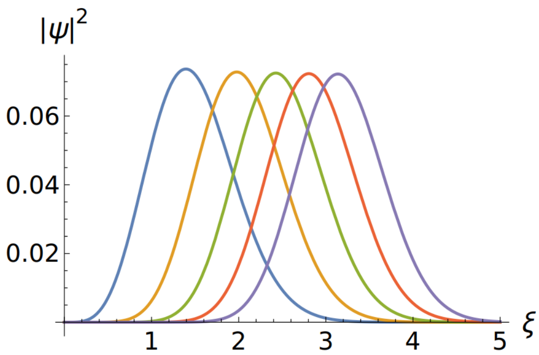


FIGURE 5.9 – $|\psi(t,\xi,\phi,\zeta)|^2$ with $\bar{B}_0=0.1$ and $\bar{m}=1$ at $\zeta=0$. The blue line represents the case $\bar{L}=2$, orange line $\bar{L}=4$, green line $\bar{L}=6$, red line $\bar{L}=8$ and purple line $\bar{L}=10$.

5.2.2 Linear Confining Potential

In this subsection we shall analyze the situation where the Coulomb-type scalar interaction, the magnetic flux and the electrostatic potential are absent, but keeping the linear confining potential and the magnetic field. That means $\eta_C=\xi_C=\Phi_B=0$ and $\eta_L\neq 0$. For this system, the function $R(\xi)$ satisfies the differential equation below, obtained directly from 5.34:

$$-\partial_\xi^2 R-\left(\frac{1}{\xi}+2\xi\right)\partial_\xi R+\left[\frac{L^2}{\alpha^2\xi^2}-\frac{gB_0L}{\alpha\sqrt{m\omega_0}\xi}\right.$$
$$\left.-\xi^2+\frac{1}{m\omega_0}\left(m+\eta_L\frac{\xi}{\sqrt{m\omega_0}}\right)^2+\frac{\beta}{m\omega_0}\right]R=0. \tag{5.46}$$

$R(\xi)$ can be expressed by means of an unknown function $F(\xi)$ as follows:

$$R(\xi) = \xi^{\sqrt{\frac{L^2}{\alpha^2}}} \exp\left(\frac{1}{2}\xi^2\left(\frac{|\eta_L|}{m\omega_0} - 1\right) + \frac{\xi\eta_L\sqrt{m\omega_0}}{\omega_0\eta_L}\right) F(\xi). \tag{5.47}$$

Substituting the ansatz into 5.46, we can see that $F(\xi)$ satisfies the equation

$$\xi\partial_\xi^2 F + \partial_\xi F\left(\bar{\gamma} + \delta\xi + \epsilon\xi^2\right) + \left(\xi\bar{\alpha} - q\right) F = 0, \tag{5.48}$$

with

$$\bar{\gamma} = 1 + 2\frac{L}{\alpha}, \tag{5.49a}$$

$$\delta = 2\frac{m}{\sqrt{m\omega_0}}\frac{|\eta_L|}{\eta_L}, \tag{5.49b}$$

$$\epsilon = 2\frac{|\eta_L|}{m\omega_0}, \tag{5.49c}$$

$$\bar{\alpha} = 2\frac{L}{\alpha}\frac{|\eta_L|}{m\omega_0} + 2\frac{|\eta_L|}{m\omega_0} - \frac{\beta}{m\omega_0} - 2, \tag{5.49d}$$

$$q = -\frac{B_0 g L}{\alpha\sqrt{m\omega_0}} - \frac{\left(2\frac{L}{\alpha} + 1\right)}{\alpha\omega_0\sqrt{m\omega_0}}\frac{|\eta_L|}{\eta_L}. \tag{5.49e}$$

This is the biconfluent Heun equation, BCH, and it was explored in detail in section 4.1, therefore we are going to use those results here. Since we are looking for polynomial solutions, we impose the condition 4.37 for $r = 0$ and obtain the following energy spectrum:

$$\mathcal{E}_{k,L,n_0} = \pm\sqrt{-2|\eta_L|\left(\frac{|L|}{\alpha} + n_0 + 1\right) + \frac{B_0^2 g^2}{4} + 2mk - m\omega_0}. \tag{5.50}$$

This expression is valid for all values of η_L and is independent of its sign. The energy is real for $B_0^2 g^2/4 + 2mk > 2|\eta_L|\left(|L|/\alpha + n_0 + 1\right) + m\omega_0$, assuming positive and negative solutions; otherwise, the energy is complex. As discussed in the previous section, complex energy can be interpreted as a gain or loss term, depending on its sign. We saw that the plus sign leads to a divergent wave function and this possibility is ruled out. We impose a restriction on the values of k to obtain real values of energy, which gives

$$k > \frac{|\eta_L|}{m}\left(\frac{|L|}{\alpha} + n_0 + 1\right) + \frac{\omega_0}{2} - \frac{B_0^2 g^2}{8m}. \tag{5.51}$$

We can also observe if $\eta_L \to 0$, the quantized behavior of the energy is lost . This happens due to the presence of the confinement potential and something very similar happens in the article (MEDEIROS; MELLO, 2012) when they consider a linear potential for a relativistic charged particle in cosmic string spacetime in the presence of a magnetic field and a scalar potential.

The BCH has two conditions to be obtained from the Heun polynomials. The first provides the quantization of energy. As suggest in (MEDEIROS; MELLO, 2012), the second condition leads to an expression for the scalar potential coupling constant, η_L, related to mass and angular momentum. If we consider a linear polynomial, the condition $c_{n+1} = 0$ leads us to Eq. 4.48. For $r = 0$, we find the following condition for the confinement parameter:

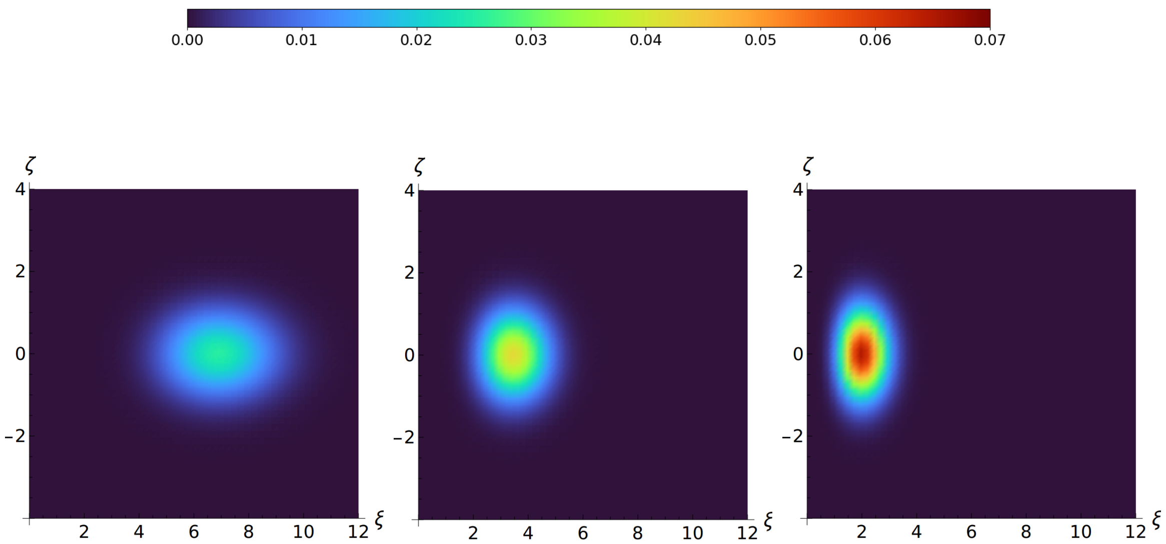
$$\eta_{L_{1,L}} = \frac{mB_0 gL}{\alpha}\left(1 + 2\sqrt{\frac{L^2}{\alpha^2}}\right)^{-1} + \frac{B_0 gLm}{\alpha}$$
$$\mp \left[m^2 + \frac{B_0^2 g^2 L^2}{2\alpha^2}\left(1 + 2\sqrt{\frac{L^2}{\alpha^2}}\right)^{-1} + \frac{m^2}{2}\left(2\sqrt{\frac{L^2}{\alpha^2}} + 1\right)\right]. \tag{5.52}$$

The relation given in Eq.5.52 gives the value of the parameter $\eta_{L_{1,L}}$ that permit us to construct a first degree polynomial solution of H_B.

The effect of linear confinement can be explored by implementing the definition 5.45 with

$$\bar{\eta}_L = \frac{|\eta_L|}{m\omega_0}. \tag{5.53}$$

Here, we explore $\bar{\eta}_L$ with three different values, 0.1, 0.5 and 0.8. As we increase the value of the confinement parameter, the probability density tends to disperse and move away from the cosmic string, see Figs 5.10 and 5.11.



(a) (b) (c)

FIGURE 5.10 – $|\psi(t, \xi, \phi, \zeta)|^2$ with $\bar{B}_0 = 0.1$, $\bar{L} = 2$ and $\bar{m} = 1$ (a) is the case $\bar{\eta}_L = 0.1$, (b) $\bar{\eta}_L = 0.5$ and (c) $\bar{\eta}_L = 0.8$.

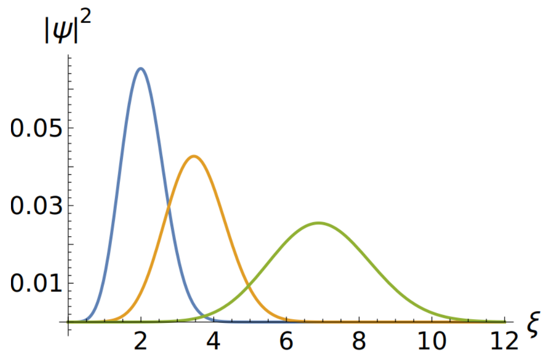


FIGURE 5.11 – $|\psi(t,\xi,\phi,\zeta)|^2$ with $\bar{B}_0 = 0.1$, $\bar{L} = 2$ and $\bar{m} = 1$ at $\zeta = 0$. The green line represents the case $\bar{\eta}_L = 0.1$, orange line $\bar{\eta}_L = 0.5$ and blue line $\bar{\eta}_L = 0.8$.

