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EFFECTIVE ACTIONS VIA WORLDLINE FORMALISM

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Resumo

Nesta dissertação de mestrado estudamos ações efetivas de um loop obtidas via Formalismo de linhas de universo para os casos específicos de um campo escalar acoplado a um potencial escalar, bem como QED escalar e espinorial. Exploramos e descrevemos também o efeito Schwinger, uma consequência direta da existência de uma parte imaginária não-nula na ação efetiva.

Nosso estudo baseia-se principalmente em uma abordagem analítica considerando um regime semiclássico, onde calculamos a taxa de criação de pares de elétrons no caso específico de campos elétricos e magnéticos paralelos, uma extensão do caso discutido em (GORDON; SEMENOFF, 2015), e nosso principal método de interesse, que é um método numérico, o chamado de método Worldline Numerics, Worldline Monte Carlo ou Loop Cloud, para o qual implementamos numericamente o método e o testamos em uma configuração de campo magnético de fundo constante.

Esses métodos são usados na literatura para estudar a física do efeito Casimir (GIES; MOYAERTS, 2003), não tendo que depender sempre de formas simples de superfícies de contorno, e o efeito Schwinger, mesmo no caso de campos dinâmicos (SCHÜTZHOLD *et al.*, 2008), que têm a importância teórica de diminuir o limite de Schwinger, a intensidade de campo mínima teórica necessária para observar os fenômenos em um experimento.

Abstract

In this master thesis we study one-loop effective actions obtained via Worldline Formalism for the specific cases of a scalar field coupled to a scalar potential as-well as scalar and spinorial QED. We explore and describe also the Schwinger effect, a direct consequence of a non-vanishing imaginary part in the one-loop effective action.

Our study is mainly based on a analytical approach considering a semi-classical regime, where we calculate electrons pair creation rate in the specific case of parallel electric and magnetic fields, an extention of the presented in (GORDON; SEMENOFF, 2015), and our main method of interest, which is a numerical method, the so-called Worldline Numerics, Worldline Monte Carlo or Loop Cloud method, for which we implemented numerically the method and tested it on a constant magnetic background setting.

These methods are used throughout the literature to study the physics of the Casimir effect (GIES; MOYAERTS, 2003), not having to rely always on simple boundary surfaces shapes, and the Schwinger effect, even in the case of dynamical fields (SCHÜTZHOLD *et al.*, 2008), which have the theoretical importance of lowering the Schwinger limit, the theoretical minimum field intensity necessary to observe the phenomena in a experiment.

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x	4-position of the considered particle, position (Chapter 3, section 3)
e	electric charge of the considered particle
m	mass of the considered particle
E	Electric field
B	Magnetic field
γ	pair production rate
$\mathcal{L}$	Lagrangian density
$F_{\mu\nu}$	electromagnetic tensor
ψ	fermionic field
ϕ	scalar field
$\|x\rangle$	position eigenstate
$\|p\rangle$	momentum eigenstate
G	Green's function
$\|A\rangle$	vacuum state of the interacting theory
Γ	effective action
$\hat{H}$	identified Hamiltonian
Π	canonical momentum of the associated Hamiltonian
J^{μ}	current density
γ^{μ}	Dirac matrices
S	action of a system
s	Schwinger proper time
T	time order (Chapter 2), proper-time (Chapter 3 and 4)
A_{μ}	4-electromagnetic potential

1 Introduction

The quantum description of physical systems had its beginning in 1900, when Max Planck proposed the quantization of energy to solve the problem of the radiation of black bodies(PLANCK, 1900). That problem had no solution in classical physics that could explain the behavior of the power distribution for the whole spectra, giving rise to the ultraviolet catastrophe.

In this early development, it was not clear if this quantization was just a mathematical method that led to the right answer or actually something with real physical meaning. It was only with Einstein's solution for the photoelectric effect that it became more evident that energy carried by electromagnetic radiation was indeed discrete.

Einstein went even further and argued that the electromagnetic radiation was constituted by particles that carried the energy (EINSTEIN, 1905). This was the birth of the wave-particle duality. Louis de Broglie (BROGLIE, 1925) proposed that such a duality held even for massive particles and proposed a relation between the particle's momenta and its wavelength. By these means, de Broglie was able to derive the Bohr model of the atom and its formula for the energy levels.

This period, commonly referred to as the old quantum theory, ends with the proposal of a wave equation for describing particles by Schrödinger (SCHROEDINGER, 1926), and parallel to it, the matrix mechanics proposed by Heisenberg (HEISENBERG, 1925). These two approaches, which later proved to be equivalent(PIZA, 2003), are the foundations of what nowadays is called quantum mechanics.

Quantum mechanics turned out to be a fantastic theory. It provided the level of understanding of the behavior of particles needed to us to advance in fields like optics and electronics, having deep technological impacts. It is used today for understanding and predicting properties of materials that may not even be found in nature, but that can be useful in industrial and scientific applications, and is the theoretical framework of quantum computing, which is a candidate for the next technological revolution and promises to revolutionize areas as cryptography and cybersecurity.

In spite of its enormous impacts on physics and society, quantum mechanics still has its limitations. The other major theory that was developed in the twentieth century, Einstein's Relativity, in specific the special theory of relativity, changed the previous existing notion of an absolute time, the so-called Newtonian time (RINDLER, 2003). It was believed to exist as an absolute quantity that flowed steadily and on the same pace for all observers in the universe. Special relativity implied that displacements in time, despite having a different behavior as space displacements, are frame dependent in the same sense that the spacial ones. So, instead of being treated as different objects, space and time turned to be different components of a more fundamental entity, the spacetime, which for special relativity is called the Minkowski spacetime.

In the same sense that one can define and measure distances in the Euclidean space, one might define and measure distances in the Minkowski spacetime. The Euclidean space is invariant under a set of transformations called Galilean transformations. In the same way, Minkowski spacetime is invariant under a set of transformations called Lorentz transformations, which consists of rotations and the so-called Lorentz boosts.

The Schrödinger equation, which is the fundamental equation governing the dynamics of quantum mechanics, is not invariant under Lorentz transformations. Therefore, in light of special relativity, it cannot be a fundamental property of nature, meaning that there should be a more fundamental theory whose non-relativistic limit is quantum mechanics. Other problems such as the incapacity of explaining effects such as particle annihilation also suggests something is missing.

There have been different attempts to create a relativistic version of quantum mechanics. However, up to this point only one theory could manage to unify both theories, explains annihilation and is in agreement with experiments. In fact, that is considered to be the most successful physical theory ever created with astonishing experimental accuracy. That theory is the so-called Quantum Field Theory(QFT)(DIRAC, 1927), in which the elementary particles are interpreted to be the quanta of fields, more fundamental entities which permeate the whole spacetime.

That leads us to the present time, in which the most precise description of the nature of particles is given by the quantum theory of fields. Three out of four of the fundamental interactions of nature are well described by QFT, the electromagnetic, weak and strong interactions. A quantum theory of gravity, on the other hand, proved to be a more difficult task and to the present day we do not have a theory giving such unified description. That being said, we do have candidates to that unifying theory such as String Theory and Loop Quantum Gravity, but those candidates couldn't be experimentally tested.

It is in the context of quantum field theory that we center our discussion in this master thesis. In fact, effective theories are part of all areas of physics and are an important tool for our understanding of complex phenomena and for practical calculations under some physical limits.

Let us introduce the concept of an effective theory by considering a simple example. In Newtonian physics, the gravity effect in a body with mass m by another body of mass M is described by Newton's Law of Universal Gravitation

$$\vec{F}(r_1, r_2) = -\frac{GMm}{||\vec{r_1} - \vec{r_2}||^3}(\vec{r_1} - \vec{r_2}), \tag{1.1}$$

where M and m are the masses of the two bodies interacting by means of the gravitational field, $\vec{r_1}$ is the position of the body of mass m, $\vec{r_2}$ is the position of the body of mass M and G is the universal gravitational constant.

Consider now the problem of the free fall of a body near the Earth's surface. In this case, the distance $||\vec{r_1} - \vec{r_2}||$ is equal to $R_E + h$, where h is the height of the falling body with respect to the Earth's surface. So,

$$F(r_1, r_2) = -\frac{GMm}{(R_E + h)^2} = -\frac{GMm}{R_E^2(1 + \frac{h}{R_E})^2}, \tag{1.2}$$

and since $\left|\frac{h}{R_E}\right| < 1$, we can consider a expansion of the denominator as the square of a geometric series.

Therefore,

$$F(r_1, r_2) = -\frac{GMm}{R_E^2}\left[1 - 2\frac{h}{R_E} + 3\frac{h^2}{R_E^2} - 4\frac{h^3}{R_E^3} + ...\right]. \tag{1.3}$$

Thus, in the regime where $h << R_E$, we can approximate the force as being constant

$$\vec{F}(r_1, r_2) = m\vec{g}, \tag{1.4}$$

where

$$\vec{g} = -\frac{GMm}{R_E^2}\frac{(\vec{r_1} - \vec{r_2})}{||\vec{r_1} - \vec{r_2}||}. \tag{1.5}$$

Thus, this constant gravitational force is an effective theory of Newtonian gravity in the small length scales. In day-to-day observations of free falling objects, for example, if one was to choose a model to the gravitational force, the constant gravity theory suits our needs with high accuracy. Indeed, we were to model an object's trajectory by a free fall it would make no sense to consider the higher order perturbations, since the effect of air resistance would be far more relevant for the deviation of the predicted trajectory.

In our study, we consider in general two fields interacting, but consider one to be stronger than the other, establishing some hierarchy. This way, we integrate out the influence of one of the fields defining an effective Lagrangian for the other (PESKIN; SCHROEDER, 1995). We will deal with those effective actions by means of the World-line Formalism(AFFLECK; MANTON, 1982), a method historically first applied in a string theoretic framework(BERN; KOSOWER, 1991) .

In chapter 2, we introduce some vital concepts for the methods under discussion as the Schwinger proper-time, give a derivation of the Euler-Heisenberg Lagrangian, which is a good example of effective Lagrangian and its link to the Schwinger effect. The latter is a physical phenomena predicted theoretically by means of these effective theories and will be the object of application of one of the discussed methods.

In chapter 3, we explicitly discuss the Worldline Formalism, deriving the Effective Actions for a variety of cases of the coupling of a field with an external potential. Later, we present two methods of solving the worldline integrals, one semi-classical analytical method, namely Worldline Instantons, and a numerical one, the Worldline Numerics Method. As an application, we extend the analysis in (GORDON; SEMENOFF, 2015) for a scalar field in a constant electric field for the case of a Dirac Spinor under the influence of both constant electric and magnetic fields. We verify, in the same way they conclude in the paper, that for these fields configuration the semi-classical approach yields to the exact result.

In chapter 4, we discuss our results implementing a Worldline Numerics algorithm. We discuss the particularities of the chosen approach and benchmark our results with the theoretical results, validating our developed program. As a benchmark, we chose to study a scalar field under the influence of a constant magnetic field. We display results for $D = 3$ and $D = 4$, where D is the number of considered dimensions in the Euclidean Spacetime.

In chapter 5, follows our conclusions, where we summarize our results and consider the next steps of our research.

2 Proper-time method, Euler-Heisenberg Lagrangian and the Schwinger Effect

As stated in the introduction, the Worldline formalism gained more popularity in (BERN; KOSOWER, 1991) in the context of string theory. The formalism was introduced as a limiting case when the string tension tends to infinity. It was later in (STRASSLER, 1992) when this formalism was directly derived on a Quantum Field Theory context, without any notion of strings needed.

It was done by making use of the Schwinger proper-time parameter, originally developed by Schwinger to compute an effective action, the so-called Euler-Heisenberg action, as means to discuss the instability of the vacuum in the presence of a strong electromagnetic field. In the present chapter, we review the Schwinger method, arriving in the Euler-Heisenberg Lagrangian and discussing a bit the vacuum instability. In the next chapter, we will use a worldline method, the so-called Worldline Instantons, to compute pair production rates in the case of Dirac spinors subject to an external electromagnetic field composed of parallel electric and magnetic fields.