5.2.3 Scalar Coulomb-type Potential

Let us now admit that $\eta_L = \xi_C = \Phi_B = 0$, but take into account the effects of the scalar Coulomb-type potential, i.e., $\eta_C \neq 0$. In this case we can rewrite Eq. 5.34 as follows:

$$-\partial_\xi^2 R - \left(\frac{1}{\xi} + 2\xi\right)\partial_\xi R + \left[-\frac{gB_0 L}{\sqrt{m\omega_0}\alpha\xi} + \frac{L^2}{\alpha^2\xi^2} - \xi^2 + \frac{1}{m\omega_0}\left(m + \frac{\eta_C\sqrt{m\omega_0}}{\xi}\right)^2 + \frac{\beta}{m\omega_0}\right]R = 0. \tag{5.54}$$

Now, proposing the ansatz

$$R(\xi) = \xi^{\sqrt{\eta_C^2 + \frac{L^2}{\alpha^2}}} \exp\left[-\frac{1}{2}\xi\left(\frac{2\sqrt{\beta + m^2 + 2m\omega_0}}{\sqrt{m\omega_0}} + \xi\right)\right]F(\xi), \tag{5.55}$$

and substituting into 5.54, we obtain

$$\xi\partial_\xi^2 F + (1 + a - b\xi)\,\partial_\xi F + nF = 0, \tag{5.56}$$

with

$$a = 2\xi\sqrt{\frac{\beta}{m\omega_0} + \frac{m}{\omega_0} + 2}, \tag{5.57a}$$

$$b = 2\sqrt{\eta_C^2 + \frac{L^2}{\alpha^2}}, \tag{5.57b}$$

$$n = \frac{B_0 g L}{\alpha\sqrt{m\omega_0}} - \left(2\sqrt{\eta_C^2 + \frac{L^2}{\alpha^2}} + 1\right)\sqrt{\frac{\beta}{m\omega_0} + \frac{m}{\omega_0} + 2} - 2\frac{\eta_C m}{\sqrt{m\omega_0}}. \tag{5.57c}$$

This is the associated Laguerre differential equation with the solution

$$F(\xi) = B_1 L^{a}_{\frac{n}{b}}(b\xi), \tag{5.58}$$

where B_1 is constant.

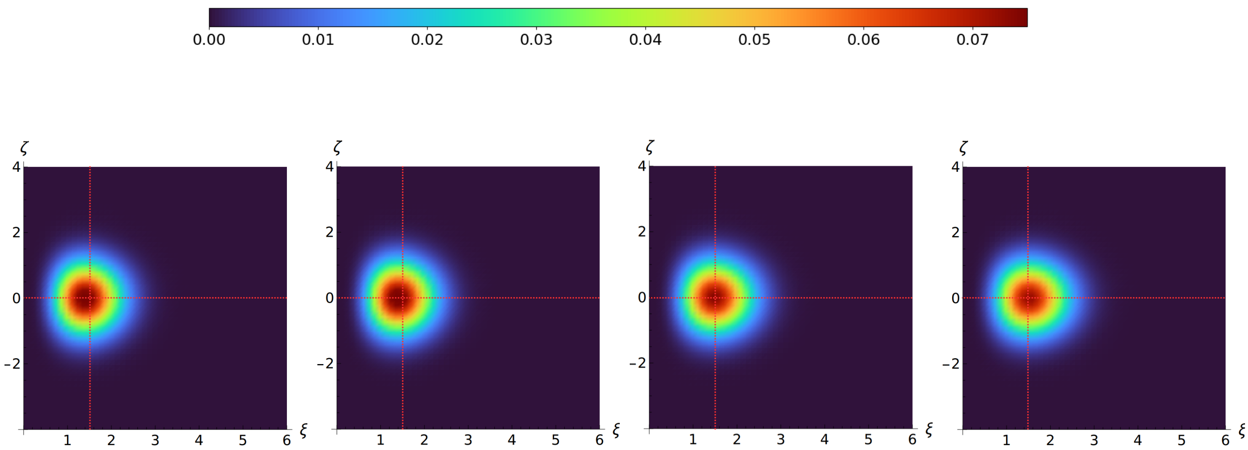
This equation was described in more details in section 5.2.1. Using the quantization condition for associated Laguerre polynomials, its provides the following energy spectrum:

$$\mathcal{E}_{k,L,n_0} = \pm \sqrt{-\frac{(B_0 g L - 2\alpha\eta_C m)^2}{4\alpha^2\left(\sqrt{\eta_C^2 + \frac{L^2}{\alpha^2}} + n_0 + \frac{1}{2}\right)^2} + \frac{B_0^2 g^2}{4} + 2km + m^2 - m\omega_0}. \tag{5.59}$$

Comparing with the energy found in the Landau levels, Eq. 5.42, we see that the presence of the Coulomb scalar potential changes the energy spectrum and, in the non-potential limit, the equation reduces to the case described in the Landau energy levels. Here we can also see that complex energies are possible and we impose the same restriction over k to obtain real values of the energy,

$$k > \frac{(B_0 g L - 2\alpha\eta_C m)^2}{8m\alpha^2\left(\sqrt{\eta_C^2 + \frac{L^2}{\alpha^2}} + n + \frac{1}{2}\right)^2} + \frac{\omega_0}{2} - \frac{B_0^2 g^2}{8m} - \frac{m}{2}. \tag{5.60}$$

The presence of a scalar potential of the Coulombian type causes a small change in the probability density. Its presence generates a small change in the position of the peak. As η_C increases, the probability density shifts in the opposite direction to the cosmic string, see Figs 5.12 and 5.13.



(a) (b) (c) (d)

FIGURE 5.12 – $|\psi(t,\xi,\phi,\zeta)|^2$ with $\bar{B}_0 = 0.1$, $\bar{L} = 2$ and $\bar{m} = 1$ (a) is the case $\eta_C = 0.5$, (b) $\eta_C = 1$, (c) $\eta_C = 10$ and (d) $\eta_C = 100$.

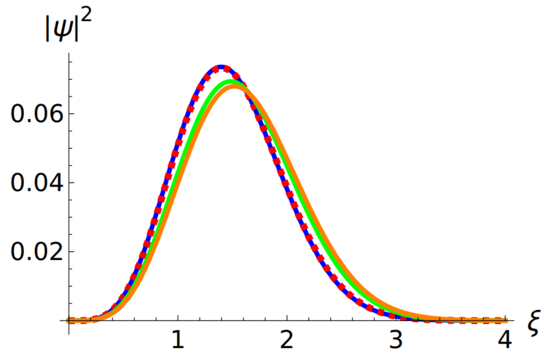


FIGURE 5.13 – $|\psi(t,\xi,\phi,\zeta)|^2$ with $\bar{B}_0 = 0.1$, $\bar{L} = 2$ and $\bar{m} = 1$ at $\zeta = 0$. The blue line represents the case $\eta_C = 0.5$, red dashed line $\eta_C = 1$ and green line $\eta_C = 10$ and orange line $\bar{\eta}_L = 100$.

5.2.4 Cornell-type Potential

In this system, we shall explore a more general problem, which is the behavior of the KGO under the presence of a staticmagnetic field, given by 5.19, and the Cornell-type scalar potential 5.24 in the space-time of the cosmic string. The radial differential equation is

$$-\partial_\xi^2 R - \left(\frac{1}{\xi} + 2\xi\right)\partial_\xi R + \left[-\xi^2 + \frac{L^2}{\alpha^2\xi^2} - \frac{gB_0 L}{\sqrt{m\omega_0}\alpha\xi} + \frac{1}{m\omega_0}\left(m + \frac{\eta_C\sqrt{m\omega_0}}{\xi} + \frac{\eta_L \xi}{\sqrt{m\omega_0}}\right)^2 + \frac{\beta}{m\omega_0}\right] R = 0. \tag{5.61}$$

In order to solve the ODE, we can write the function $R(\xi)$ as

$$R(\xi) = \xi^{\sqrt{\eta_C^2 + \frac{L^2}{\alpha^2}}} \exp\left(\frac{1}{2}\xi^2\left(\frac{|\eta_L|}{m\omega_0} - 1\right) + \frac{\xi\sqrt{m\omega_0}}{\omega_0}\frac{|\eta_L|}{\eta_L}\right) F(\xi), \tag{5.62}$$

where F is the new function to be determined. Substituting this form for $R(\xi)$ into Eq. 5.61, we obtain

$$\xi\partial_\xi^2 F + \partial_\xi F\left(\bar{\gamma} + \delta\xi + \epsilon\xi^2\right) + (\xi\alpha - q) F = 0, \tag{5.63}$$

with

$$\bar{\gamma} = 1 + 2\sqrt{\eta_C^2 + \frac{L^2}{\alpha^2}}, \tag{5.64a}$$

$$\delta = \frac{2m\,|\eta_L|}{\sqrt{m\omega_0}\eta_L}, \tag{5.64b}$$

$$\epsilon = 2\frac{|\eta_L|}{m\omega_0}, \tag{5.64c}$$

$$\bar{\alpha} = -\frac{\beta}{m\omega_0} + \frac{2}{m\omega_0}\left(\sqrt{\eta_C^2 + \frac{L^2}{\alpha^2}}\,|\eta_L| + |\eta_L| - m\omega_0\right) - \frac{2\eta_C\eta_L}{m\omega_0}, \tag{5.64d}$$

$$q = -\frac{B_0 g L}{\alpha\sqrt{m\omega_0}} - \frac{m}{\sqrt{m\omega_0}}\left(2\sqrt{\eta_C^2 + \frac{L^2}{\alpha^2}} + 1\right)\frac{|\eta_L|}{\eta_L} + 2\frac{\eta_C m}{\sqrt{m\omega_0}}. \tag{5.64e}$$

This is the BCH differential equation with the solution 4.29 and the constants $\bar{\gamma}, \delta, \epsilon, \alpha$ and q are defined above.