2.1 The proper time method

This method is based on the introduction of a parameter called the Schwinger proper time to represent the propagator of the theory. Consider a fermionic field described by the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\,\partial\!\!\!/ - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi. \tag{2.1}$$

The equation of motion for the classical field is given by the solution to the Euler-Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = 0. \tag{2.2}$$

Making use of this equation for $\bar{\psi}$, we reach to the minimally coupled Dirac equation:

$$\gamma^\mu(i\partial_\mu + eA_\mu(x))\psi(x) - m\psi(x) = 0. \tag{2.3}$$

We will study the Green function of the equations of motion, which is the propagator of the fermionic field. The Green function of a differential operator D is defined to satisfy:

$$DG(x,x') = \delta(x-x'). \tag{2.4}$$

For a true vacuum state of the theory, where there is no background field, the propagator is given by:

$$G(x,y) = \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)e^{ip\cdot(x-y)}}{p^2-m^2+i\epsilon}. \tag{2.5}$$

However, instead of working directly with this standard expression, we will consider another view by introducing states of a Hilbert space. We will work in an analogy with non-relativistic quantum mechanics, working with eigenvectors of "momentum" and "position" as in a one particle space of states [1]. We have that:

$$\langle p|x\rangle = e^{ip\cdot x}, \tag{2.6}$$

where $|x\rangle$ and $|p\rangle$ are eigenstates of the position and momentum operators respectively. Replacing the exponentials in the expression of the propagator for the expression above gives:

$$G(x,y) = \int \frac{d^4p}{(2\pi)^4}\langle y|p\rangle \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\langle p|x\rangle , \tag{2.7}$$

and since

$$\frac{i(\hat{\not{p}}+m)}{\hat{p}^2-m^2+i\epsilon}|p\rangle = \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}|p\rangle \quad \textit{and} \quad \int \frac{dp^4}{(2\pi)^4}|p\rangle\langle p| = 1,$$

the following expression for the propagator is obtained:

$$G(x,y) = \langle y|(\hat{\not{p}}+m)\frac{i}{\hat{\not{p}}^2-m^2+i\epsilon}|x\rangle . \tag{2.8}$$

Now, a mathematical identity whose usage will be of great importance for the method

[1] It is important to keep in mind that it is not actually representing a particle, the actual representation of a particle in quantum field theory is much more complicated.

will be introduced. Let A be a real valued operator. The following identity holds:

$$\frac{1}{A+i\epsilon} = \int_0^\infty ds\, e^{is(A+i\epsilon)}. \tag{2.9}$$

That's just an integral representation of the inverse operator of $A+i\epsilon$ where the integral is made over a real parameter s. Substituting that identity for the operator $\frac{i}{\hat{\not p}^2 - m^2 + i\epsilon}$ we get an integral representation for the propagator.

$$G(x,y) = \int ds\, e^{-im^2 s} e^{-\epsilon s} \langle y| (\hat{\not p} + m) e^{-is\hat{H}} |x\rangle , \tag{2.10}$$

where $\hat{H} = \hat{\not p}^2$.

That representation has a very interesting interpretation once we define $\hat{H}$ and think of it as a Hamiltonian operator. The exponential resembles the time evolution operator e^{-itH}. Thus, one can think of the exponential as the operator of evolution in the s variable. Due to that interpretation this parameter s is called the Schwinger proper time.

Now, we can make the same steps and reach to a similar form for the propagator in the presence of the electromagnetic field. Here, we will obtain this propagator by performing the minimal coupling between the fermionic field and the electromagnetic field. So, we will make the formal replacement:

$$p_\mu \to p_\mu - eA_\mu.$$

This leads us to:

$$G_A(x,y) = \langle y| \frac{i}{\hat{\not p} - e\,\hat{\not A}\,(x) - m + i\epsilon} |x\rangle = \langle y| (\hat{\not p} - e\,\hat{\not A}\,(x) + m) \frac{i}{(\hat{\not p} - e\,\hat{\not A}\,(x))^2 - m^2 + i\epsilon} |x\rangle , \tag{2.11}$$

and introducing the Schwinger proper time in the previous expression we get:

$$G_A(x,y) = \int ds\, \langle y| i(\hat{\not p} - e\,\hat{\not A}\,(x) + m) e^{is[(\hat{\not p} - e\hat{\not A}(x))^2 - m^2 + i\epsilon]} |x\rangle . \tag{2.12}$$

So,

$$G_A(x,y) = \int ds e^{-s\epsilon} e^{-ism^2} \langle y| i(\hat{\not p} - e\,\hat{\not A}\,(x) + m) e^{-i\hat{H}s} |x\rangle , \tag{2.13}$$

where the operator $\hat{H}$ was defined as:

$$\hat{H} = -(\hat{\not{p}} - e\,\hat{\not{A}}\,(x))^2. \tag{2.14}$$

By effectively calculating this operator and rearranging the terms we get:

$$(\hat{\not{p}} - e\,\hat{\not{A}}\,(x))^2 = (\hat{p} - e\hat{A}(x))^2 - \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}. \tag{2.15}$$

So, finally:

$$\hat{H} = -(\hat{p} - e\hat{A}(x))^2 + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}, \tag{2.16}$$

where $\sigma^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$

That way, the Green function can be interpreted as an amplitude associated to a non-relativistic particle from a initial position x to a final position y after a temporal evolution in the parameter s. With this expression for the Green function in hands, the next analysis to be made is one regarding the Lagrangian of the theory.

Even though we already have a Lagrangian for the fermionic field coupled to the electromagnetic field, it will be beneficial to work with an effective Lagrangian instead of the full theory Lagrangian.

By utilizing an effective theory, one can simplify the problem, turning it into a more easily tractable one. The following equation exemplifies this objective for the case of a field theory:

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{i\int dx^4 \mathcal{L}_{eff}[\psi, \bar{\psi}]\right\} = \int \mathcal{D}\phi\mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{i\int dx^4 \mathcal{L}[\phi, \psi, \bar{\psi}]\right\}. \tag{2.17}$$

The equation (2.17) exemplifies the role of an effective Lagrangian for the case of a theory such that its full description depends on two different fields, one scalar and a spinorial. By making use of an effective Lagrangian, we build an equation that depends only on the spinorial degrees of freedom. It's said that the scalar field ϕ has been integrated out of the theory. Its influence still exists, but it is now incorporated into the Lagrangian itself.

Therefore, we wish to obtain an effective lagrangian for quantum electrodynamics in the presence of the electromagnetic field. Once it is done, it will be used to study the stability of the vacuum.

The full lagrangian of the theory is given by equation (2.1). For the construction of the effective lagrangian we will assume that the fermionic field is a solution of the classical equation of motion of the free theory:

$$(i\not\partial - m)\psi = 0. \tag{2.18}$$

This is the Dirac equation. It is a well known result that this theory has the following conserved current:

$$j^\mu = \bar{\psi}\gamma^\mu\psi.$$

We obtain this current by applying the Noether's Theorem, which states that for any continuous symmetry a theory might have there is one associated conserved quantity, which we call a current. Since the current is given by $J^\mu = \langle A|\,\bar{\psi}\gamma^\mu\psi\,|A\rangle$, we will substitute the actual current by the expected value:

$$\bar{\psi}\gamma^\mu\psi \to \langle A|\,\bar{\psi}\gamma^\mu\psi\,|A\rangle$$

where the state $|A\rangle$ denotes the vacuum state of the coupled theory, not the free one. By doing so, we are left if the effective Lagrangian:

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu J^\mu \quad , \quad J^\mu = \langle A|\,\bar{\psi}\gamma^\mu\psi\,|A\rangle\,. \tag{2.19}$$

Now, the value J^μ will be explicitly calculated, resulting in an expression dependent on the electromagnetic tensor $F_{\mu\nu}$ and the gauge field A_μ. By working with the spinorial indices and rearranging the terms without forgetting that the spinorial components anti-commute we get:

$$J^\mu = \langle A|\,\bar{\psi}\gamma^\mu\psi\,|A\rangle = \langle A|\,\bar{\psi}_\alpha\gamma^\mu_{\alpha\beta}\psi_\beta\,|A\rangle = -\langle A|\,\psi_\beta\bar{\psi}_\alpha\gamma^\mu_{\alpha\beta}\,|A\rangle\,. \tag{2.20}$$

The last expression is sum over the diagonal terms of the matrix $\psi\bar{\psi}\gamma^\mu$, it is, the trace of the operator:

$$J^\mu = -\operatorname{Tr}\langle A|\,\psi\bar{\psi}\gamma^\mu\,|A\rangle\,. \tag{2.21}$$

Recall that:

$$G_A(x,y) = \langle A|\,T\psi(x)\bar{\psi}(y)\,|A\rangle\,, \tag{2.22}$$

where T is the time order operator.

Following the procedure of (SCHWARTZ, 2014), in the limit when $x = y$ we formally replace it on the expression for the current:

$$J^\mu = -\operatorname{Tr} \langle x| G_A \gamma^\mu |x\rangle . \tag{2.23}$$

Now we can write the current, and therefore the effective Lagrangian, in terms of the integral formulation of the Green function using the Schwinger proper time:

$$J^\mu = -\operatorname{Tr} \int ds\, e^{-im^2 s} e^{-s\epsilon} \langle x| \gamma^\mu (\hat{\not p} - e \not A\,(x) + m) e^{-is\hat{H}} |x\rangle . \tag{2.24}$$

Using the fact that the trace of the product of an odd number of γ^μ matrices is zero we get:

$$J^\mu = -\operatorname{Tr} \int ds\, e^{-im^2 s} e^{-s\epsilon} \langle x| \gamma^\mu (\hat{\not p} - e \not A\,(x)) e^{-is\hat{H}} |x\rangle . \tag{2.25}$$

Recalling that $\hat{H} = -(\hat{p} - e\hat{A}(x))^2 + \frac{e}{2} F_{\mu\nu}\sigma^{\mu\nu}$:

$$J^\mu = -\frac{i}{2e}\frac{\partial}{\partial A_\mu} \int ds\, e^{-im^2 s} e^{-s\epsilon} \operatorname{Tr} \langle x| e^{-is\hat{H}} |x\rangle . \tag{2.26}$$

Integrating and substituting in the expression of the Lagrangian:

$$\mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{i}{2} \int ds\, e^{-im^2 s} e^{-s\epsilon} \operatorname{Tr} \langle x| e^{-is\hat{H}} |x\rangle . \tag{2.27}$$

So, in order to finally get an expression for the effective lagrangian one must calculate $\langle y| e^{-i\hat{H}s} |x\rangle$. From now on, the result depends on the electromagnetic field. For the following derivation it is assumed that the electric field is constant and non zero and that there's no magnetic field. Let's use the following commutation relations:

$$[x_\mu, \Pi_\nu] = i g_{\mu\nu} \quad , \quad [\Pi_\mu, \Pi_\nu] = -ieF_{\mu\nu}. \tag{2.28}$$

where Π_μ is the canonical momentum associated to the Hamiltonian $\hat{H}$. Consider also the Heisenberg equations:

$$\frac{dx_\mu}{ds} = i[H, x_\mu] = 2\Pi_\mu \quad , \quad \frac{d\Pi_\mu}{ds} = i[H, \Pi_\mu] = 2eF_{\mu\nu}\Pi^\nu. \tag{2.29}$$

For a constant $F_{\mu\nu}$ and writing the equations on an operator form:

$$\frac{d\mathbf{\Pi}}{ds} = 2e\mathbf{F}\cdot\mathbf{\Pi} \Rightarrow \mathbf{\Pi}(s) = e^{2e\mathbf{F}s}\mathbf{\Pi}(0), \tag{2.30}$$

$$\frac{d\mathbf{x}}{ds} = i[H,\mathbf{x}] = 2e^{2e\mathbf{F}s}\mathbf{\Pi}(0) \Rightarrow \mathbf{x}(s)-\mathbf{x}(0) = \left(\frac{e^{2e\mathbf{F}s}-1}{e\mathbf{F}}\right)\mathbf{\Pi}(0). \tag{2.31}$$

Thus,

$$\mathbf{\Pi}(0) = \frac{1}{2}e^{-e\mathbf{F}s}\frac{e\mathbf{F}}{\sinh(\mathbf{F}s)}[\mathbf{x}(s)-\mathbf{x}(0)], \tag{2.32}$$

$$\mathbf{\Pi}(s) = \frac{1}{2}e^{e\mathbf{F}s}\frac{e\mathbf{F}}{\sinh(\mathbf{F}s)}[\mathbf{x}(s)-\mathbf{x}(0)]. \tag{2.33}$$

So,

$$\hat{H} = -\mathbf{\Pi}(s)\cdot\mathbf{\Pi}(s) - \frac{e}{2}\,\mathrm{Tr}(\sigma\mathbf{F}) = -[\mathbf{x}(s)-\mathbf{x}(0)]\left(\frac{e^2\mathbf{F}^2}{4\sinh^2(e\mathbf{F}s)}\right)[\mathbf{x}(s)-\mathbf{x}(0)] - \frac{e}{2}\,\mathrm{Tr}(\sigma\mathbf{F}). \tag{2.34}$$

But, $\frac{d}{ds}\langle y,s|x,0\rangle = \frac{d}{ds}\langle y|\,e^{-is\hat{H}}\,|x\rangle = -i\langle y|\,e^{-is\hat{H}}\hat{H}\,|x\rangle$.