In order to find the energy spectrum we impose the condition 4.37 for $r = 0$ and we get

$$\mathcal{E}_{k,L,n_0} = \pm\sqrt{-2\,|\eta_L|\left(n_0 + \sqrt{\eta_C^2 + \frac{L^2}{\alpha^2}} + 1\right) + 2\eta_C\eta_L + \frac{B_0^2 g^2}{4} + 2mk - m\omega_0}. \tag{5.65}$$

We can see that the energy is unchanged by the discrete symmetry $\eta_L \to -\eta_L$ together with $\eta_C \to -\eta_C$. This solution also admit complex values for the energy. Here we make a restriction to the constant k in order to obtain real values. This condition is

$$k > \frac{|\eta_L|}{m}\left(n_0 + \sqrt{\eta_C^2 + \frac{L^2}{\alpha^2}} + 1\right) + \frac{\omega_0}{2} - \frac{B_0^2 g^2}{8m} - \frac{\eta_C\eta_L}{m}. \tag{5.66}$$

The Heun function has two conditions to be fullfield in order to obtain the Heun polynomials. Imposing the second constraint 4.48 we obtain a expression for the linear potential:

$$\begin{aligned}\eta_{L_{1,L}} = {} & \left(\frac{mB_0 g L}{\alpha} - 2\eta_C m^2\right)\left(1 + 2\sqrt{\eta_C^2 + \frac{L^2}{\alpha^2}}\right)^{-1} + \frac{B_0 g L m}{\alpha} - 2m^2\eta_C \\ & \mp\left[\left(\frac{B_0^2 g^2 L^2}{2\alpha^2} - 2\frac{B_0 g L}{\alpha}\eta_C m + 2\eta_C^2 m^2\right)\left(1 + 2\sqrt{\eta_C^2 + \frac{L^2}{\alpha^2}}\right)^{-1} \right. \\ & \left. + \frac{m^2}{2}\left(2\sqrt{\eta_C^2 + \frac{L^2}{\alpha^2}} + 1\right) + m^2\right].\end{aligned} \tag{5.67}$$

Here, we have chosen to restrict the parameter associated with the linear term of the

Cornell potential. Instead, we could also find a constraint for the Coulomb-type parameter η_C, but the equation resulting from the 4.48 is more complicated to solve for η_C.

5.2.5 Electrostatic potential

In this subsection, the environment is composed only of the magnetic field and the electrostatic potential, $\eta_C = \eta_L = \Phi_B = 0$ and $\xi_C \neq 0$, the radial ODE is

$$-\partial_\xi^2 R - \left(\frac{1}{\xi} + 2\xi\right)\partial_\xi R + \left[-\xi^2 + \frac{1}{\xi^2}\left(\frac{L^2}{\alpha^2} - \frac{g^2\xi_C^2}{m\omega_0}\right)\right. \tag{5.68}$$

$$\left. - \frac{1}{\sqrt{m\omega_0}\xi}\left(\frac{gB_0L}{\alpha} - 2g\xi_C\mathcal{E}\right) + \frac{m}{\omega_0} + \frac{\beta}{m\omega_0}\right] R = 0. \tag{5.69}$$

We propose the ansatz

$$R(\xi) = \xi^{L/\alpha} \exp\left(-\frac{1}{2}\xi\left(2\sqrt{\frac{\beta}{m\omega_0} - \frac{g^2\xi_C^2}{m\omega_0} + \frac{m}{\omega_0} + 2} + \xi\right)\right) F(\xi). \tag{5.70}$$

Thus, the equation becomes

$$\xi\partial_\xi^2 F + (1 + a - b\xi)\partial_\xi F + nF = 0, \tag{5.71}$$

with the following constants

$$a = -2\sqrt{\frac{\beta}{m\omega_0} - \frac{g^2\xi_C^2}{m\omega_0} + \frac{m}{\omega_0} + 2}, \tag{5.72a}$$

$$b = 2\frac{L}{\alpha}, \tag{5.72b}$$

$$n = \frac{B_0 g L}{\alpha\sqrt{m\omega_0}} - \left(1 + 2\frac{L}{\alpha}\right)\sqrt{\frac{\beta}{m\omega_0} - \frac{g^2\xi_C^2}{m\omega_0} + \frac{m}{\omega_0} + 2} - 2\frac{\mathcal{E}g\xi_C}{\sqrt{m\omega_0}}. \tag{5.72c}$$

As found in the section 5.2.1, the Eq. 5.71 is the associated Laguerre differential equation with the solution given by the Laguerre polynomials 5.58. The condition to obtain the Laguerre polynomials is $n/b = n_0$, being n_0 a non-negative integer. This condition leads us to the following energy spectrum:

$$\mathcal{E}_{k,L,n_0} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{5.73}$$

with

$$a = 4\alpha^2 g^2 \xi_C^2 + 4\alpha^2 \left(\frac{L}{\alpha} + n_0 + \frac{1}{2}\right)^2, \tag{5.74}$$

$$b = -4B_0 g L \alpha g \xi_C, \tag{5.75}$$

$$c = B_0^2 g^2 L^2 - \alpha^2 \left(\frac{L}{\alpha} + n_0 + \frac{1}{2}\right)^2 \left[B_0^2 g^2 - 4\left(g^2\xi_C^2 - 2km - m^2 + m\omega_0\right)\right]. \tag{5.76}$$

Here, we note that the structure of the expression for the energy is different from those obtained for the previous cases, in particular the fact that here we can obtain a complex value for the energy and no longer just a pure complex as in the previous cases. Despite the difference in the limit of the non-electrostatic potential, the energy is reduced to the energy eigenvalues of the Landau levels found previously, Eq. 5.42.

As energy also has complex values, we can impose a restriction on the values of k to keep it real. This condition is the following:

$$k < -\frac{B_0^2 g^2 L^2 \alpha^2 g^2 \xi_C^2}{8m\left(\alpha^2 g^2 \xi_C^2 + \alpha^2\left(\frac{L}{\alpha} + n_0 + \frac{1}{2}\right)^2\right)\alpha^2\left(\frac{L}{\alpha} + n_0 + \frac{1}{2}\right)^2} + \frac{B_0^2 g^2 L^2}{8m\alpha^2\left(\frac{L}{\alpha} + n_0 + \frac{1}{2}\right)^2} + \frac{\omega_0}{2} - \frac{B_0^2 g^2}{8m} + \frac{g^2\xi_C^2}{2m} - \frac{m}{2}. \tag{5.77}$$

To observe the effect of the presence of the electrostatic potential, we define a dimensionless parameter $\bar{\xi}_C$ as follows

$$\bar{\xi}_C = g\xi_C. \tag{5.78}$$

Eq. 5.78 with the definition 5.45, we can analyze the behavior of the probability density under a change in the electrostatic potential. Similar to the effects in the other cases, as we increase the values of the ξ_C potentials, the probability density moves away from the cosmic string, see Figs 5.14 and 5.15.

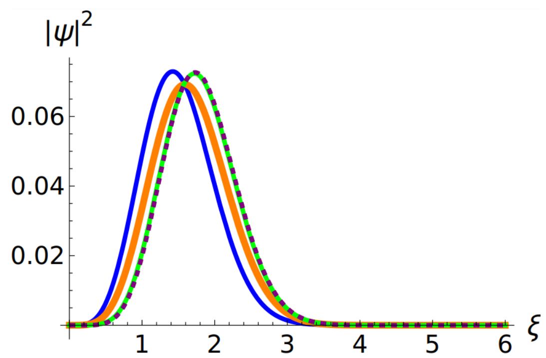
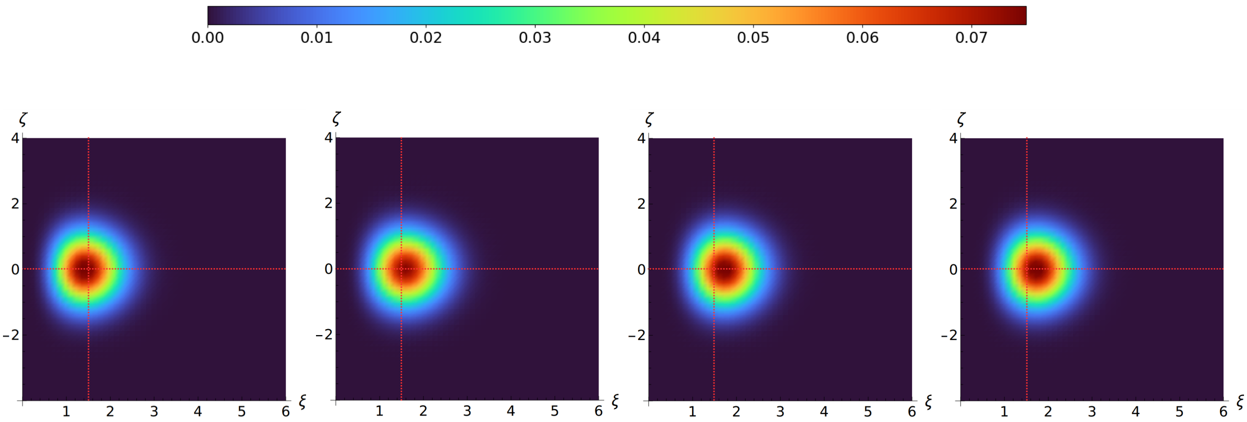


FIGURE 5.14 – $|\psi(t,\xi,\phi,\zeta)|^2$ with $\bar{B}_0 = 0.1$, $\bar{L} = 2$ and $\bar{m}_{=}1$ at $\zeta = 0$. The blue line represents the case $\bar{\xi}_C = 0.1$, orange line $\bar{\xi}_C = 1$, and green line $\bar{\xi}_C = 10$ and purple dashed line $\bar{\xi}_C = 100$.



(a) (b) (c) (d)

FIGURE 5.15 – $|\psi(t,\xi,\phi,\zeta)|^2$ with $\bar{B}_0 = 0.1$, $\bar{L} = 2$ and $\bar{m} = 1$ (a) is the case $\bar{\xi}_C = 0.1$, (b) $\bar{\xi}_C = 1$, (c) $\bar{\xi}_C = 10$ and (d) $\bar{\xi}_C = 100$.