Substituting the expression 2.34 for $\hat{H}$ and factorising it on a beneficial form to get $\mathbf{x}(s)$ on the left and $\mathbf{x}(0)$ on the right of the expression, we get:

$$-i\partial_s\langle y;0|x;s\rangle = -\left[(y-x)\left(\frac{e^2\mathbf{F}^2}{4\sinh^2(e\mathbf{F}s)}\right)(y-x) + \frac{i}{2}\,\mathrm{Tr}[e\mathbf{F}\coth es\mathbf{F}] + \frac{e}{2}\,\mathrm{Tr}\{\sigma\mathbf{F}\}\right]\langle y;0|x;s\rangle. \tag{2.35}$$

This last expression is a differential equation on $\langle y;0|x;s\rangle$. Solving it and substituting the solution on the effective Lagrangian expression:

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{e^2}{32\pi^2}\int ds\frac{1}{s}e^{-im^2s}\frac{\mathrm{Re}[\cos(esX)]}{\mathrm{Im}[\cos(esX)]}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{2.36}$$

$$X = \sqrt{\frac{1}{2}F_{\mu\nu}^2 + \frac{i}{2}F^{\mu\nu}\tilde{F}_{\mu\nu}} \quad , \quad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}. \tag{2.37}$$

This effective Lagragian is the so called Euler-Heisenberg Lagrangian. It was first derived by Sauter (SAUTER, 1931). Now that we finally got the effective Lagrangian it will be discussed how it leads to a vacuum instability.

2.2 The Schwinger Effect

How to relate the Euler-Heisenberg Lagrangian to the production of pairs of real particles is not somehow direct. In order to make this connection, one should study it as a scattering problem. The major role in a scattering process is attributed to the scattering matrix S. It encodes the probability amplitudes for scattering processes of asymptotically free states. It is known that in a stable vacuum state we would get $\langle A| S |A\rangle = 1$, where S is the scattering matrix.

The S-matrix is given by:

$$\langle A| S |A\rangle = e^{i\Gamma}, \tag{2.38}$$

where Γ is the effective action.

So, one can associate $|\langle A| S |A\rangle|^2$ to the probability of the theory to remain on the vacuum state. In other words, that's the probability for the vacuum not decay. The decay of the vacuum state is performed to another state in which we have real particles. That's where the pair production comes up.

Performing the square of the absolute value,

$$|\langle A| S |A\rangle|^2 = e^{i(\Gamma-\Gamma^*)} = e^{-2\,\mathrm{Im}(\Gamma)}. \tag{2.39}$$

Finally, one gets the probability of pair formations:

$$P = 1 - e^{-\gamma}, \tag{2.40}$$

where

$$\gamma = 2\,\mathrm{Im}\left[\int dx^4 - \frac{e^2}{32\pi^2}\int ds \frac{1}{s} e^{-im^2 s}\frac{\mathrm{Re}[\cos(esX)]}{\mathrm{Im}[\cos(esX)]}F_{\mu\nu}\tilde{F}^{\mu\nu}\right]. \tag{2.41}$$

The trivial conclusion is that the vacuum is unstable if the integral has a non zero imaginary part.

The imaginary part of this integral is non vanishing only in the presence of poles on the integrand. The function $\cos(esX)$ cannot have a vanishing imaginary part in the pure magnetic field. In other words, this tells us that the theory is stable in this configuration. In the case of non zero electric fields, one may solve the integral by using the residues theorem. It results in an imaginary component. Thus, the theory is unstable in the presence of electric fields. For the pure, constant and uniform electric field case, the pair production rate is given by:

$$\gamma = \frac{e^2E^2}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-n\pi m^2/eE}. \tag{2.42}$$

In this last expression we see the exponential dependence in $\frac{1}{E}$. So, as a function of E, that function is not analytic in the origin. Therefore, it cannot be expanded in series around $E = 0$. That is an evidence of the non perturbative characteristic of the phenomena.

In summary, the conclusion is that the vacuum state, defined as the state of lowest energy, turns out to be unstable and decays to a new vacuum in which we have the presence of real particles. This effect can be thought as the virtual pairs in the quantum vacuum, when subjected to an electric field, turning into real ones. Considering a classical point view of this phenomena, this effect can be visualized as the electric field providing a difference of potential necessary to the virtual pairs to overcome the potential barrier and, instead of being annihilated, they get separate apart.

In the Dirac sea picture, the particle would tunnel from a negative energy state, leaving behind a hole, to a positive energy state.

This effect was initially predicted by Sauter (SAUTER, 1931), but it was Schwinger (SCHWINGER, 1951) the first one to calculate the production rate of pairs by introducing the proper-time formalism.

From (2.42), it can be noted that this effect can only be detectable for extremely intense electric fields, of the order of $E \approx 10^{18} V/m$, the so called Schwinger limit. Due to the necessity of such an intense electric field this effect has never been able to be detected experimentally.

Nowadays, there are experiments dedicated to study, among other physical phenomena, the Schwinger effect. These attempts of observing the Schwinger effect are based on the expected reduction of this limiting value of field intensity for the case of fields varying in time (SCHÜTZHOLD *et al.*, 2008). Therefore, techniques based on the usage of high energy lasers are being used to provide a detectable effect (BLASCHKE A. V. PROZORKEVICH; ROBERTS, 2008).

3 The Worldline Formalism

This chapter introduces the worldline formalism, which appeared first in a work by Feynman, who expressed the QED S-matrix in terms of the path integral of a relativistic particle, but got relevance only after the 1980's work of Affleck (AFFLECK; MANTON, 1982) and got consolidated as an important tool after the work of Bern and Kosower (BERN; KOSOWER, 1991) in the 1990's. Before introducing the method, we briefly discuss the concept of Euclidean spacetime, which is stage of the Worldline method as applied in this work. After an introduction of the method, in which we derive the expressions for a few different cases, we discuss two methods of solving the worldline integrals such as a semiclassical approach to it and numerical methods for solving the path integral.

3.1 Quantum Theory in Euclidean Spacetime

A field theory is said to be an Euclidean Field Theory when the space in which the dynamical variables, the fields, are defined is a Riemannian manifold. A Riemannian manifold is a differential manifold, it is, locally diffeomorphic to $\mathbf{R}^k$ for some finite k, endowed with a Riemannian metric, a positive definite symmetric 2-form on the manifold.

In contrast, in a so called pseudo-Riemannian manifold we don't require the positive definiteness of the metric tensor. A Lorentzian Spacetime is a particular case in which, in its diagonalized form, the metric tensor has only one negative eigenvalue. The Minkowski spacetime is the simplest example of such a manifold, while the Euclidean space itself is the simplest example of a Riemannian manifold.

The basic framework of relativistic field theory is defined in the Minkowski spacetime. However, by considering an analytical continuation of the propagators and n-point functions to complex values of time we may be able to work on an Euclidean spacetime. Such transformation is performed by the formal replacement:

$$t \to it,$$

which is called a Wick rotation. After this procedure, instead of working in (3+1)-

spacetime dimensions we are left with a 4-dimensional Euclidean space.

By performing the Wick rotation, we explicit the strong analogy existing between quantum field theory and statistical field theory. As an illustration of this, consider the Lagrangian for a scalar field in d dimensions, one temporal and d-1 spatial, given below:

$$\mathcal{L} = \int d^d x (\partial_\mu \phi)^2 - V(x), \tag{3.1}$$

with the potential depending only on the spatial coordinates. Performing a Wick rotation, the following transformations will take place:

$$dt \to -idt,$$

$$(\partial_t \phi)^2 \to (i\partial_t \phi)^2 = -(\partial_t \phi)^2,$$

and the Lagrangian in the new coordinates is given by:

$$\mathcal{L}[\phi] = i \int d^d x (\nabla \phi)^2 + V(x), \tag{3.2}$$

where a negative imaginary unit factor arose from the metric substitution and the global minus sign was brought out of the integral, which arose from the derivative.

Now, defining

$$\mathcal{F}[\phi] = \int d^d x (\nabla \phi)^2 + V(x),$$

the functional of the theory, is given by:

$$\mathcal{Z}[\phi] = \int \mathcal{D}\phi e^{iS[\phi]} = \int \mathcal{D}\phi e^{-\mathcal{F}[\phi]}. \tag{3.3}$$

Thus, after the Wick rotation the functional assumes the exact form of a partition function of a statistical system. If the functional $\mathcal{F}[\phi]$ is positive definite the analogy is exact in the sense that the exponential corresponds precisely to the Gibbs factor.

3.2 The Effective action derivation

We will now consider the derivation of the effective actions in the worldline formalism. We will first introduce it for the simplest case of a scalar field in the presence of a background scalar potential and then consider the scalar field in the presence of a classical electromagnetic field, finally, it is considered the case of Dirac spinors.

3.2.1 Scalar field on external scalar potential

Here we concentrate on scalar fields under the influence of an external scalar potential. In Euclidean spacetime the lagrangian density of this theory can be written as:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{m^2}{2}\phi^2 + \frac{V(x)}{2}\phi^2, \tag{3.4}$$

where m denotes the mass of the scalar quantum field ϕ and $V(x)$ the scalar potential, which depends on the spacetime coordinates.

The vacuum persistence amplitude for this theory is:

$$\langle 0|0\rangle = \int \mathcal{D}\phi e^{-S[\phi,V]}\,, \tag{3.5}$$

and, by definition, this is equal to:

$$e^{-\Gamma[V]} = \int \mathcal{D}\phi e^{-S[\phi,V]}\,, \tag{3.6}$$

where Γ is the effective action.

Recall that,

$$S = \int dx\,\mathcal{L} = \int dx\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{m^2}{2}\phi^2 + \frac{V(x)}{2}\phi^2\,. \tag{3.7}$$

Performing an integration by parts in the kinetic term and disregarding the surface term, we get:

$$S = \int dx\phi\left[-\frac{1}{2}\partial_\mu^2 + \frac{m^2}{2} + \frac{V(x)}{2}\right]\phi\,. \tag{3.8}$$

Now the path integral has a Gaussian form and can be analytically solved. This yields, considering a normalization from the free field case:

$$e^{-\Gamma[V]} = det^{\frac{-1}{2}}\left(\frac{-\partial^2 + m^2 + V(x)}{-\partial^2 + m^2}\right)\,, \tag{3.9}$$

and therefore

$$\Gamma[V] = \frac{1}{2}\,\mathrm{Tr}\ln\left(\frac{-\partial^2 + m^2 + V(x)}{-\partial^2 + m^2}\right)\,. \tag{3.10}$$

Now we introduce a Schwinger proper-time parameter by means of the mathematical identity:

$$\ln\left(\frac{A}{B}\right) = \int_0^\infty dT\frac{e^{-BT} - e^{-AT}}{T}\,. \tag{3.11}$$

Substituting this in (3.10) we get:

$$\Gamma[V] = -\frac{1}{2}\,\mathrm{Tr}\int_0^\infty \frac{dT}{T}e^{-m^2T}\left[e^{-(-\partial^2+V(x))T} - e^{-(-\partial^2)T}\right] \tag{3.12}$$

The next step consists now of representing the functional trace in a quantum mechanical spacetime position basis. In other words, considering the sum (integral) over the diagonal elements of the matrix representation of the operator in this basis.

$$\mathrm{Tr}[\mathcal{O}] = \int dx \, \langle x| \, \mathcal{O} \, |x\rangle \ . \tag{3.13}$$

By doing so, (3.12) assumes the form:

$$\Gamma[V] = -\frac{1}{2}\int dx \int_0^\infty \frac{dT}{T} e^{-m^2 T} \left[\langle x| \, e^{-(-\partial^2 + V(x))T} \, |x\rangle - \langle x| \, e^{-(-\partial^2)T} \, |x\rangle \right] \ . \tag{3.14}$$

Solving for $\langle x| \, e^{-(-\partial^2)T} \, |x\rangle$ is straightforward. We introduce a momentum basis by the identity $I = \int \frac{dp}{(2\pi)^D} |p\rangle \langle p|$, where I is the identity operator.

$$\langle x| \, e^{-(-\partial^2)T} \, |x\rangle = \int dp \, \langle x| \, e^{-(-\partial^2)T} \, |p\rangle \, \langle p|x\rangle \ . \tag{3.15}$$

From where we recall that $\hat{p} = -i\hbar\partial$ and $\langle p|x\rangle = e^{-\frac{i}{\hbar}px}$. Then, omitting the $\hbar$ since we are using natural units,

$$\partial^2 \, |p\rangle = -p^2 \, |p\rangle \Rightarrow e^{\partial^2} \, |p\rangle = e^{-p^2} \, |p\rangle \ .$$

Therefore,

$$\langle x| \, e^{-(-\partial^2)T} \, |x\rangle = \int \frac{dp}{(2\pi)^D} \, e^{-Tp^2} = (4\pi T)^{-\frac{D}{2}} \ . \tag{3.16}$$

Now we shall study the matrix element $\langle x| \, e^{(\partial^2 - V(x))T} \, |x\rangle$. For that our strategy will be to first discretize the problem in the proper-time parameter defining $\epsilon = \frac{T}{N}$, where N is the considered number of proper-time steps. Then, we will consider the continuous limit as $N \to \infty$.

Translating this into mathematics, we are considering the following:

$$e^{(\partial^2 - V(x))T} = \lim_{N\to\infty} \prod_{i=1}^{N} e^{(\partial^2 - V(x))\epsilon} \ . \tag{3.17}$$

Then, assuming

$$\langle x| \, e^{(\partial^2 - V(x))T} \, |x\rangle = \lim_{N\to\infty} \langle x| \prod_{i=1}^{N} e^{(\partial^2 - V(x))\epsilon} \, |x\rangle \ , \tag{3.18}$$

we can study the problem for a finite number of steps N and take the limit afterwards.

Proceeding this way and introducing N momentum state and $N-1$ space state identities:

$$\begin{aligned}
&\langle x|\prod_{i=1}^{N} e^{(\partial^2-V(x))\epsilon}|x\rangle = \\
&= \mathcal{N}\int dp_1\ldots dp_N dx_1 ... dx_{N-1}\langle x|\, e^{(\partial^2-V(x))\epsilon}|p_1\rangle\,\langle p_1|x_1\rangle\, ...\,\langle x_{N-1}|\, e^{(\partial^2-V(x))\epsilon}\,|p_N\rangle\,\langle p_N|x\rangle \\
&= \mathcal{N}\int dp_1 ... dp_N dx_1 ... dx_{N-1} e^{-p_1^2\epsilon}e^{-V(x)\epsilon}e^{ip_1(x-x_1)} ... e^{-p_N^2\epsilon}e^{-V(x_{N-1})\epsilon}e^{ip_N(x_{N-1}-x)} \qquad (3.19)
\end{aligned}$$

Rearranging the terms, we get:

$$\begin{aligned}
&\langle x|\prod_{i=1}^{N} e^{(\partial^2-V(x))\epsilon}|x\rangle = \mathcal{N}\int dp_1 ... d_{p_N} dx_1 ... dx_{N-1}\exp\sum_j\left[-p_j^2\epsilon - ip_N(x_j-x_{j-1}) - V(x_j)\epsilon\right] \\
&= \mathcal{N}\int dp_1 ... d_{p_N} dx_1 ... dx_{N-1}\exp\left[-i\sum_j\left[p_N(x_j-x_{j-1}) - i\epsilon(p_j^2+V(x_j))\right]\right]. \qquad (3.20)
\end{aligned}$$

Now, we can make sense of this by interpreting $H(p_j, x_j, t_j) = p_j^2 + V(x_j)$ as the Hamiltonian of a quantum mechanical particle of mass 1/2. Since $p_j(x_j - x_{j-1}) = p_j\frac{(x_j-x_{j-1})}{\epsilon}\epsilon = p_j\dot{x_j}\epsilon$, we have a discretized version of a quantum mechanical action:

$$S_n = \sum_{j=0}^{N} L_j\epsilon = \sum_{j=0}^{N}\left[p_j\dot{x_j} - iH(p_j, x_j, t_j)\right]\epsilon\,. \qquad (3.21)$$

One can proceed noting that we can factorize $-p_j^2\epsilon - ip_N(x_j - x_{j-1}) = -(p_j\sqrt{\epsilon} + i\frac{(x_j-x_{j-1})}{2\sqrt{\epsilon}})^2 - \frac{(x_j-x_{j-1})^2}{4\epsilon}$. Thus, the momentum integrals reduce to Gaussian integrals and then:

$$\langle x|\prod_{i=1}^{N} e^{(\partial^2-V(x))\epsilon}|x\rangle = \mathcal{N}\int dx_1 ... dx_{N-1}\exp\left[-\sum_j\left[\frac{(x_j-x_{j-1})^2}{4\epsilon} + V(x_j)\epsilon\right]\right].$$

In the limit in which $N\to\infty$, we get the definition of a path integral over the trajectory $x(t)$. Therefore

$$\sum_j\left[\frac{(x_j-x_{j-1})^2}{4\epsilon^2} + V(x_j)\right]\epsilon \to \int_0^T d\tau\left[\frac{\dot{x}^2}{4} + V(x)\right].$$

Finally, we get the following version of the effective action:

$$\Gamma[V] = -\frac{1}{2}\int dx\int_0^\infty \frac{dT}{T}e^{-m^2T}\left[\mathcal{N}\int_{x(0)=x}^{x(T)=x}\mathcal{D}[x(t)]e^{-\int_0^T d\tau\left[\frac{\dot{x}^2}{4}+V(x)\right]} - (4\pi T)^{-\frac{D}{2}}\right]. \qquad (3.22)$$

The normalization constant, which we have introduced and absorbed all the constant factors generated in the computations, can be determined by taking the free field limit

$(V(x) \to 0)$. This gives us

$$\mathcal{N} \int \mathcal{D}[x(t)] e^{-\int_0^T d\tau \left[\frac{\dot{x}^2}{4}\right]} = (4\pi T)^{-\frac{D}{2}} . \tag{3.23}$$

Thus,

$$\Gamma[V] = -\frac{1}{2}(4\pi)^{-\frac{D}{2}} \int dx \int_0^\infty \frac{dT}{T^{D/2+1}} e^{-m^2 T} \left[\left\langle e^{-\int_0^T V(x) d\tau} \right\rangle - 1 \right] , \tag{3.24}$$

where we defined

$$\left\langle e^{-\int_0^T V(x) d\tau} \right\rangle := \frac{\int_{x(0)=x}^{x(T)=x} \mathcal{D}[x(t)] e^{-\int_0^T d\tau \left[\frac{\dot{x}^2}{4} + V(x)\right]}}{\int_{x(0)=x}^{x(T)=x} \mathcal{D}[x(t)] e^{-\int_0^T d\tau \left[\frac{\dot{x}^2}{4}\right]}} .$$

Finally, we can shift the loops and transform the spacetime integral over the initial and final points of the trajectories in a integration over the spacetime coordinates of the loops, yielding:

$$\Gamma[V] = -\frac{1}{2}(4\pi)^{-\frac{D}{2}} \int dx_{CM} \int_0^\infty \frac{dT}{T^{D/2+1}} e^{-m^2 T} \left[\left\langle e^{-\int_0^T V(x_{CM} + x(\tau)) d\tau} \right\rangle - 1 \right] , \tag{3.25}$$

where now we have $\int_0^T x_\mu(\tau) d\tau = 0$. We can see that the problem of finding the effective action of the quantum field theory defined in Eq. (3.4) is mapped to a one-dimensional field theory. In this framework we have the field x_μ in the one-dimensional space defined by the proper-time parameter.

The trajectory in spacetime parameterized by the proper-time parameter τ is the so-called worldline.

In summary, to have a representation of the effective action we must study the term $\left\langle e^{-\int_0^T V(x_{CM} + x(\tau)) d\tau} \right\rangle$. This term is interpreted as an expectation value of the factor $\exp\left(-\int_0^T V(x)\right)$ on the space of closed trajectories, with weights $\exp\left(-\int_0^T \frac{\dot{x}^2}{4}\right)$.

An important application of this method for scalar potentials is the modelling of the Casimir effect, as in (GIES; MOYAERTS, 2003). The boundaries in spacetime are interpreted as the external field.

3.2.2 Scalar field on external electromagnetic potential

For the case of a scalar field in a electromagnetic background, we consider, as we did for the scalar case, the Lagrangian density, but now we start with its formulation on Minkowski space-time, since there are more subtleties in the transformation. This is obtained via the minimal coupling $\partial_\mu \to D_\mu := \partial_\mu + iA_\mu$. The Lagrangian reads for a complex ϕ field:

$$\mathcal{L} = (D^\mu\phi)^* D_\mu\phi - m^2\phi^*\phi\,. \tag{3.26}$$

And therefore the action is given by:

$$S_\phi = \int d^4x (D^\mu\phi)^* D_\mu\phi - m^2\phi^*\phi\,. \tag{3.27}$$

We can work this out to a Gaussian form by expanding the D^μ into its definition, integrating by parts and disregarding for the surface term. Explicitly, it goes as:

$$S_\phi = \int d^4x (D^\mu\phi)^* D_\mu\phi - m^2\phi^*\phi = \int d^4x (\partial^\mu - iA^\mu)\phi^* D_\mu\phi - m^2\phi^*\phi\,. \tag{3.28}$$

As $\phi*$ and A^μ commute, integrating by parts we have:

$$\int d^4x (\partial^\mu - iA^\mu)\phi^* D_\mu\phi - m^2\phi^*\phi = \int d^4x \phi^* (-\partial^\mu - iA^\mu) D_\mu\phi - m^2\phi^*\phi\,,$$

which yields:

$$S_\phi = \int d^4x (D^\mu\phi)^* D_\mu\phi - m^2\phi^*\phi = -\int d^4x \phi^* (D^\mu D_\mu\phi + m^2)\phi\,. \tag{3.29}$$

Performing a Wick rotation and working on Euclidean time we get:

$$\int d^4x \to -i\int d^4x\,. \tag{3.30}$$

For the covariant derivatives it goes as:

$$D^\mu D_\mu = (\partial_t + iA_t)^2 - (\partial_i + iA_i)^2 = (i\partial_\tau + iA_t)^2 - (\partial_i + iA_i)^2\,,$$

and therefore

$$D^\mu D_\mu = -(\partial_\tau + A_t)^2 - (\partial_i + iA_i)^2\,. \tag{3.31}$$

Thus, defining

$$\begin{aligned} A^E_\tau &= -iA_t, \\ A^E_j &= A_j \quad j = 1,2,3\,, \end{aligned} \tag{3.32}$$

we get

$$D^\mu D_\mu = -(\partial_\tau + iA^E_\tau)^2 - (\partial_i + iA^E_i)^2 =: -\Box_A\,. \tag{3.33}$$

We defined the gauge field in Euclidean spacetime. From now on we drop the subscript E used to differ between Euclidean and Minkowskian space-times during the derivation and instead of τ, use the subscript 4.

In summary, we get that

$$\exp(iS) = \exp\left[-\int d^4x\phi^*(-\Box_A + m^2)\phi\right].$$

And therefore, the effective action reads:

$$\Gamma[A] = \frac{1}{2}\,\mathrm{Tr}\ln\left[\frac{-(\partial + iA)^2 + m^2}{-\partial^2 + m^2}\right]. \tag{3.34}$$

We follow now the same procedure as before, making use of the mathematical identity stated in Eq. (3.11), introducing a proper-time parameter.

Making the identification with the problem of the amplitude of a particle to loop through spacetime, we get:

$$\Gamma[V] = (4\pi)^{-\frac{D}{2}}\int dx_{CM}\int_0^\infty \frac{dT}{T^{D/2+1}}e^{-m^2T}\left[\left\langle e^{i\int_0^T A(x_{CM}+x(\tau))\cdot\dot{x}d\tau}\right\rangle - 1\right]. \tag{3.35}$$

3.2.3 Spinors on external electromagnetic potential

We turn ourselves now to the problem of a Dirac Spinor coupled to an Abelian gauge field A_μ. As we did before, we start introducing the Euclidean Lagrangian density of the underlying theory:

$$\mathcal{L} = \bar{\psi}[-i\not{D} + m]\psi\,, \tag{3.36}$$

where $\not{D} = \gamma^\mu D_\mu$.

The functional of the theory is therefore:

$$Z[A] = \int \mathcal{D}\bar{\psi}D\psi\exp\left(-\int dx\bar{\psi}[-\not{D} + m]\psi\right). \tag{3.37}$$

Due to the anti-commutative nature of the spinorial fields, the path integrals performed here have to incorporate this information. This is done by considering the fields ψ and $\bar{\psi}$ to be Grassmann valued fields.

Grassmann numbers are elements of a Grassmann or exterior algebra, which is the algebra whose generators obey anti-commuting relations, over a field that we will consider to be complex numbers. The generators of the algebra are called Grassmann variables.

Consider a Grassmann algebra $\mathcal{G}$ with N generators θ_i. We have by definition that

$$\{\theta_i, \theta_j\} = 0\,.$$

In particular, the generators are nilpotent of degree 2. This means that the most general elements of the algebra we can form have a factor or none of a generator, yielding a maximum of 2^N linear independent monomials. In particular, we conclude then that the generated Grassmann algebra has 2^N dimensions.