5.2.6 Aharonov–Bohm Effect and Non-Commutative geometry

Now we consider only the magnetic field and flux acting, which means that $\Phi_B \neq 0$. From Eq. 5.34, the associated differential equation is

$$-\partial_\xi^2 R - \left(\frac{1}{\xi} + 2\xi\right)\partial_\xi R + \left[-\xi^2 + \frac{1}{\xi^2}\left(\frac{1}{\alpha^2}\left(L - \frac{g\Phi_B}{2\pi}\right)^2\right)\right. \tag{5.79}$$

$$\left. - \frac{1}{\sqrt{m\omega_0}\xi}\left(\frac{gB_0}{\alpha}\left(L - \frac{g\Phi_B}{2\pi}\right)\right) + \frac{m}{\omega_0} + \frac{\beta}{m\omega_0}\right] R = 0. \tag{5.80}$$

Proposing the following ansatz:

$$R(\xi) = \xi^{\frac{1}{\alpha}\left(L - \frac{g\Phi_B}{2\pi}\right)} \exp\left(-\frac{\xi\sqrt{\beta + m^2 + 2m\omega_0}}{\sqrt{m\omega_0}} - \frac{\xi^2}{2}\right) F(\xi), \tag{5.81}$$

we get a well-known ODE, the associated Laguerre differential equation

$$\xi\partial_\xi^2 F + (1 + a - b\xi)\partial_\xi F + nF = 0, \tag{5.82}$$

with

$$a = 2\xi\sqrt{\frac{\beta}{m\omega_0} + \frac{m}{\omega_0} + 2}, \tag{5.83}$$

$$b = \frac{2}{\alpha}\left(L - \frac{g\Phi_B}{2\pi}\right), \tag{5.84}$$

$$n = \frac{B_0 g}{\alpha\sqrt{m\omega_0}}\left(L - \frac{g\Phi_B}{2\pi}\right) - \frac{\left(1 + \frac{2}{\alpha}\left(L - \frac{g\Phi_B}{2\pi}\right)\right)}{\sqrt{m\omega_0}}\sqrt{\beta + m^2 + 2m\omega_0}. \tag{5.85}$$

The solution is described by the Eq. 5.39 with the above constants. Imposing the previous section's condition, $n/a = n_0$, where n_0 is a non-negative integer, the energy spectrum is obtained as

$$\mathcal{E}_{k,L,n_0} = \pm\sqrt{-\frac{B_0^2 g^2 \left(L - \frac{g\Phi_B}{2\pi}\right)^2}{4\alpha^2\left(\frac{1}{\alpha}\left(L - \frac{g\Phi_B}{2\pi}\right) + n_0 + \frac{1}{2}\right)^2} + \frac{B_0^2 g^2}{4} + 2km + m^2 - m\omega_0}. \tag{5.86}$$

We see that the effect of the magnetic flux always appears as a change in the angular momentum L, as we saw in the equation of motion 5.34. This dependence on the relativistic energy level of the geometric quantum phase gives us a relativistic analog of the Aharonov-Bohm effect for (AHARONOV; BOHM, 1959) bound states. In the limit $\Phi_B \to 0$, the energy spectrum reduces to the Landau levels, Eq. 5.42.

Here, complex energy values are possible and the whole discussion is the same as in subsection 5.2.1. To restrict for real energy we can impose the following condition upon the constant k:

$$k > \frac{B_0^2 g^2 \left(L - \frac{g\Phi_B}{2\pi}\right)^2}{8m\alpha^2\left(\frac{1}{\alpha}\left(L - \frac{g\Phi_B}{2\pi}\right) + n_0 + \frac{1}{2}\right)^2} + \frac{\omega_0}{2} - \frac{B_0^2 g^2}{8m} - \frac{m}{2}. \tag{5.87}$$

As we have done with the angular magnetic field, it is interesting to interpret the magnetic field as a non-commutative in momentum space. Then, we shall map the non-commutative moment to the magnetic field in the z-direction with the magnetic flux.

Taking the equation of motion of the non-commutative Klein-Gordon oscillator, Eq. 5.17, and setting $\Omega_2 = 0$, we get

$$\begin{aligned} 0 = & - m^2\psi + 3m\omega_0\psi + m^2\omega_0^2\rho^2\psi + m^2\omega_0^2 z^2\psi \\ & + \left(\frac{1}{\alpha^2\rho^2}\right)\left(-i\frac{1}{2}\Omega_3\rho + i\frac{1}{2}\Omega_1 z\right)^2\psi + 2m\omega_0\rho\partial_1\psi \\ & - 2\left(\frac{1}{\alpha^2\rho^2}\right)\left(-i\frac{1}{2}\Omega_3\rho + i\frac{1}{2}\Omega_1 z\right)\partial_2\psi + 2m\omega_0 z\partial_3\psi \\ & - \partial_0^2\psi + \partial_1^2\psi + \frac{1}{\rho}\partial_1\psi + \left(\frac{1}{\alpha^2\rho^2}\right)\partial_2^2\psi + \partial_3^2\psi. \end{aligned} \tag{5.88}$$

Comparing the above equation with 5.80, we obtain the following two equations, which must be satisfied for the mapping between the magnetic field and flux with the non-commutative to be possible:

$$\frac{1}{\alpha^2\rho^2}\Omega_1 z - \frac{\Omega_3}{\alpha^2\rho} = \frac{2}{\alpha^2\rho^2}\frac{g\Phi_B}{2\pi} + \frac{gB_0}{\alpha\rho}, \tag{5.89}$$

$$\frac{1}{4}\frac{\Omega_3^2}{\alpha^2} - \frac{1}{2}\frac{\Omega_3\Omega_1 z}{\alpha^2\rho} + \frac{1}{4}\frac{\Omega_1^2 z^2}{\alpha^2\rho^2} = \frac{1}{4}g^2B_0^2 + \frac{g^2B_0\Phi_B}{2\alpha\pi\rho} + \frac{g^2\Phi_B^2}{4\alpha^2\pi^2\rho^2}. \tag{5.90}$$

The above equations are satisfied for the following non-commutative parameter:

$$\Omega_1 = \frac{g\Phi_B}{\pi z}, \tag{5.91}$$

$$\Omega_3 = -\alpha g B_0. \tag{5.92}$$

This result show us that not only the magnetic field but also the magnetic flux can be describe as a non-commutative in momentum space.

5.2.7 Complete Solution

Here we analyze the more general scenario, we consider the presence of all potential and magnetic flux. Thus, the differential equations is the complete 5.34 and we start proposing the following ansatz:

$$R(\xi) = \exp\left(\frac{1}{2}\xi^2\left(\frac{|\eta_L|}{m\omega_0} - 1\right) + \frac{\xi m|\eta_L|}{\eta_L\sqrt{m\omega_0}}\right)\xi^{\sqrt{\eta_C^2 + \frac{1}{\alpha^2}\left(L - \frac{g\Phi_B}{2\pi}\right)^2}} F(\xi). \tag{5.93}$$

Substituting in the equation of motion, we obtain the Heun Bicofluent equation

$$\xi\partial_\xi^2 F + \partial_\xi F\left(\bar{\gamma} + \delta\xi + \epsilon\xi^2\right) + (\xi\alpha - q)F = 0, \tag{5.94}$$

where

$$\bar{\gamma} = 1 + 2\sqrt{\eta_C^2 + \frac{1}{\alpha^2}\left(L - \frac{g\Phi_B}{2\pi}\right)^2}, \tag{5.95a}$$

$$\delta = 2\frac{m}{\sqrt{m\omega_0}}\frac{|\eta_L|}{\eta_L}, \tag{5.95b}$$

$$\epsilon = 2\frac{|\eta_L|}{m\omega_0}, \tag{5.95c}$$

$$\bar{\alpha} = 2\frac{|\eta_L|}{m\omega_0}\sqrt{\eta_C^2 + \frac{1}{\alpha^2}\left(L - \frac{g\Phi_B}{2\pi}\right)^2} + 2\frac{|\eta_L|}{m\omega_0} + \frac{g^2\xi_C^2}{m\omega_0} - 2\frac{\eta_C\eta_L}{m\omega_0} - \frac{\beta}{m\omega_0} - 2, \tag{5.95d}$$

$$q = -\frac{B_0 g}{\alpha\sqrt{m\omega_0}}\left(L - \frac{g\Phi_B}{2\pi}\right) - \frac{m|\eta_L|}{\eta_L\sqrt{m\omega_0}}\left(2\sqrt{\eta_C^2 + \frac{1}{\alpha^2}\left(L - \frac{g\Phi_B}{2\pi}\right)^2} + 1\right) + 2\frac{\mathcal{E}g\xi_C}{\sqrt{m\omega_0}} + 2\frac{m\eta_C}{\sqrt{m\omega_0}}. \tag{5.95e}$$

Here we do the same steps that was done in the subsection 5.2.2. We impose the first BCH's condition to obtain the Heun polynomials, Eq. 4.37, which leads to the energy eigenvalues:

$$\mathcal{E}_{k,L,n_0} = \pm\left[-2\left|\eta_L\right|\left(n_0+\sqrt{\eta_C^2+\frac{1}{\alpha^2}\left(L-\frac{g\Phi_B}{2\pi}\right)^2}+1\right)\right.$$
$$\left.+2\eta_C\eta_L-g^2\xi_C^2+\frac{B_0^2g^2}{4}+2mk-m\omega_0\right]^{\frac{1}{2}}. \tag{5.96}$$

This energy spectrum is similar to the one obtained in the Cornell potential's subsection 5.2.4. The first difference is the presence of a quadratic term associated with the electrostatic potential; the second difference is due to the magnetic flux which, in this case, still generates a change in momentum, as discussed in 5.2.6.

To restrict our energy to be real, we obtain the following condition for the constant k:

$$k > \frac{|\eta_L|}{m}\left(n_0+\sqrt{\eta_C^2+\frac{1}{\alpha^2}\left(L-\frac{g\Phi_B}{2\pi}\right)^2}+1\right)+\frac{g^2\xi_C^2}{2m}+\frac{\omega_0}{2}-\frac{\eta_C\eta_L}{m}-\frac{B_0^2g^2}{8m}, \tag{5.97}$$

and the second condition of the BCH 4.36 leads to the expression for the parameter of the linear potential:

$$\left|\eta_{L_{1,L}}\right| = \left[-2\left(1+2\sqrt{\eta_C^2+\frac{1}{\alpha^2}\left(L-\frac{g\Phi_B}{2\pi}\right)^2}\right)\right]^{-1}\left\{-2\frac{mB_0g}{\alpha}\left(L-\frac{g\Phi_B}{2\pi}\right)+4m\mathcal{E}g\xi_C+4m^2\eta_C\right.$$
$$+\left(2\sqrt{\eta_C^2+\frac{1}{\alpha^2}\left(L-\frac{g\Phi_B}{2\pi}\right)^2}+1\right)\left(-2\frac{mB_0g}{\alpha}\left(L-\frac{g\Phi_B}{2\pi}\right)+4m\mathcal{E}g\xi_C+4m\omega_0\eta_C\right)$$
$$\pm\left[m^2\left(2\sqrt{\eta_C^2+\frac{1}{\alpha^2}\left(L-\frac{g\Phi_B}{2\pi}\right)^2}+1\right)\left(2+\left(2\sqrt{\eta_C^2+\frac{1}{\alpha^2}\left(L-\frac{g\Phi_B}{2\pi}\right)^2}+1\right)\right)\right.$$
$$\left.\left.+\frac{B_0^2g^2}{\alpha^2}\left(L-\frac{g\Phi_B}{2\pi}\right)^2+4\left(L-\frac{g\Phi_B}{2\pi}\right)\left(\frac{B_0g\mathcal{E}g\xi_C}{\alpha}+\frac{mB_0g\eta_C}{\alpha}\right)+\left(2\mathcal{E}g\xi_C+2m\eta_C\right)^2\right]\right\}. \tag{5.98}$$

The above equations provide the energy spectrum of the KGO in a cosmic string spacetime in the presence of magnetic field parallel to the string with a Cornell potential and flux magnetic and this complete or section about magnetic fields.