Our objective here is to define a way to do integrations with those variables so that we can make sense of the path integrals. This was done by Berezin (BEREZIN, 1966) by imposing that integrals over Grassmann numbers should retain some properties of the integrals of real numbers over the whole line. Namely, linearity and invariance to translations.

$$\int_{-\infty}^{\infty} dx[af(x) + bg(x)] = a\int_{-\infty}^{\infty} dxf(x) + b\int_{-\infty}^{\infty} dxg(x)$$

$$\int_{-\infty}^{\infty} dxf(x + x_0) = \int_{-\infty}^{\infty} dxf(x)$$

Based on this, we can start defining the integral over a Grassmann number x_i. Consider the most general function solely of this $_i$, which by the nilpotency of degree 2 discussed before is

$$f(x_i) = a + bx_i \quad a, b \in \mathbb{C}\,. \tag{3.38}$$

Due to the linearity:

$$\int dx_i f(x_i + y) = a\int dx_i + b\int dx_i x_i + by\int dx_i\,, \tag{3.39}$$

where $y \in \mathcal{G}$ is fixed. Therefore,

$$by\int dx_i = \int dx_i f(x_i + y) - \int dx_i f(x_i) = 0 \quad \forall b \in \mathbb{C} \quad \text{and} \quad y \in \mathcal{G}\,, \tag{3.40}$$

where the second equality comes from the translational invariance.

Based on this outcomes, Berezin defined that

$$\int dx_i = 0\,. \tag{3.41}$$

Then, the missing step is to define $\int dx_i x_i$. This should be a non-vanishing constant, otherwise all integrals would trivially be equal to zero. Berezin then sets the arbitrary value

$$\int dx_i x_i = 1\,. \tag{3.42}$$

Finally, we set that while doing a multivariate integral over multiple Grassmann variables we follow from the innermost to the outermost, it is:

$$\int d\theta_1 d\theta_2 f(\theta_1,\theta_2) = \int d\theta_1 \left(\int d\theta_2 f(\theta_1,\theta_2)\right) . \tag{3.43}$$

And also,

$$\int d\theta_1 d\theta_2 \theta_2 \theta_1 = -\int d\theta_1 d\theta_2 \theta_1 \theta_2 = 1 . \tag{3.44}$$

With this, we are able to solve Berezin integrals, in particular the ones of the Gaussian type we will be working with.

The exponential function, as other analytical functions, can be defined in general contexts in terms of its Taylor series. By doing so, we can define $e^{ba} \in \mathcal{G}$ as

$$e^{ba} = 1 + ba \quad b \in \mathbb{C}, \quad a \in \mathcal{G} ,$$

again as a consequence of the nilpotence.

From that, for fixed $i, j \in \mathbb{N}$ (not using Einstein's summation convention in the next equality):

$$\int d\theta_i d\theta_j e^{-\theta_i A_{ij} \theta_j} = \int d\theta_i d\theta_j [1 - A_{ij}\theta_i\theta_j] = (1-\delta_{ij})A_{ij} , \tag{3.45}$$

which generalizes in the case we are integrating over $2N$ generators, namely $\theta_1, \theta_2, ..., \theta_N$ and $\bar{\theta}_1, \bar{\theta}_2, ..., \bar{\theta}_N$ to:

$$\int d\bar{\theta}_1 d\theta_1 ... d\bar{\theta}_N d\theta_N e^{-\bar{\theta}_i A_{ij}\theta_j} = \int d\bar{\theta}_1 d\theta_1 ... d\bar{\theta}_N d\theta_N \left[1 - \bar{\theta}_i A_{ij}\theta_j + (\bar{\theta}_i A_{ij}\theta_j)(\bar{\theta}_k A_{kl}\theta_l) - ...\right] , \tag{3.46}$$

where all terms missing one of the generators will vanish and also the terms with order higher than one in any of the θ_i's or $\bar{\theta}_i$'s, the first because of the way we defined the integral, and the second because of the nilpotence.

Thus, the integral reduces to:

$$\int d\bar{\theta}_1 d\theta_1 ... d\bar{\theta}_N d\theta_N e^{-\bar{\theta}_i A_{ij}\theta_j} = \frac{(-1)^n}{n!} \int d\bar{\theta}_1 d\theta_1 ... d\bar{\theta}_N d\theta_N \sum_{\sigma_i} \sum_{\sigma_j} \prod_{k=1}^{n} \bar{\theta}_{\sigma_j(k)} A_{\sigma_j(k)\sigma_i(k)} \theta_{\sigma_i(k)} . \tag{3.47}$$

The term $\bar{\theta}_{\sigma_j(k)} A_{\sigma_j(k)\sigma_i(k)} \theta_{\sigma_i(k)}$ has a even degree on Grassmann variables, so they commute with each other. So, for each $_j$ we can reorder the product, yielding to $n!$ equal terms.

$$\int d\bar{\theta}_1 d\theta_1 ... d\bar{\theta}_N d\theta_N e^{-\bar{\theta}_i A_{ij}\theta_j} = \frac{(-1)^n}{n!} \int d\bar{\theta}_1 d\theta_1 ... d\bar{\theta}_N d\theta_N \sum_{\sigma_i} \left(n! \prod_{k=1}^{n} \bar{\theta}_k A_{k\sigma_i(k)} \theta_{\sigma_i(k)} \right) . \tag{3.48}$$

Now, we can reorder the $\bar{\theta}_i$'s and θ_i's to the proper order of integration:

$$\begin{aligned}\prod_{k=1}^{n} \bar{\theta}_k A_{k\sigma_i(k)} \theta_{\sigma_i(k)} &= \prod_{k=1}^{n} A_{k\sigma(k)} \prod_{k=1}^{n} \bar{\theta}_k \prod_{k=1}^{n} \theta_{\sigma(n+1-k)} = (-1)^{sign(\sigma_i)} \prod_{k=1}^{n} A_{k\sigma(k)} \prod_{k=1}^{n} \bar{\theta}_k \prod_{k=1}^{n} \theta_{n+1-k} \\ &= (-1)^{\mathrm{sign}(\sigma_i)} \prod_{k=1}^{n} A_{k\sigma(k)} \prod_{k=1}^{n} \bar{\theta}_k \theta_k = (-1)^n (-1)^{sign(\sigma_i)} \prod_{k=1}^{n} A_{k\sigma(k)} \prod_{k=1}^{n} \theta_k \bar{\theta}_k \,.\end{aligned} \tag{3.49}$$

Finally, as

$$\int d\bar{\theta}_1 d\theta_1 ... d\bar{\theta}_N d\theta_N \prod_{k=1}^{n} \theta_k \bar{\theta}_k = \int d\bar{\theta}_1 d\theta_1 ... d\bar{\theta}_N d\theta_N \prod_{k=1}^{n} \theta_{n+1-k} \bar{\theta}_{n+1-k} = 1 \,, \tag{3.50}$$

we have

$$\int d\bar{\theta}_1 d\theta_1 ... d\bar{\theta}_N d\theta_N e^{-\bar{\theta}_i A_{ij} \theta_j} = \sum_{\sigma_i} (-1)^{sign(\sigma_i)} \prod_{k=1}^{n} A_{k\sigma(k)} =: \det(A) \,. \tag{3.51}$$

Note that the final result is similar, but different from what we would get by performing a Gaussian integral over real or complex variables.

Thus, considering our particular problem of the generating functional of the Spinorial QED, we get:

$$e^{-\Gamma[A]} = \det\left(-i \not{D} + m\right) , \tag{3.52}$$

or, in its normalized version,

$$e^{-\Gamma[A]} = \frac{\det\left(-i \not{D} + m\right)}{\det\left(-i \not{\partial} + m\right)} \,. \tag{3.53}$$

Therefore, for the Spinorial case, we obtain an effective action expression similar to the one obtained for the scalar one:

$$\Gamma[A] = -\operatorname{Tr}\ln\left(\frac{-i \not{D} + m}{-i \not{\partial} + m}\right) = -\operatorname{Tr}\ln\left(\frac{\not{D}^2 + m}{-\not{\partial}^2 + m^2}\right) , \tag{3.54}$$

where we define $\not{D}^2 = -\frac{1}{2}(\partial + iA)^2 + \sigma_{\mu\nu} F^{\mu\nu}$.

We can follow then as we did for the scalar QED case, the only difference is that in the worldline Lagrangian we get an extra term, the so-called spin factor and will change sign, which as we saw is due to the Gaussian integration using Grassmann numbers.

$$\begin{aligned}\Gamma[A] &= -(4\pi)^{-\frac{D}{2}} \int dx_{CM} \int_0^\infty \frac{dT}{T^{D/2+1}} e^{-m^2 T} \\ &\times \left[\left\langle e^{i \int_0^T A(x_{CM}+x(\tau))\cdot\dot{x} d\tau} \frac{1}{2} \operatorname{Tr} P_T e^{\frac{1}{2}\int_0^T d\tau \sigma_{\mu\nu} F^{\mu\nu}(x_{CM}+x(\tau))} \right\rangle - 1 \right].\end{aligned} \tag{3.55}$$

3.3 Worldline Instantons

3.3.1 Review

The worldline instantons approach to the worldline formalism relies on the fact that in many occasions it is not feasible to solve the path integral analytically. If the form of the potential makes the integrand too complicate, we can try to find a semiclassical solution.

Consider the following path integral representing the amplitude of a particle to evolve from the space eigenstate $|x_i\rangle$ at t_i to $|x_f\rangle$ in t_f in the framework of non-relativistic quantum mechanics.

$$\langle x_f| e^{-iH(t_f-t_i)} |x_i\rangle = \int_{x'(t_i)=x_i}^{x'(t_f)=x_f} \mathcal{D}[x'(t)] e^{\frac{i}{\hbar} S[x'(t), \dot{x}'(t)]} . \tag{3.56}$$

Instead of trying to solve it exactly, one might consider an expansion of $S[x(t)]$ centered in the classical trajectory, which is given by the principle of least action. Let's call $x_c(t)$ the classical solution and work with $y(t)$ defined as $x(t) = x_c(t) + y(t)$.

The Taylor expansion of the action is given by the expression:

$$S[x_c(t)+y(t)] = S[x_c(t)] + \int dt_1 \frac{\delta S}{\delta x(t_1)}\bigg|_{x=x_c} y(t_1) + \frac{1}{2}\int dt_1 dt_2 \frac{\delta^2 S}{\delta x(t_1)\delta x(t_2)}\bigg|_{x=x_c} y(t_1)y(t_2) + \mathcal{O}(y^3) \tag{3.57}$$

The linear term in $y(t)$ vanishes as it is reduced to the Euler-Lagrange equations evaluated on $x_c(t)$.

We then obtain the following equation:

$$\langle x_f| e^{-iH(t_f-t_i)} |x_i\rangle = \int_{x'(t_i)=x_i}^{x'(t_f)=x_f} \mathcal{D}[y(t)] e^{\frac{i}{\hbar}(S[x_c(t)] + \frac{\delta^2 S}{\delta x^2} y^2 + \mathcal{O}(y^3))} . \tag{3.58}$$

We can now consider a change of variables consisting of a simple rescaling $y(t) = \sqrt{\hbar}\tilde{y}$.

$$\langle x_f| e^{-iH(t_f-t_i)} |x_i\rangle = \int_{x'(t_i)=x_i}^{x'(t_f)=x_f} \mathcal{D}[y(t)] e^{i(S[x_c(t)] + \frac{\delta^2 S}{\delta x^2} y^2 + \mathcal{O}(\sqrt{\hbar} y^3))} . \tag{3.59}$$

Thus, in the small $\hbar$ limit, we can approximate the expression by:

$$\langle x_f| \, e^{-iH(t_f-t_i)} \, |x_i\rangle = \int_{x'(t_i)=x_i}^{x'(t_f)=x_f} \mathcal{D}[y(t)] e^{i(S[x_c(t)]+\frac{\delta^2 S}{\delta x^2} y^2)} [1+\mathcal{O}(\sqrt{\hbar})] \, . \tag{3.60}$$

Therefore, the transition amplitude can be approximated in the semi-classical limit as a normalization constant times the complex exponential of the action evaluated on the classical trajectory times a Gaussian integral on the perturbations of the classical trajectory.

This is the essence of the Worldline instantons method for solving the worldline path integral. In case the classical solution obey the boundary conditions we can proceed as above. In this case, we should be careful in considering all the possible solutions to the classical equations of motion and the quantum perturbations on the vicinity of them.