6 Conclusion

In this work, we investigate the influence of topological defects due to cosmic strings on the Klein-Gordon Oscillator in the presence of a constant electric and magnetic field with a Coulomb and Cornell potential. We start by finding the energy-momentum tensor for a static cosmic string in the z direction and, making a weak field approximation, we solve Einstein's field equation and obtain the conical metric created by the string. Then we introduce the cosmic string in the Lagrangian density of the complex scalar field in a curved space-time and introduce the oscillation by means of a non-minimal coupling, obtaining the KG oscillator in the space-time of the cosmic string. The related equation of motion for the KGO is solved by separation of variables and the associated angular quantum number L is quantized by means of the boundary condition on the angular coordinates and the energy spectrum is found by the roots of the Bessel function.

A brief introduction to gauge theory by Utiyama is shown to enable the coupling of interaction fields, electric and magnetic fields. As an application, a study of KGO under local phase transformation is presented.

The electric field is introduced by means of a minimum coupling, and the first case analyzed is the radial electric field. We find the bicofluent Heun equation for the radial differential equation, explore the equation using the Frobenius method and find two conditions for creating the Heun polynomials. These conditions allow us to find the quantization of the separation constant k and the energy $\mathcal{E}$. The second scenario is the electric field in the z direction. We find the Bessel equation for the radial equation and obtain the energy spectrum using the boundary conditions. A restriction on the electric field is discovered by imposing that the Hermite polynomials are solutions for the z-part.

We also explore the magnetic field in two different scenarios. The first is the angular magnetic field, in which we present two finite difference methods for solving the partial differential equation of ρ and z, the first being the explicit method and the second is the SOR method. We present the results using the first method, finding the probability density. We identified that a change in the parameter associated with the string means a change in the angular momentum; as we increase these parameters, the probability density tends to move away from the string. We also observed that the magnetic field plays a significant role in the solution. At low values of the field $\bar{B}_0 < 1$, the probability has a single peak, while at high values it splits into two peaks. The change in the parameter associated with the energy, $\bar{\gamma}$, means a change in the peak of the probability density in the radial direction.

We also explore the magnetic field parallel to the cosmic string with an electrostatic potential, a Cornell potential and a magnetic flux. The equation of motion is analyzed for each scenario individually, keeping the magnetic field. The first case was characterized by the extinction of all potentials, the radial equation obtained was the Laguerre equation and the quantization of energy was found by imposing the condition to obtain the associated Laguerre polynomials. In this way we obtain the Landau levels; the energy takes on complex values and a restriction for the possible values of the constant k was found in order to keep it real. It is worth noting that the expression for the energy found was the same as the one presented in the paper (CUZINATTO *et al.*, 2022).

The second scenario is due to the presence of the confinement potential. In this case, the radial equation obtained was the biconfluent Heun equation with solutions given by the Bicofluent Heun functions. The two conditions for obtaining the Heun polynomials were imposed and we obtained the energy eigenvalues and a condition for the confinement potential. The presence of the linear potential changed the form of the energy compared to that obtained in the case of the landau levels.

The third system was constructed with the presence of only the Coulombian scalar potential, the Laguerre equation was found, the condition for obtaining polynomials was imposed and, consequently, the energy spectrum was found. The energy spectrum is altered by the potential, maintaining the correspondence with the case of the Landau levels in the non-potential limit.

In the fourth case, we analyzed the complete Cornell potential and obtained the BCH equation as a radial equation. We imposed the conditions for the Heun functions to be polynomials and obtained the quantization of the energy and an expression for the confinement potential. In the limit of the non-Coulombian potential, we returned to the results found in the case of the confinement potential.

The fifth system was characterized only by the presence of an electrostatic potential, which was introduced by means of minimum coupling. The radial equation obtained was the associated Laguerre equation and, by imposing the conditions to obtain the polynomials, the energy spectrum was found and, in the non-potential limit, led us to the case of Landau levels.

The sixth case is characterized by the presence of a magnetic flux, which was included through minimum coupling. The radial equation found was also that of Laguerre and by imposing the conditions to obtain the Laguerre polynomials, we obtain the energy spectrum. We observed that the magnetic flux always appears as a change in the angular momentum L. This dependence on the relativistic energy level of the geometric quantum phase gives us a relativistic analog of the Aharonov-Bohm effect. A description of the magnetic field and magnetic flux as non-commutative in momentum space was also presented.

Finally, the last scenario was characterized by the inclusion of all potentials and the magnetic flux. The radial equation obtained the BCH and the first condition led us to the quantization of energy similar to the case of confinement. Heun's second condition led us to the fixing of the linear confinement parameter which, when considering the electrostatic potential.

In this work, we were able to explore the Klein-Gordon Oscillator in the presence of cosmic strings in space-time with different interaction fields. In summary, the results obtained here show that the presence of a cosmic string produces a significant modification in the energy spectrum and probability density associated with the KGO in the presence of electric and magnetic fields.

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Appendix A - Finite Differences Method (FDM)

Finite differences is a discrete method widely used to find numerical solutions to ODEs and PDEs. The method is based on replacing the derivatives of the differential equation with approximations using finite differences (GILAT; SUBRAMANIAM, 2011). The strategy is to divide the domain into N subintervals of length h (in general, the subintervals can have different lengths, $h(x)$), which is commonly called a mesh, and the solutions can be found discretely.

First, a grid is defined in the domain of the function, so that it is separated into discrete subintervals of size $h = (b - a)/N$[1], where a and b are the boundary points (or end points) that define the length of the interval and N is an integer that determines the size of the parts. This discretization of the domain can be seen in figure A.1.

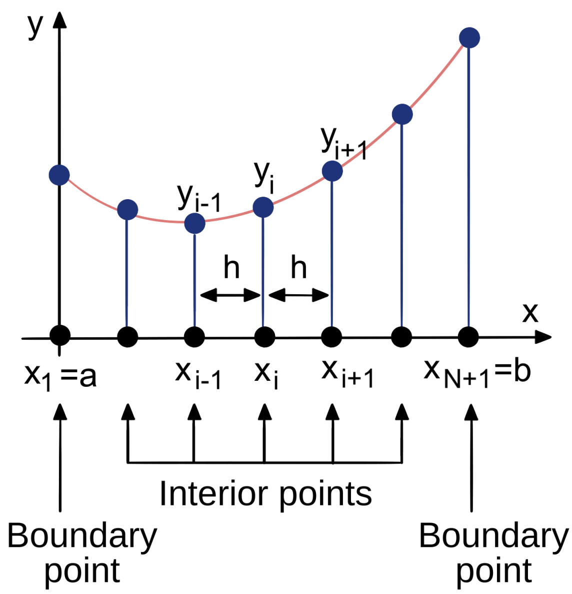

FIGURE A.1 – An illustration of the discretization of the domain into N parts of width h of a generic function $y(x)$.

[1]Intervals can have different lengths for different directions, such as $k = (d - c)/M$ in the y-direction

The idea behind FDM is to use the Taylor series approximation to evaluate the derivative of the function with respect to points in the neighborhood. Therefore, a function f at point x_{i+1} can be expressed in terms of a neighboring point x_i as

$$f(x_{i+1}) = f(x_i) + \frac{df}{dx}\Big|_{x=x_i} h + \frac{1}{2!}\frac{d^2 f}{dx^2}\Big|_{x=x_i} h^2 + \frac{1}{3!}\frac{d^3 f}{dx^3}\Big|_{x=x_i} h^3 + \cdots \tag{A.1}$$

Set up to only two terms plus a truncation error $O(h)$, we get

$$f(x_{i+1}) = f(x_i) + \frac{df}{dx}\Big|_{x=x_i} h + O(h). \tag{A.2}$$

Here we can find a expression for the derivative in the point x_i in term of the function in the points x_i and x_{i+1},

$$\frac{df}{dx}\Big|_{x=x_i} = \frac{f(x_i) - f(x_{i+1})}{h} + O(h). \tag{A.3}$$

This formula for the first derivative is called the forward finite difference method because it calculates the derivative of the ith term using the ith and $(i+1)$th terms of the function.

In the same way, we can find the formula for regressive finite differences. By doing the same procedure as before, but replacing the neighborhood point of x_{i+1} with x_{i-1}, this leads us to the regressive finite difference method expression

$$\frac{df}{dx}\Big|_{x=x_i} = \frac{f(x_i) - f(x_{i-1})}{h} + O(h). \tag{A.4}$$

An interesting case, and more precise in certain cases, is the central finite difference method with two points. It expresses the first derivative using two points, we use the Taylor series for the equation x_{i+1} and for x_{i-1}, but now for the second order of the error, which means

$$f(x_{i+1}) = f(x_i) + \frac{df}{dx}\Big|_{x=x_i} h + \frac{1}{2!}\frac{d^2 f}{dx^2}\Big|_{x=x_i} h^2 + O(h^2), \tag{A.5}$$

$$f(x_{i-1}) = f(x_i) - \frac{df}{dx}\Big|_{x=x_i} h + \frac{1}{2!}\frac{d^2 f}{dx^2}\Big|_{x=x_i} h^2 - O(h^2). \tag{A.6}$$

Making the difference between them we obtain a expression in term of the two neighborhood points, x_{i-1} and x_{i+1},

$$f(x_{i+1}) - f(x_{i-1}) = 2\frac{df}{dx}\Big|_{x=x_i} h + O(h^2), \tag{A.7}$$

and the derivative is obtain as

$$\frac{df}{dx}\Big|_{x=x_i} = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2). \tag{A.8}$$

This second method is called the central difference formula and is more efficient if the point x_i is a maximum or minimum point of the function, as it considers the point in front and behind.