In fact, the integral we intend to approximate is a slightly modification of the aforementioned result. Namely,

$$I = \int_{x'(t_i)=x_i}^{x'(t_f)=x_f} \mathcal{D}[x'(t)] H[x(t)] e^{-\frac{S[x'(t),\dot{x}'(t)]}{\hbar}} \, , \tag{3.61}$$

where $H[x(t)]$ is a functional of the trajectory $x(t)$ and the exponent is in the form we face it when dealing with Euclidean space-time.

We again proceed by approximating it as a Gaussian integral and assume $H[x(t)]$ to vary slowly in the significant support of the exponential, allowing us to take it to outside the integration as the constant $H[x_{cl}]$. This gives as a total contribution of:

$$I \approx H[x_{cl}] e^{-\frac{S[x_cl]}{\hbar}} \frac{1}{\sqrt{\det\left(\frac{\delta^2 S}{\delta x^2}\right)}} \, . \tag{3.62}$$

This is the path integral version of a well-known asymptotic approximation method called Laplace's Method. In the following we will use this method to derive the effective action of spinorial QED in the case of a constant electromagnetic background, and as a consequence, the pair production rate, which for this case we will show can be described exactly with no higher order corrections.

3.3.2 Application: pair production rate on spinorial QED

In Ref. (GORDON; SEMENOFF, 2015) the Worldline instanton method was applied for the scalar QED in a constant electric background. Here we extent their analysis to spinorial QED.

Consider the effective action of spinorial QED in its unrenormalized form.

$$\Gamma[A_\mu] = -\int_0^\infty \frac{dT}{T} \int_{x(0)=x(T)} \mathcal{D}[x(t)]\Phi[x] e^{-\int_0^1 d\tau \left[\frac{T\dot{x}^2(t)}{4} + ie\dot{x}\cdot A + \frac{m^2}{T}\right]}, \tag{3.63}$$

where the spin factor $\Phi[x]$ is

$$\Phi[x] = \frac{1}{2} \operatorname{Tr} \mathcal{P} \exp\left\{\frac{i}{4} e \int_0^T d\tau \sigma_{\mu\nu} F^{\mu\nu}(x(\tau))\right\}. \tag{3.64}$$

We shall consider now the particular case of a constant non-vanishing electromagnetic field. We consider E and B both pointing towards the z-direction, as this configuration can be obtained from a general configuration through a Lorentz boost.

Let $\mathcal{S}$ be the action of the worldline particle in Euclidean time:

$$S = \int_0^1 d\tau \left[\frac{T}{4}\dot{x}^2 + i\dot{x}\cdot A\right] + \frac{m^2}{T^2}. \tag{3.65}$$

Here, the scalar product is the usual Euclidean one performed in four dimensions.

We shall begin by choosing the gauge. In this calculation we are going to work in the Fock-Schwinger gauge. In this gauge, it is imposed the following relation for A_μ:

$$x_\mu A^\mu = 0. \tag{3.66}$$

Since here a constant electromagnetic field is assumed, and thus a constant electromagnetic tensor, this condition is translated into:

$$A_\mu = \frac{1}{2} x^\nu F_{\nu\mu}, \tag{3.67}$$

so that, we have the following worldline action.

$$S = \int_0^1 d\tau \left[\frac{T}{4}\dot{x}^2 + \frac{ie}{2}\dot{x^\mu} x^\nu F_{\nu\mu}\right] + \frac{m^2}{T^2}. \tag{3.68}$$

We consider the dynamical variables of this action to be x_μ and T. Now, we impose that the variation of the action under these variables vanish in order to find the classical equations of motion. By doing so, one reaches to the following equations:

$$\begin{aligned} \frac{T}{4}\ddot{x}_\rho + \frac{ie}{2}\dot{x^\nu} F_{\nu\mu}\delta^\mu_\rho - \frac{ie}{2}\dot{x^\mu} F_{\rho\mu}\delta^\nu_\rho &= 0, \\ \int_0^1 \frac{(\dot{x_\mu})^2}{4} - \frac{m^2}{T^2} &= 0. \end{aligned} \tag{3.69}$$

Consider now the electromagnetic tensor for this field configuration:

$$F_{\mu\nu} = \begin{bmatrix} 0 & B & 0 & 0 \\ -B & 0 & 0 & 0 \\ 0 & 0 & 0 & iE \\ 0 & 0 & -iE & 0 \end{bmatrix} . \tag{3.70}$$

This leads us to the following set of differential equations:

$$\frac{T}{2}\ddot{x}_1 - ieB\dot{x}_2 = 0 \,, \tag{3.71}$$

$$\frac{T}{2}\ddot{x}_2 + ieB\dot{x}_1 = 0 \,, \tag{3.72}$$

$$\frac{T}{2}\ddot{x}_3 + eE\dot{x}_4 = 0 \,, \tag{3.73}$$

$$\frac{T}{2}\ddot{x}_4 - eE\dot{x}_3 = 0 \,. \tag{3.74}$$

The first two equations lead to two independent set of solutions which are not periodic, so that they cannot obey the boundary conditions. This make us conclude that:

$$x_{1,2}(\tau) = 0 \,. \tag{3.75}$$

The other two equations have a set of periodic solutions.

$$x_3(\tau) = \frac{A\,T}{2eE}\cos\left(\frac{2eE}{T}\tau\right) \,, \tag{3.76}$$

$$x_4(\tau) = \frac{A\,T}{2eE}\sin\left(\frac{2eE}{T}\tau\right) \,, \tag{3.77}$$

where A is a constant. Given the periodic boundary conditions we conclude that:

$$\frac{2eE}{T} = 2\pi n \Rightarrow T = \frac{eE}{n\pi} \quad , \quad n \in \mathbf{Z}^* \,. \tag{3.78}$$

Using this result on Eq. (3.69), we can determine the value of the constant A:

$$A = \frac{2m}{T} = \frac{2n\pi}{eE} \,. \tag{3.79}$$

So, the classical solutions are:

$$x_{1,2} = 0 \,, \tag{3.80}$$

$$x_3(\tau) = \frac{m}{eE}\cos\left(2\pi n\tau\right) \,, \tag{3.81}$$

$$x_4(\tau) = \frac{m}{eE}\sin\left(2\pi n\tau\right) \,, \tag{3.82}$$

$$T = \frac{eE}{\pi n} \,. \tag{3.83}$$

Such solutions correspond to circular trajectories on the Euclidean spacetime and due to their finite action and localized trajectories, they are the so called worldline instantons.

Substituting this set of solutions on the worldline action we get:

$$S_{cl} = \frac{\pi n m^2}{eE} \,. \tag{3.84}$$

This is the exact exponent that appears in the rate of pair production of the Schwinger effect in the case of a constant electric field as we deduced in a previous chapter.

Our next step is to perform a semiclassical calculation of the path integral. For that, we will perform a transformation of variables such that our dynamical variables will be replaced by the instanton solution plus perturbations around it.

$$x_\mu \to x_{0,\mu} + \delta x_\mu, \qquad T \to T_0 + \delta T \,, \tag{3.85}$$

where the subscript 0 indicates a classical instanton solution. Consider as well the expansion of the perturbations of the spatial coordinates in trigonometric functions as follows:

$$\delta x_\mu = x_\mu + \sum_{k=1}^{\infty} [\sqrt{2}\cos(2\pi k\tau) a_{k\mu} + \sqrt{2}\sin(2\pi k\tau) b_{k\mu}] \,, \tag{3.86}$$

with coefficients $a_{k\mu}$, $b_{k\mu}$ and where this new x_μ stands for the zero mode of the expansion, and not for the spatial coordinate as a whole. This particular set of orthonormal functions was chosen due to the periodic boundary conditions of the problem.

Substituting that on the action, we get:

$$\begin{aligned} S \;=\; & \frac{\pi n m^2}{eE} + \frac{m^2\pi^3 n^3}{e^3E^3}\delta T^2 + \frac{4\pi^2 n^2 m}{2eE}\delta T\left(\frac{a_{n3}+b_{n4}}{\sqrt{2}}\right) + 2\pi neE\left(\frac{a_{n3}-b_{n4}}{\sqrt{2}}\right)^2 \\ & + 2\pi neE\left(\frac{a_{n4}+b_{n3}}{\sqrt{2}}\right)^2 + \frac{E}{4\pi n}\sum_{k=1,k\neq n,\mu=3,4}^{\infty}\left[(a_{k\mu}^2+b_{k\mu}^2) - \frac{2n}{k}(a_{k3}b_{k4}-a_{k3}b_{k4})\right] \\ & + \sum_{k=1,\mu=1,2}^{\infty} T\pi^2k^2[(a_{k\mu})^2+(b_{k\mu})^2] + \sum_{k=1,\mu=1,2}^{\infty}(2k\pi ieB)(a_{k1}b_{k2}-a_{k2}b_{k1}) \\ & + \delta T\sum_k \pi^2k^2(a_{k\mu}^2+b_{k\mu}^2) + \sum_{k=3}^{\infty} m^2\left(\frac{\pi n}{eE}\right)^{k+1}(-\delta T)^k \,. \end{aligned} \tag{3.87}$$

The action obtained in Eq. (3.87) resembles the action obtained by Gordon and Semenoff in Ref. (GORDON; SEMENOFF, 2015), with the addition of the two extra terms present on the third line of (3.87). Those are the terms coupled to the magnetic field and the integration on those degrees of freedom that will provide a modification to the pair production rate compared to the pure electric case.

The last line of Eq. (3.87) contains only terms of order higher than two on the fluctuations, so that will be neglected. Now, we shall proceed to the calculation of the Gaussian path integrals. It is important to note that the action doesn't depend on the mode $(a_{n4} - b_{n3})$. That is related to the invariance of the action on translations of the parameter τ, while the instantons solutions do depend on such transformations. As a result, we have the appearance of a zero mode on the fluctuation in the same fashion that occurred when discussing the quantum tunneling on the double well potential.

We shall handle this problem by making use of the Faddev-Popov method. Considering the following resolution of the identity for a function $g(t)$ with ω roots in the domain of integration $[0,1]$:

$$1 = \frac{1}{\omega}\int_0^1 dt\delta(g(t))\left|\frac{d}{dt}g(t)\right|. \tag{3.88}$$

The function $g(t)$ will be chosen such that the integration over the the zero mode will be well defined. Consider the following:

$$g(t) = \int_0^1 d\tau[\sin(2\pi n\tau)x_3(\tau - t) - \cos(2\pi n\tau)x_4(\tau - t)]. \tag{3.89}$$

Separating the x_3 and x_4 in one instanton solution and the fluctuations, expanding the fluctuations as (3.86) and then calculating the integral:

$$\begin{aligned} g(t) &= \frac{1}{\sqrt{2}}\left[\left(\frac{m}{\sqrt{2}eE} + a_{n3}\right)\sin(2\pi nt) + b_{n3}\cos(2\pi nt) - a_{n4}\cos(2\pi nt)\right. \\ &\quad \left. - \left(\frac{m}{\sqrt{2}eE} + b_{n4}\right)\sin(2\pi nt)\right]. \end{aligned} \tag{3.90}$$

The roots of $g(t)$ as defined above are such:

$$\tan(2\pi nt) = \frac{a_{n4} - b_{n3}}{\frac{\sqrt{2}m}{eE} + a_{n4} + b_{n3}}, \tag{3.91}$$

thus, since the period of this function is $\frac{1}{2n}$ and it has one root per period we set $\omega = 2n$.

Finally, the Jacobian of $|g'(t)|$ reads:

$$|g'(t)| = 2\pi n\int_0^1 d\tau\left[\cos(2\pi n\tau)x_3(\tau) + \sin(2\pi n\tau)x_4(\tau)\right] = 2n\left|\pi\frac{m}{eE} + \pi\frac{a_{4n} + b_{n3}}{\sqrt{2}}\right|. \tag{3.92}$$

Therefore, the contribution of the whole procedure is:

$$\delta\left(\frac{a_{n4}-b_{n3}}{\sqrt{2}}\right)\left|\pi\frac{m}{eE}+\pi\frac{a_{3n}+b_{n4}}{\sqrt{2}}\right| . \tag{3.93}$$

The delta function suppresses the integration over the zero mode. So, in leading order, disregarding for the fluctuations dependent term, the contribution of the Faddeev-Popov method is:

$$\pi\frac{m}{eE} . \tag{3.94}$$

Now that we got rid of the zero mode, we are allowed to perform the gaussian integration. We shall begin by integrating the degrees of freedom associated with the same frequency as the instanton solution and as well δT, that is coupled to them.