Similarly to what was done for the first derivative, the second derivative will be carried out. By writing the equation A.5 and A.6 in terms of a point x_i until the third derivative and the truncation error are of the order of h^4,

$$f(x_{i+1}) = f(x_i) + \frac{df}{dx}\Big|_{x=x_i} h + \frac{1}{2!}\frac{d^2 f}{dx^2}\Big|_{x=x_i} h^2 + \frac{1}{3!}\frac{d^3 f}{dx^3}\Big|_{x=x_i} h^3 + O(h^4), \tag{A.9}$$

$$f(x_{i-1}) = f(x_i) - \frac{df}{dx}\Big|_{x=x_i} h + \frac{1}{2!}\frac{d^2 f}{dx^2}\Big|_{x=x_i} h^2 - \frac{1}{3!}\frac{d^3 f}{dx^3}\Big|_{x=x_i} h^3 + O(h^4). \tag{A.10}$$

The sum of the above equation gives us

$$f(x_{i+1}) + f(x_{i-1}) = 2f(x_i) + 2\frac{1}{2!}\frac{d^2 f}{dx^2}\Big|_{x=x_i} h^2 + O(h^4), \tag{A.11}$$

and the second derivative can be express in term of three points of the function as

$$\frac{d^2 f}{dx^2}\Big|_{x=x_i} = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} + O(h^2). \tag{A.12}$$

This is the three-point formula, x_{i+1}, x_{i+1} and x_i, which estimates the second derivative at the point x_i, also called the three-point central difference and has a truncation error of $O(h^2)$.

A.1 Explicit Method

In order to solve the 5.8 equation, an intuitive way is to use the explicit method, which consists of rewriting our PDE using FDM and then isolating a term related to the function at a point in terms of the function of its neighboring points. The idea is to define the boundary condition and calculate the function in a loop until the information comes from the boundary to the interior points and the solution converges. We start defining our domain as

$$\xi \in (0, \rho_0) \quad \text{and} \quad \zeta \in (-z_0, z_0), \tag{A.13}$$

where ρ_0 is the maximum radius of our solution and z_0 is the maximum height.

By dividing our radial domain into N_ξ parts of length $\Delta\xi$, we can express the radial coordinate in discrete form as

$$\xi_i = i\Delta\xi, \quad i = 0, 1, 2, \ldots, N_\xi, \quad \text{where } \Delta\xi = \frac{\rho_0}{N_\xi}, \tag{A.14}$$

And similarly, separating the coordinate ζ by spaces of length $\Delta\zeta$, we get

$$\zeta_j = j\Delta\zeta, \quad j = 0, 1, 2, \ldots, N_\zeta, \quad \text{where } \Delta\zeta = \frac{2z_0}{N_\zeta}. \tag{A.15}$$

Here, we assume a uniform partition, which means that $\Delta\xi$ and $\Delta\zeta$ are constant.

The function $F(\xi, \zeta)$ can be expressed in discrete form using the following notation:

$$F(\xi_i, \zeta_j) = F(i\Delta\xi, j\Delta\zeta) = F_{i,j}. \tag{A.16}$$

The lower indices are related to the coordinates ξ and ζ, respectively.

Using the equation A.8 and A.12 to express the derivatives of the function F in terms of the neighboring points, we have the following expressions for the first derivative and the second derivative

$$\partial_\xi F = \frac{F_{i+1,j} - F_{i-1,j}}{2\Delta\xi}, \tag{A.17}$$

$$\partial_\zeta F = \frac{F_{i,j+1} - F_{i,j-1}}{2\Delta\zeta}, \tag{A.18}$$

$$\partial_\xi^2 F = \frac{F_{i+1,j} - 2F_{i,j} + F_{i-1,j}}{(\Delta\xi)^2}, \tag{A.19}$$

$$\partial_\zeta^2 F = \frac{F_{i,j+1} - 2F_{i,j} + F_{i,j-1}}{(\Delta\zeta)^2}. \tag{A.20}$$

Applying the above expressions into Eq. 5.8 and isolating the $F_{i,j}$ term, we obtain

$$F_{i,j} = \frac{1}{\beta_j^i}\left[\frac{F_{i+1,j} + F_{i-1,j}}{(\Delta\xi)^2} + \frac{1}{\xi_i}\left(\frac{F_{i+1,j} - F_{i-1,j}}{2\Delta\xi}\right) - i\bar{B}_0\left(\zeta_j\frac{F_{i+1,j} - F_{i-1,j}}{2\Delta\xi} - \xi_i\frac{F_{i,j+1} - F_{i,j-1}}{2\Delta\zeta}\right)\right.$$
$$\left. +2\left(\xi_i\frac{F_{i+1,j} - F_{i-1,j}}{2\Delta\xi} + \zeta_j\frac{F_{i,j+1} - F_{i,j-1}}{2\Delta\zeta}\right) + \frac{F_{i,j+1} + F_{i,j-1}}{(\Delta\zeta)^2}\right], \tag{A.21}$$

where

$$\beta_j^i = \frac{2}{(\Delta\xi)^2} + \frac{1}{2\xi_i}i\bar{B}_0\zeta_j + \frac{L^2}{\alpha^2\xi_i^2} + \frac{2}{(\Delta\zeta)^2} + \left(\frac{1}{4}\bar{B}_0^2 - 1\right)\left(\zeta_j^2 + \xi_i^2\right) + \bar{\gamma}. \tag{A.22}$$

This expression shows us how to calculate the value of a function at the point ξ_i and ζ_j using the values of the neighboring functions, $F_{i+1,j}$, $F_{i-1,j}$, $F_{i,j+1}$ and $F_{i,j-1}$. This dependency of the A.21 calculation can be seen in the figure A.2

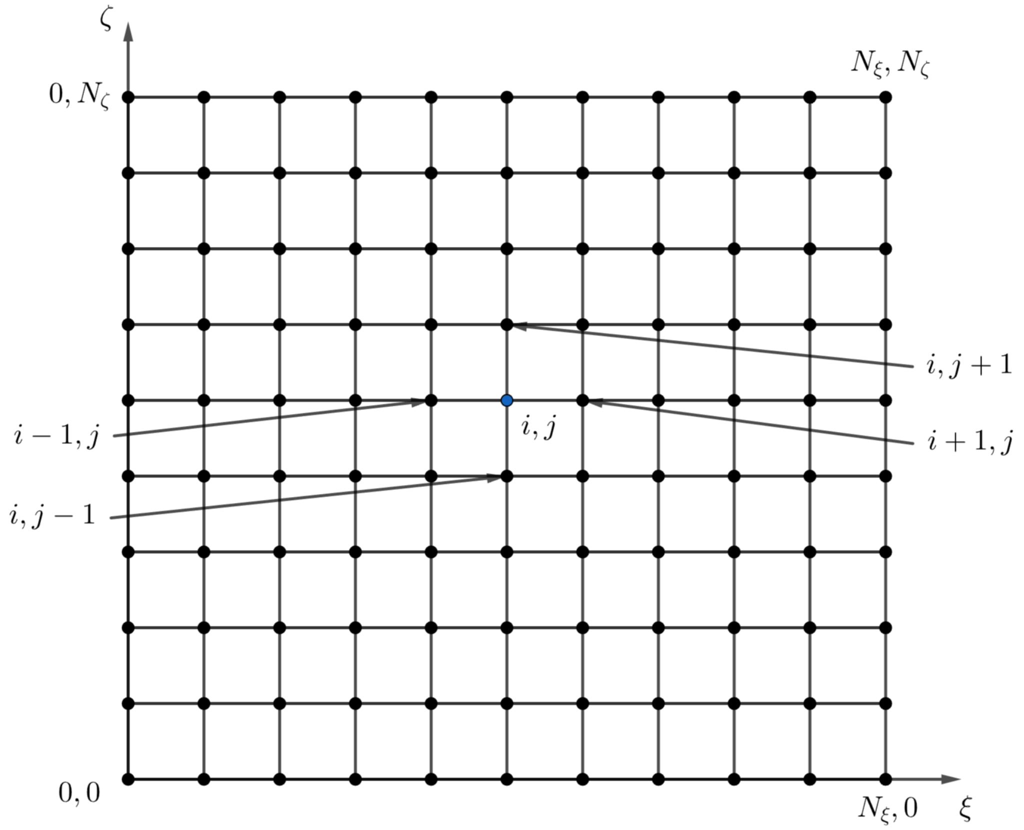

FIGURE A.2 – 2D Grid representation of Eq. A.21, the point (i, j) depends on the points in its neighborhood.

We can now write a sketch code to solve the PDE using the A.21. It starts by initializing the parameters and constants associated with the system, and then we define the division for the coordinates ξ and ζ and their respective intervals $\Delta\xi$ and $\Delta\zeta$. We also create the solution function matrices F^{new} and F^{old}, and fill them in with an initial guess for the solution, usually zero, and apply the boundary values. Now we start the interactions, first we calculate all the values of F^{new} using the A.21 and then we calculate the error (the error calculation will be described in Eq. A.23), check whether the error is smaller than the desired accuracy or whether the interaction counter n is larger than the maximum set of interactions $N_{\max}$; if this is true, we stop the interaction steps and save the solution to a file; otherwise, we increase the number of interactions by one and copy F^{new} to F^{old}. We calculate the components of the function F^{new} again. This happens in a loop until one of the conditions is met, precision error $<$ or interaction number n greater than or equal to the maximum interaction number $N_{\max}$.