These are the following degrees of freedom:

$$\left[\delta T, \left(\frac{a_{n3}+b_{n2}}{\sqrt{2}}\right), \left(\frac{a_{n3}-b_{n2}}{\sqrt{2}}\right), \left(\frac{a_{n4}-b_{n3}}{\sqrt{2}}\right)\right] . \tag{3.95}$$

The matrix associated with the quadratic form is:

$$A=\begin{bmatrix} \frac{2m^2\pi^3n^3}{e^3E^3} & \frac{2\pi^2n^2m}{eE} & 0 & 0 \\ \frac{2\pi^2n^2m}{eE} & 0 & 0 & 0 \\ 0 & 0 & 4\pi neE & 0 \\ 0 & 0 & 0 & 4\pi neE \end{bmatrix} . \tag{3.96}$$

Using the well known result for multi-dimensional Gaussian integration, the contribution of the integration over these modes is:

$$\frac{(2\pi)^2}{[\det A]^{\frac{1}{2}}}=\frac{i}{2\pi n^3 m} . \tag{3.97}$$

Next, we proceed integrating on the other degrees of freedom. There will be two major contributions, one from the components coupled to the electric field only and other to the components also coupled to the magnetic field.

We find the following matrices for the quadratic forms respectively:

$$M=\begin{bmatrix} 1 & 0 & 0 & -\frac{n}{k} \\ 0 & 1 & \frac{n}{k} & 0 \\ 0 & \frac{n}{k} & 1 & 0 \\ -\frac{n}{k} & 0 & 0 & 1 \end{bmatrix} , \tag{3.98}$$

$$N = \begin{bmatrix} 1 & 0 & 0 & \frac{iB}{E}\frac{n}{k} \\ 0 & 1 & -\frac{iB}{E}\frac{n}{k} & 0 \\ 0 & -\frac{iB}{E}\frac{n}{k} & 1 & 0 \\ \frac{iB}{E}\frac{n}{k} & 0 & 0 & 1 \end{bmatrix} . \tag{3.99}$$

For the components associated to the first matrix, we should remember that we have already integrated over the components of the frequency of the instanton. So, its contribution is:

$$\prod_{k=1,k\neq n}^{\infty} \left((2\pi)\frac{2\pi n}{eE} \right)^2 (2\pi k)^{-4} \frac{1}{(detM)^{\frac{1}{2}}}, . \tag{3.100}$$

Analogously, for the components of matrix N:

$$\prod_{k=1}^{\infty} \left((2\pi)\frac{2\pi n}{eE} \right)^2 (2\pi k)^{-4} \frac{1}{(detN)^{\frac{1}{2}}} . \tag{3.101}$$

Here, we define the infinite products using the Zeta function regularization. The net contribution is:

$$\left(2\pi \frac{2\pi n}{E} \right)^{4\zeta(0)-2} (2\pi n)^4 \left(\prod_{k=1,k\neq n} \frac{1}{1-\frac{n^2}{k^2}} \right) \left(\prod_{k=1} \frac{1}{1+\frac{B^2}{E^2}\frac{n^2}{k^2}} \right) . \tag{3.102}$$

The productory $\left(\prod_{k=1,k\neq n} \frac{1}{1-\frac{n^2}{k^2}} \right)$ reads:

$$\left(\prod_{k=1,k\neq n} \frac{1}{1-\frac{n^2}{k^2}} \right) = \lim_{\alpha\to n} \left(\prod_{k=1} \frac{1-\frac{\alpha^2}{n^2}}{1-\frac{\alpha^2}{k^2}} \right), . \tag{3.103}$$

Then, using the identity for the sinusoidal function:

$$\sin(\pi\alpha) = \pi\alpha \prod_{k=1}^{\infty} \left(1 - \frac{\alpha^2}{k^2} \right) , \tag{3.104}$$

it yields us to

$$\left(\prod_{k=1,k\neq n} \frac{1}{1-\frac{n^2}{k^2}} \right) = \lim_{\alpha\to n} \frac{\pi\alpha \left(1 - \frac{\alpha^2}{n^2} \right)}{\sin(\pi\alpha)} = 2(-1)^{n+1} . \tag{3.105}$$

For the productory involving B, the calculation is more straightforward. The following identity is directly applied:

$$\sinh\left(\pi n \frac{B}{E} \right) = \pi n \frac{B}{E} \prod_{k=1}^{\infty} \left(1 + \frac{B^2 n^2}{E^2 k^2} \right) . \tag{3.106}$$

Thus, the contribution is finally:

$$2(-1)^{n+1}\frac{E^4}{16\pi^4}\frac{\pi n\frac{B}{E}}{\sinh\left(\pi n\frac{B}{E}\right)}\,. \tag{3.107}$$

The total contribution so far is given by the product of this value, the Faddeev-Popov determinant, the term $\frac{1}{T}$ and the integral over the modes with same frequency as the instanton.

This is:

$$i\frac{E^2}{16\pi^3 n^2}(-1)^{n+1}\frac{\pi n\frac{B}{E}}{\sinh\left(\pi n\frac{B}{E}\right)}\,. \tag{3.108}$$

This is the "scalar part", the same prefactor we obtained for a scalar field. The missing part in our calculation is the spin factor contribution and the global sign that appears in the effective action.

Consider in the following the Dirac matrices to be in the Chiral representation:

$$\gamma^i = \begin{bmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{bmatrix} \quad , \quad \gamma^4 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}, \tag{3.109}$$

in which the index i runs from 1 to 3, the space coordinates, the σ^i's are the Pauli matrices and I_2 is the identity matrix of rank 2.

For such representations we have that $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ is:

$$\sigma^{\mu\nu} = i\varepsilon^{\mu\nu\rho}\begin{bmatrix} \sigma^\rho & 0 \\ 0 & \sigma^\rho \end{bmatrix} \quad 1 \leq \nu, \mu \leq 3\,. \tag{3.110}$$

and,

$$\sigma^{\mu 4} = 2i(1-\delta^{\mu 4})\begin{bmatrix} \sigma^\mu & 0 \\ 0 & -\sigma^\mu \end{bmatrix} = -\sigma^{4\mu} \quad 1 \leq \mu \leq 4\,. \tag{3.111}$$

Therefore,

$$\sigma^{\mu\nu}F_{\mu\nu} = 2Bi\begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix} - 2E\begin{bmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{bmatrix} \tag{3.112}$$

That's a constant term and we can trivially integrate it, thus the spin factor takes the form:

$$\Phi[x] = \frac{1}{2}\,\mathrm{Tr}\,\mathcal{P}\exp\left\{\frac{eT}{2}\mathbf{diag}[-B-iE, B+iE, -B+iE, B-iE]\right\}. \tag{3.113}$$

Since that is the exponentiation of a diagonal matrix, we have:

$$\Phi[x] = \frac{1}{2}\,\mathrm{Tr}\,\mathcal{P}\mathbf{diag}[e^{\frac{eT}{2}(-B-iE)}, e^{\frac{eT}{2}(B+iE)}, e^{\frac{eT}{2}(-B+iE)}, e^{\frac{eT}{2}(B-iE)}]\,. \tag{3.114}$$

Taking the trace:

$$\Phi[x] = \frac{1}{2}\left(e^{\frac{eT}{2}(-B-iE)} + e^{\frac{eT}{2}(B+iE)} + e^{\frac{eT}{2}(-B+iE)} + e^{\frac{eT}{2}(B-iE)}\right) . \tag{3.115}$$

Grouping the terms and factorizing we have:

$$\Phi[x] = \frac{1}{2}\left(e^{\frac{-eTB}{2}}(e^{\frac{-eTiE}{2}} + e^{\frac{eTiE}{2}}) + e^{\frac{eTB}{2}}(e^{\frac{eTiE}{2}} + e^{\frac{-eTiE}{2}})\right) = \frac{(e^{\frac{-eTiE}{2}} + e^{\frac{eTiE}{2}})}{2}\left(e^{\frac{-eTB}{2}} + e^{\frac{eTB}{2}}\right) . \tag{3.116}$$

Finally, considering the following identities:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad , \quad \cosh(x) = \frac{e^{x} + e^{-x}}{2} . \tag{3.117}$$

We get the following result

$$\Phi[x] = 2\cos\left(\frac{eET}{2}\right)\cosh\left(\frac{eBT}{2}\right) . \tag{3.118}$$

However, when solving the classical equations to reach to the instantons solutions we got the classical values for the variable $T = \frac{2\pi n}{eE}$, with n corresponding to the winding number of the instanton.

So, for each instanton the spin factor contribution assumes its final form as:

$$\Phi[x] = 2(-1)^n \cosh\left(en\pi\frac{B}{E}\right), \tag{3.119}$$

and the n-th instanton contribution, accounting for all factors is:

$$i\frac{e^2E^2}{8\pi^3n^2}\pi n\frac{B}{E}\coth\left(n\pi\frac{B}{E}\right) . \tag{3.120}$$

Thus,

$$\gamma = 2\,\mathrm{Im}\{\Gamma\} = \frac{(eE)(eB)}{(2\pi)^2}\sum_{i=1}^{\infty}\frac{1}{n}\coth\left(n\pi\frac{B}{E}\right)\exp\left(-\frac{n\pi m^2}{eE}\right), \tag{3.121}$$

and the probability to the vacuum decay through pair production is

$$P = 1 - e^{-\gamma} \approx \gamma . \tag{3.122}$$

The last approximation comes from the small γ regime, which is the case of interest most of the times, as a consequence of the Schwinger limit being so intense as discussed in the last section of chapter 2.

It is interesting to note that this expression is a good way to visualize the non-linear behavior of QED in this regime. A pure magnetic background leads to a stable vacuum, on the other hand, a pure electric background is unstable and we have a non-vanishing pair production rate. It is interesting, however, to see that in the case we have the presence of both fields, in the discussed configuration, the magnetic field modifies the production rate, playing a role in the vacuum decay mechanism.

3.4 Worldline Numerics

Analytical methods are powerful in the sense that they convey us a complete description of the modelled system. However, their applicability is restrict to scenarios where we can, at least approximately, explicitly find these solutions. For more complex systems numerical computations are very important tools.

A numerical approach can give us the ability to study systems for which there are no known analytical solutions or the calculations are unbearable to perform analytically, giving us physical insight on such systems.

Besides the insight, it also provides us quantitative estimates for different scenarios systematically.

In this section, we review a numerical approach to the worldline method, so-called Worldline Numerics, Worldline Monte-Carlo or Loop Cloud Method. In this method, we calculate the amplitude of temporal evolution on the propertime parameter by generating a finite ensemble of worldlines that are representative of the whole space of possible trajectories, the loops. After calculating the loop average, we perform the proper-time integration and retrieve the effective action density, or effective lagrangian.

3.4.1 Review

Consider here the case of a scalar field subject to a external scalar potential $V(x)$. As we showed before, the worldline representation of the effective action is

$$\Gamma[V] = -\frac{1}{2}(4\pi)^{-\frac{D}{2}} \int dx_{CM} \int_0^\infty \frac{dT}{T^{D/2+1}} e^{-m^2T} \left[\left\langle e^{-\int_0^T V(x_{CM}+x(\tau))d\tau} \right\rangle - 1 \right] . \qquad (3.123)$$

The Wilson Loop is obtained by solving the worldline path integral over closed loops. In the case we can't perform it analytically, one has to find a way to compute it numerically on a finite computational time. Given this, we know it is impossible to consider the whole space of closed loops, since it is unaccountably infinite, nor to consider an arbitrarily large ensemble.

The idea then is to generate a finite ensemble of such loops that can approximately describe the originally infinite ensemble effectively. We do this by means of a strategy called importance sampling, in which we reproduce in our finite ensemble the relative importance of different loops in the original infinite one.

Let's look again to our Wilson loops average term:

$$\left\langle e^{-\int_0^T V(x)d\tau} \right\rangle := \frac{\int_{x(0)=x}^{x(T)=x} \mathcal{D}[x(t)] e^{-\int_0^T d\tau \left[\frac{\dot{x}^2}{4} + V(x)\right]}}{\int_{x(0)=x}^{x(T)=x} \mathcal{D}[x(t)] e^{-\int_0^T d\tau \left[\frac{\dot{x}^2}{4}\right]}} .$$

It can be interpreted as an average of the values $e^{-\int_0^T V(x)d\tau}$ calculated for each loop, with the factor $e^{-\int_0^T \frac{\dot{x}^2}{4} d\tau}$ playing the role of a weighting factor.

Therefore, if we generate an ensemble using the probability density function

$$p(T, x(\tau)) = \frac{e^{-\int_0^T \frac{\dot{x}^2}{4} d\tau}}{\int_{x(0)=x}^{x(T)=x} \mathcal{D}[x(t)] e^{-\int_0^T d\tau \left[\frac{\dot{x}^2}{4}\right]}} , \tag{3.124}$$

where the denominator is just a normalization function, for sufficiently large samples, the finite ensemble distribution will be a good approximation of Eq. (3.124).