Below is a flowchart of the code that expresses the step by step described:

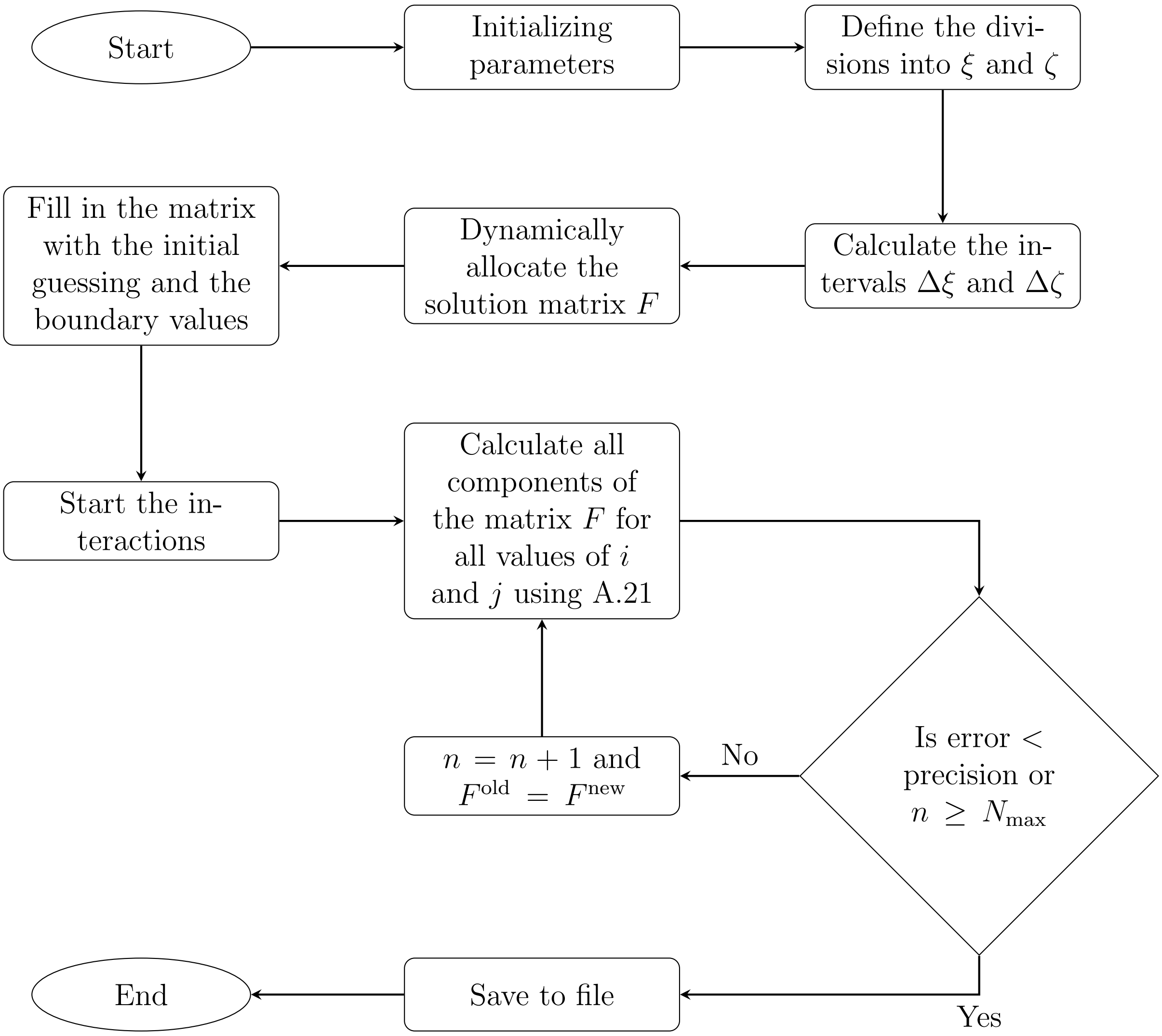

FIGURE A.3 – Flowchart of code for the explicit method

The error calculation is defined by a matrix before the interaction F^{old} and by a matrix after the interaction F^{new} as

$$\text{error} = \left|\text{Max}\left(F^{\text{old}} - F^{\text{new}}\right)\right|. \tag{A.23}$$

The idea is to evaluate the maximum value of the difference due to an interaction, when the interaction step does not change the solution function by a set value, we stop the run.

A.2 Successive Overrelaxation Method (SOR) with Chebyshev acceleration

An alternative method to the explicit method is the Successive Overrelaxation Method (SOR). SOR is an iterative method that has been introduced by (YOUNG, 1954) and it is considered as one of the efficient ways to accelerate the convergence of primary iterative methods such as Jacobi and Gauss-Seidel method to solve a linear system of equations. Some modifications to the SOR method have been made in (KINCAID; YOUNG, 1972), creating the modified SOR (MSOR) to solve a system of equations with a red-black coefficient matrix, and in (HADJIDIMOS, 1978), an accelerated super-relaxation (ASOR) method based on two parameters. Recently, two different versions of SOR have been proposed, KSOR (YOUSSEF, 2012) and KMSOR (YOUSSEF; TAHA, 2013).

Here we will present an alternative solution to the explicit method, using the usual SOR with Chebyshev acceleration in order to increase efficiency.

We begin by introducing Jacobi's ordinary iterative methods for solving linear systems. Given a real $(n \times n)$-matrix A and a real n-vector b, the problem considered is to find x belonging to $\mathbf{R}^n$ such that

$$A \cdot x = b. \tag{A.24}$$

This is a linear system, A is the matrix of coefficients, b is the right-hand side vector and x is the vector of unknowns.

Decomposing the matrix A as

$$A = L + D + U, \tag{A.25}$$

where D is the diagonal of A, L is its strict lower part and U is its strict upper part. It is assumed here that the diagonal is always non-zero. The Jacobi method for the k-th interaction is (SAAD, 2003)

$$x_{(k)} = -D^{-1}\left(L + U\right) \cdot x_{(k-1)} + D^{-1}b. \tag{A.26}$$

Jacobi method converges for A matrices that are diagonally dominant, that means

$$|a_{ii}| \geq \sum_{j \neq i} |a_{i,j}|\,, \quad \text{for all } i. \tag{A.27}$$

The matrix $-D^{-1} \cdot (L + U)$ in A.26 is called the interaction matrix and the convergence factor of the method is given by its spectral radius[2] of the interaction matrix, label by ρ_s (PRESS, 2007).

The number of interactions r required for the error factor to be of the order of 10^{-p} can be estimated by the equation (PRESS, 2007):

$$r \approx \frac{p \ln 10}{(-\ln \rho_s)}. \tag{A.28}$$

[2]The spectral radius is the maximum absolute eigenvalue of the matrix (GRADSHTEYN; RYZHIK, 2014).

An alternative to Jacobi method is the Gauss-Seidel method, which converges faster than Jacobi method and requires less computational memory when programmed (GILAT; SUBRAMANIAM, 2011). The Gauss-Seidel method for the k-th interaction is

$$x^{(k)} = -\left(L + D\right)^{-1} U \cdot x^{(k-1)} + \left(L + D\right)^{-1} b. \tag{A.29}$$

Here, the interaction matrix is $-\left(L + D\right)^{-1} U$, consequently, the spectral radius and the number of interactions required to obtain a solution with a certain accuracy are different from Jacobi method.

We have given a brief introduction to the Jacobi and Gauss-Seidel methods so that we can now explore the successive over-relaxation method. The idea of the method is to overcorrect the value of $x^{(k)}$ in the k stage of the Gauss-Seidel iteration A.29. We start adding and subtracting $x^{(k-1)}$ on the right-hand side. we get

$$\begin{aligned} x^{(k)} &= x^{(k-1)} - x^{(k-1)} - \left(L + D\right)^{-1} \left(U \cdot x^{(k-1)} + b\right) \\ &= x^{(k-1)} - \left(L + D\right)^{-1} \left(U \cdot x^{(k-1)} + \left(L + D\right) \cdot x^{(k-1)} + b\right) \\ &= x^{(k-1)} - \left(L + D\right)^{-1} \cdot \left[\left(L + D + U\right) \cdot x^{(k-1)} - b\right]. \end{aligned} \tag{A.30}$$

Now introducing overrelaxation parameter ω, we obtain

$$x^{(k)} = x^{(k-1)} - \omega \left(L + D\right)^{-1} \cdot \xi^{(k-1)}, \tag{A.31}$$

where

$$\xi^{(k-1)} = \left(L + D + U\right) \cdot x^{(k-1)} - b. \tag{A.32}$$

The method converges only for $0 < \omega < 2$. If $0 < \omega < 1$, we have an underrelaxation. Its spectral radius can be expressed in terms of Jacobi's spectral radius as

$$\rho_{\text{SOR}} = \left(\frac{\rho_{\text{Jacobi}}}{1 + \sqrt{1 - \rho_{\text{Jacobi}}^2}}\right)^2. \tag{A.33}$$

As a generic example, consider the following discreatezed generic equation of second order that the unknown function is $u_{i,j}$:

$$a_{i,j} u_{i+1,j} + b_{i,j} u_{i-1,j} + c_{i,j} u_{i,j+1} + d_{i,j} u_{i,j-1} + e_{i,j} u_{i,j} = f_{i,j}, \tag{A.34}$$

isolating $u_{i,j}$, we get

$$u_{i,j} = \frac{1}{e_{i,j}} \left(f_{i,i} - a_{i,j} u_{i+1,j} - b_{i,j} u_{i-1,j} - c_{i,j} u_{i,j+1} - d_{i,j} u_{i,j-1}\right). \tag{A.35}$$

Therefore, the SOR algorithm A.31 is

$$u_{i,j}^{\text{new}} = u_{i,j}^{\text{old}} - \omega \frac{\xi_{i,j}}{e_{i,j}}, \tag{A.36}$$

and the residue is

$$\xi_{i,j} = a_{i,j}F_{i+1,j} + b_{i,j}F_{i-1,j} + c_{i,j}F_{i,j+1} + d_{i,j}F_{i,j-1} + e_{i,j}F_{i,j} - f_{i,j}. \tag{A.37}$$

In fact, our problem of solving the partial difference equation becomes solving a linear system and the equation A.36 with A.37 completes the idea of the SOR algorithm. Now, let us introduce a trivial modification to the parameter ω that can lead us to the optimal asymptotic relaxation parameter; this modification is called Chebyshev acceleration. The main idea is that the relaxation parameter may not necessarily be a good initial choice, so we introduce Chebyshev acceleration to obtain the best parameter.

The odd-even order is used and ω changes every half scan according to the following prescription:

$$\begin{aligned}
\omega^{(0)} &= 1, \\
\omega^{(1/2)} &= 1/\left(1 - \rho_{\text{Jacobi}}^2/2\right), \\
\omega^{(n+1/2)} &= 1/\left(1 - \rho_{\text{Jacobi}}^2\omega^{(n)}/2\right), \quad n = 1/2, 1, \cdots, \infty, \\
\omega^{(\infty)} &= \omega_{\text{optimal}}.
\end{aligned} \tag{A.38}$$

The good fact about the Chebyshev acceleration is that the error norm always decreases with each iteration (PRESS, 2007).

Let us now consider our system, KGO in a cosmic string background with a static angular magnetic field. Take the expression A.21 for $\Delta = \Delta\xi = \Delta\zeta$, the equation becomes:

$$\begin{aligned}
&\left[-\frac{1}{\Delta^2} - \frac{1}{\xi_i}\frac{1}{2\Delta} + i\overline{B}_0\frac{\zeta_j}{2\Delta} - \frac{\xi_i}{\Delta}\right]F_{i+1,j} + \left[-\frac{1}{\Delta^2} + \frac{1}{\xi_i}\frac{1}{2\Delta} + \frac{\xi_i}{\Delta} - i\overline{B}_0\frac{\zeta_j}{2\Delta}\right]F_{i-1,j} \\
&\quad + \left[\frac{2}{\Delta^2} + i\frac{\zeta_j}{2\xi_i}\overline{B}_0 + \frac{\overline{L}^2}{\xi_i^2} + \frac{2}{\Delta^2} + \left(\frac{1}{4}\overline{B}_0^2 - 1\right)\left(\zeta_j^2 + \xi_i^2\right) + \overline{\gamma}\right]F_{i,j} \\
&\quad + \left[-i\frac{\overline{B}_0\xi_i}{2\Delta} - \frac{\zeta_j}{\Delta} - \frac{1}{\Delta^2}\right]F_{i,j+1} + \left[i\frac{\overline{B}_0\xi_i}{2\Delta} + \frac{\zeta_j}{\Delta} - \frac{1}{\Delta^2}\right]F_{i,j-1} = 0.
\end{aligned} \tag{A.39}$$

This equation can be expressed in the form of a A.34 as

$$a_{i,j}F_{i+1,j} + b_{i,j}F_{i-1,j} + c_{i,j}F_{i,j+1} + d_{i,j}F_{i,j-1} + e_{i,j}F_{i,j} = f_{i,j}, \tag{A.40}$$

where

$$a_{i,j} = -\frac{1}{\Delta^2} + i\frac{\overline{B}_0\zeta_j}{2\Delta} - \frac{\xi_i}{\Delta} - \frac{1}{2\xi_i\Delta}, \tag{A.41a}$$

$$b_{i,j} = -\frac{1}{\Delta^2} + \frac{1}{2\xi_i\Delta} - i\frac{\overline{B}_0\zeta_j}{2\Delta} + \frac{\xi_i}{\Delta}, \tag{A.41b}$$

$$c_{i,j} = -i\frac{\overline{B}_0\xi_i}{2\Delta} - \frac{\zeta_j}{\Delta} - \frac{1}{\Delta^2}, \tag{A.41c}$$

$$d_{i,j} = i\frac{\overline{B}_0\xi_i}{2\Delta} + \frac{\zeta_j}{\Delta} - \frac{1}{\Delta^2}, \tag{A.41d}$$

$$e_{i,j} = \frac{2}{\Delta^2} + i\frac{\zeta_j}{2\xi_i}\overline{B}_0 + \frac{\overline{L}^2}{\xi_i^2} + \frac{2}{\Delta^2} + \left(\frac{1}{4}\overline{B}_0^2 - 1\right)\left(\zeta_j^2 + \xi_i^2\right) + \overline{\gamma}, \tag{A.41e}$$

$$f_{i,j} = 0. \tag{A.41f}$$

So we can use A.36 and A.37 to write a solution for the PDE in the SOR algorithm form as

$$F_j^{i(\text{new})} = F_j^{i(\text{old})} - \omega\frac{\xi_{i,j}}{e_{i,j}}, \tag{A.42}$$

with the following residue:

$$\xi_{i,j} = a_{i,j}F_{i+1,j} + b_{i,j}F_{i-1,j} + c_{i,j}F_{i,j+1} + d_{i,j}F_{i,j-1} + e_{i,j}F_{i,j} - f_{i,j}. \tag{A.43}$$

Now we can write a sketch for the code to solve the PDE using A.42 and A.43. The algorithm starts by initializing the parameters associated with the system, then we define the division for the coordinates ξ and ζ and the intervals Δ, similar as we did in the explicit method. The function matrices F^{new} and F^{old} are created and filled in with an initial estimate and the boundary conditions are also applied following 5.9.

Here we start the interaction loop, first we calculate all the values of F^{new} using the A.42 equation and then we calculate the associated norm (the norm calculation will be described in Eq. A.44), we check if the norm has been reduced by a factor, ϵ[3], or if the interaction counter n is greater than the defined maximum interaction N_{max}; if the condition is true, the interaction is stopped and the solution is saved in a file; otherwise, the number of interactions is increased by one, $F^{(\text{new})}$ is copied into $F^{(\text{old})}$ and we calculate the new over-relaxation parameter using A.38. Then the residual and all the components of the F^{new} matrix are calculated again. This happens in a cycle until at least one of the conditions is true.

A flowchart of the above sketch that expresses the step by step described below:

[3] ϵ is a factor that can be set to be the desired fractional precision.

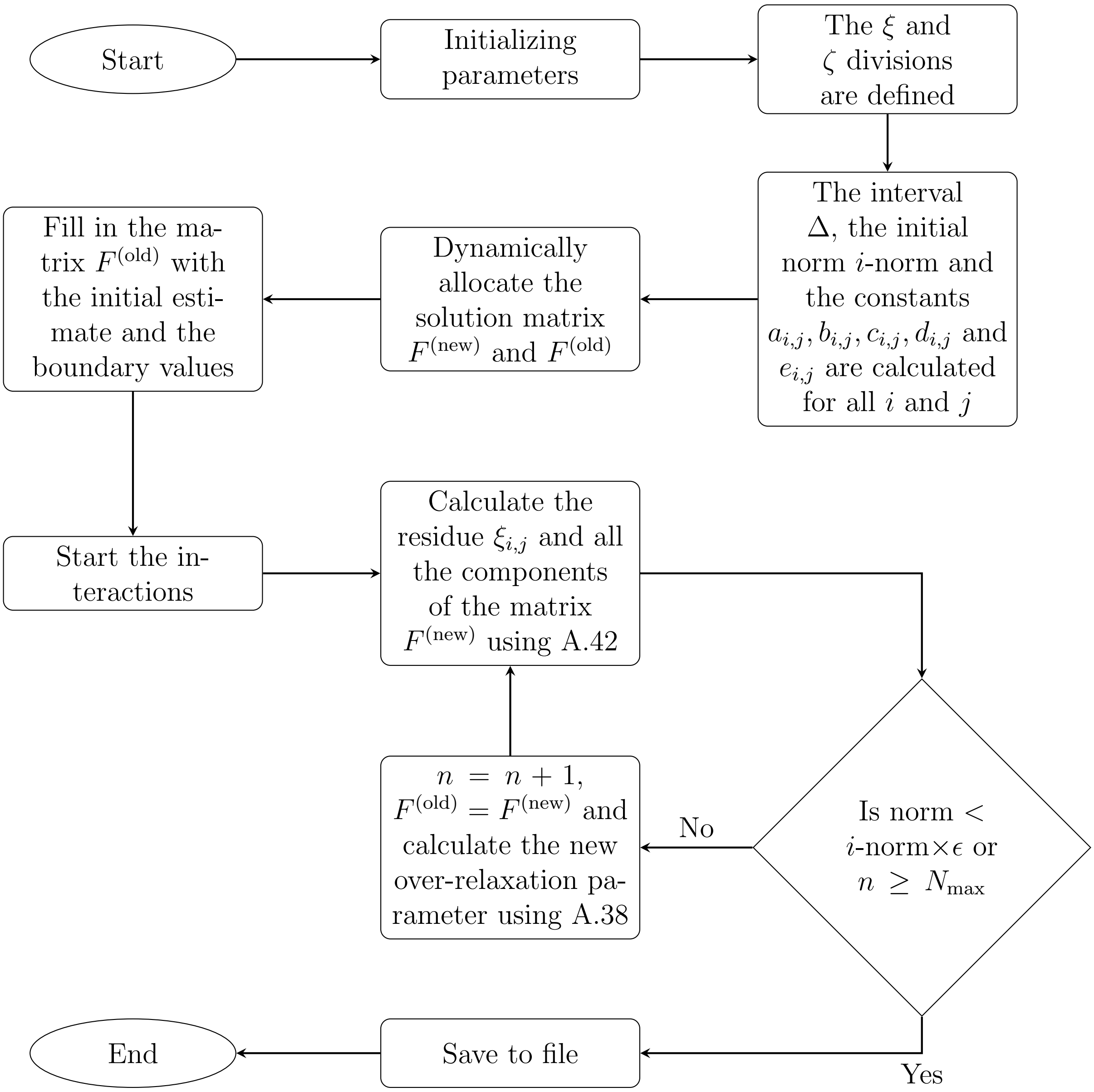

FIGURE A.4 – Flowchart of code for the Successive Overrelaxation Method with Chebyshev acceleration

The norm is calculate as

$$\text{norm} = \sum_{i,j} |\xi_{i,j}|\,, \quad \text{for all } i \text{ and } j. \tag{A.44}$$

FOLHA DE REGISTRO DO DOCUMENTO

1. CLASSIFICAÇÃO/TIPO	2. DATA	3. REGISTRO N°	4. N° DE PÁGINAS
DM	05 de fevereiro de 2024	DCTA/ITA/DM-002/2024	101

5. TÍTULO E SUBTÍTULO:

Klein-Gordon Oscillator in cosmic string spacetime in the presence of electric and magnetic field with a Coulomb and Cornell potential

6. AUTOR(ES):

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7. INSTITUIÇÃO(ÕES)/ÓRGÃO(S) INTERNO(S)/DIVISÃO(ÕES):

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8. PALAVRAS-CHAVE SUGERIDAS PELO AUTOR:

Cosmic string, Klein-Gordon Oscillator, Cornell potential, non-commutative geometry, Aharonov-Bohm effect, Coulomb potential.

9. PALAVRAS-CHAVE RESULTANTES DE INDEXAÇÃO:

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10. APRESENTAÇÃO: **(X) Nacional ( ) Internacional**

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11. RESUMO:

In this work, we analyze a Klein-Gordon Oscillator (KGO) in a spacetime produced by a one-dimensional topological defect, a cosmic string, in the presence of electric and magnetic fields with Coulomb and Cornell potentials. To develop the analysis, we separate the interaction fields into five different configurations, considering: i) free particle, ii) static radial electric field, iii) static axial electric field, iv) static angular magnetic field and v) static axial magnetic field in the presence of a Coulomb potential, a Cornell potential and a magnetic flux that presents a system analogous to the Aharonov-Bohm effect for bound states. The equations of motion are solved analytically for all scenarios, with the exception of the magnetic field in the angular direction, where a numerical solution is presented for the probability density of the particle's position. Its behavior is studied by changing the intensity of the interaction field, the linear mass density of the string and the energy. An interpretation of magnetic fields and fluxes as non-commutative in momentum space is also presented. Physical observables such as energy and linear momentum are quantized in different physical systems and their dependence on the string parameter and the interaction fields is presented.

12. GRAU DE SIGILO:

(X) OSTENSIVO ( ) RESERVADO ( ) SECRETO