In this regime, we can then approximate the Wilson Loops average by:

$$\left\langle e^{-\int_0^T V(x)d\tau} \right\rangle = \frac{1}{N_l} \sum_{i=1}^{N_l} e^{-\int_0^T V(x_i)d\tau} . \tag{3.125}$$

So, all one has to do is to generate this loop ensemble containing N_l loops and average their contributions. It is important to notice that in the current framework the distribution density of the ensemble depends on the propertime parameter T, which means that one must generate a loop ensemble for each propertime T. So, if for a given numerical integration algorithm over T one needs to compute the integrand for k different T values, then we must generate k different ensembles, one for each different T.

In reality, we can overcome this limitation of the method by performing a rescale of the loops. Consider the following change of variables:

$$t := \frac{\tau}{T} \in [0, 1] \quad y(t) := \frac{x(tT)}{\sqrt{T}} . \tag{3.126}$$

By doing this, we have:

$$\int_0^T d\tau \left(\frac{d}{d\tau} x(\tau)\right)^2 = \int_0^1 T dt \left(\frac{dt}{d\tau}\frac{d}{dt} x(tT)\right)^2 = \int_0^1 dt \left[\frac{d}{dt}\left(\frac{1}{\sqrt{T}} x(tT)\right)\right]^2$$

$$\therefore \int_0^T d\tau \left(\frac{d}{d\tau} x(\tau)\right)^2 = \int_0^1 dt \dot{y}^2 . \tag{3.127}$$

Thus, if instead working directly with the desired loop ensemble, we work with the loops $y(t)$, it is only required that we generate this ensemble of loops and, after that, we transform the y-loops into x-loops back using the desired parameter T.

By doing so, we can work with the very same loop ensembles for the different T's just needing to generate them once and then redefine as

$$x(\tau) = \sqrt{T} y \left(\frac{\tau}{T} \right) . \tag{3.128}$$

Also, as discussed before, we can perform the integration over spacetime as an integration over the center of mass coordinates. So, we generate originally the loops all with a same arbitrary center of mass, which we set to zero by simplicity, and then, when integrating over the center of mass, just shift them accordingly.

3.4.2 Loop cloud algorithms

Now that we have a description of what is the goal of the method, we discuss an algorithm for generating the unitary loop ensemble with center of mass at the origin. The algorithm to be described is called d-loops algorithm and was first described by (GIES *et al.*, 2005).

Before doing so, let us quickly define what we mean by a loop here. For sure we cannot define continuous loops by each of its points, because that would leave us to represent each loop by an infinite amount of points. Instead of it, we discretize the loops defining them by a finite amount of points. So, a loop is the continuous path that passes by the points $x(0), x(\tau_1), x(\tau_2), ...x(\tau_{N_{ppl}})$ and connects them by straight lines, where

$$0 < \tau_1 < \tau_2 < ... < \tau_{N_{ppl}-1} < \tau_{N_{ppl}} = T$$

with $x(0) = x(T) = 0$.

Given this context, we can then rewrite the probability density from which we want to generate our loops in its discrete form

$$p(y_0, y_1, ..., y_{N_{ppl}}) = \mathcal{N} \delta(y_0 + y_1 + ... + y_{Nppl-1}) \exp\left[-\frac{N_{ppl}}{4} \sum (y_i - y_{i-1})^2 \right] , \tag{3.129}$$

where $\{y_i\}$ is the set of points that represent the loop and

$$1/\mathcal{N} = \int dy_0 dy_1 ... dy_{N_{ppl}-1} \exp\left[-\frac{N_{ppl}}{4} \sum (y_i - y_{i-1})^2 \right]$$

is the normalization constant. The delta function tells us that we are only considering loops with center of mass at the orgin and we don't integrate over y_N since it is fixed equal to y_0.

Now, consider the exponent in the expression (3.129), by rearranging it for y_i we get:

$$(y_{i+1} - y_i)^2 + (y_i - y_{i-1})^2 = 2\left(y_i - \left(\frac{y_{i+1} - y_{i-1}}{2}\right)\right)^2 + \frac{y_{i+1}^2}{2} + \frac{y_{i-1}^2}{2} \tag{3.130}$$

We see that the probability distribution of y_i depends then solely on the mean position of its neighbours, following a Gaussian distribution.

We can consider then another set of N_{ppl} points $\{z_i\}$ generated in between, in the sense that

$$p(z_i, z_{i+1}|y_{i-1}, y_i, y_{i+1}) = e^{-\frac{2N_{ppl}}{4}\left[2\left(z_i - \frac{y_i - y_{i-1}}{2}\right)^2 + \left(z_{i+1} - \frac{y_{i+1} - y_i}{2}\right)^2\right]} \tag{3.131}$$

We can verify then that the probability density of the loop of $2N_{ppl}$ points defined as

$$y'_{2i} = y_i$$

$$y'_{2i+1} = z_i$$

is indeed,

$$p(y') = \mathcal{N}\delta(y'_0 + y'_1 + ... + y'_{2Nppl-1}) \exp\left[-\frac{2N_{ppl}}{4}\sum(y'_i - y'_{i-1})^2\right] . \tag{3.132}$$

In conclusion, we manage to create a loop with the same distribution that we want with the double of the number of points. Therefore, we can construct each loop in a iterative fashion. We start from an arbitrary point, let's say the origin, and define $y_0 = y_{N_0} = 0$, $N_0 = 1$. Although this is defined by two points, this is a 1-point loop.

In the next step, we generate 2-points closed loops by creating a new point between any two consecutive points, in this case, between y_0 and y_1. We generate this point in $\mathbb{R}^D$ with the following distribution

$$p(z) = e^{-\frac{1}{4}z^2} . \tag{3.133}$$

Then, we redefine the loop as: $y' = (y_0, z, y_1)$.

Now we can proceed doubling the number of points of the $(k-1)$-th step $N_{k-1} = 2^{k-1}$ generating points with the distribution

$$e^{-\frac{N_k}{4}2\left[z_i - \frac{1}{2}(y_i - y_{i-1})\right]^2} \tag{3.134}$$

We iterate this process until the total amount of points in the loop satisfies our needs. We finally calculate the center of mass of the loop and shift it accordingly to get a final loop centered in the origin.

3.4.3 Calculation of the Effective Action

With a loop generator in hands, we are able now to proceed to the effective action calculation. Once we got the loops, we are able to compute their individual contribution calculating $e^{-\int_0^T d\tau V(x_{CM}+x(\tau))}$ for a given T and center of mass coordinate X_{CM}, recalling that now we are working with $x(\tau)$, not the unitary loops $y(t)$.

The exponent calculation, an integral over τ, can be solved, depending in the case, by incorporating by hand in the algorithm the analytical expression of it when it is possible or by an implementation of any conventional numerical integration routine. After computing it for all loops in the ensemble, we average between those, yielding the value we need for a pair (x_{CM}, T).

It is worth mentioning that a great advantage of calculation the exponent of the Wilson Loops using analytical methods is that, in this way, we preserve the gauge invariance of the method, while if we solve it numerically preserve it only up to discretization errors.

Since we are now able to compute the Wilson Lopps average, we are ready to compute the whole propertime integrand, the propertime integral and finally proceed for the integration over the center of mass in spacetime. Of course, in a computer implementation what we do is to call within a spacetime integration routine a propertime integration routine which then calls a Wilson Loop average calculation routine.

4 A Case of Study: Constant Magnetic Background

In this chapter, we discuss the results we obtained regarding the implementation of the Worldline Numerics method. In order to do so, we analyse the specific case of a scalar field subject to a constant magnetic background field.

4.1 Loop Generation

The loop generation algorithm, as it was introduced in the previous chapter, is application independent, in the sense that it does not depend in any way of the considered fields nature, either the quantum field we are considering or the external classical background field. Thus, this part of the worldline numerics algorithm is very independent of the rest. Indeed, while the other parts of the implemented algorithm were written in C/C++, the generation of the loops was made via a MATLAB script.

There are some loop generation algorithms in the literature, as described in (GIES; MOYAERTS, 2003). The chosen algorithm to be implemented in this work is the d-loops algorithm described in the previous section. It has been shown that it has a better computational performance when compared with other standard loop generating algorithms (GIES *et al.*, 2005).

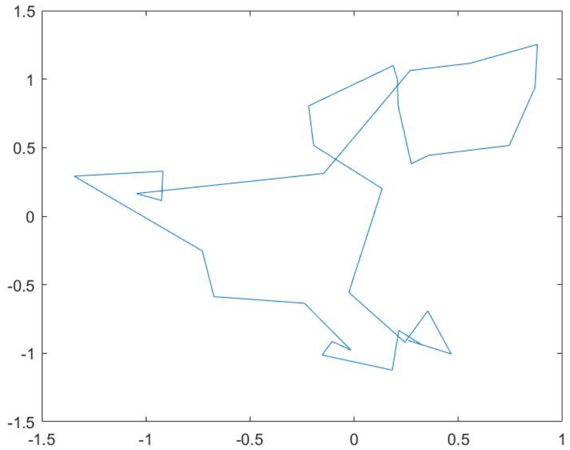

FIGURE 4.1 – Euclidean spacetime plot of a 32 points loop generated using the d-loops algorithm.

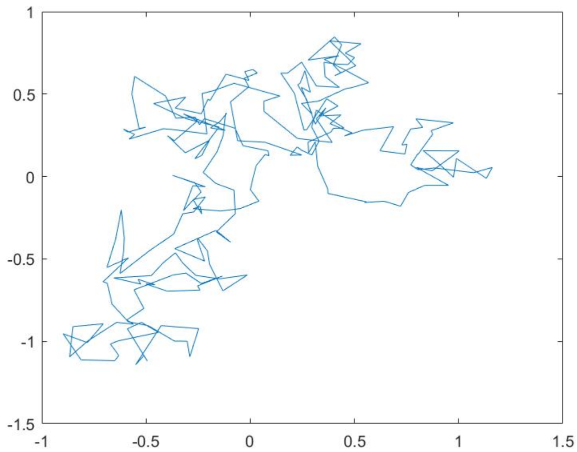

FIGURE 4.2 – Euclidean spacetime plot of a 256 points loop generated using the d-loops algorithm.

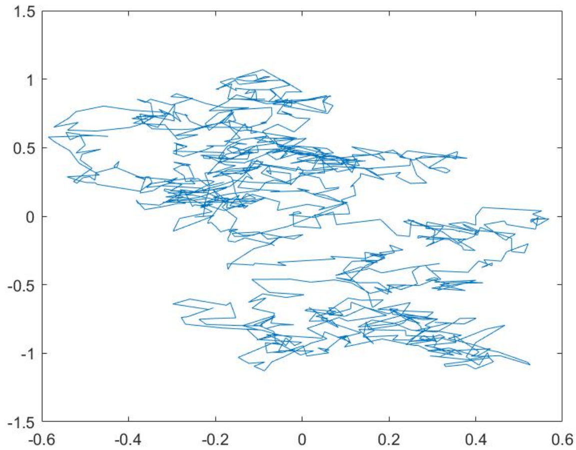

FIGURE 4.3 – Euclidean spacetime plot of a 1024 points loop generated using the d-loops algorithm.

Figs. 4.1, 4.2 and 4.3 display 3 different loops, generated in two Euclidean spacetime dimensions, by the same algorithm, just changing the parameter of the number of points composing the loop. With the increasing number of points, the loops, whose points were generated following the discrete version of our desired probability density, turn into better approximations of the original continuous loop ensemble, since our finite difference model tends to the continuous case.

4.2 Wilson Loop Averages

From this point on, since the holonomy factors depend upon the external field and on whether we are dealing with scalars or spinors fields, we will consider in the analysis the specific case of scalar fields in the presence of a constant background magnetic field.

Let's recall then that the effective action for a scalar field in the presence of a classical abelian gauge field is

$$\Gamma[A] = -(4\pi)^{-\frac{D}{2}} \int dx_{CM} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \left[\left\langle e^{i \int_0^T A(x_{CM}+x(\tau))\cdot \dot{x} d\tau} \right\rangle - 1 \right] .$$

The quantity of current interest is then $\left\langle e^{i \int_0^T A(x_{CM}+x(\tau))\cdot \dot{x} d\tau} \right\rangle$. To compute the individual loops contribution, we first fix the gauge. We consider the Fock-Schwinger gauge, which for this specific case translates to:

$$A = \frac{B}{2}(-y, x, 0, 0) .$$

Now, working on the exponent:

$$\int_0^T A(x_{CM} + x(\tau)) \cdot \dot{x} d\tau = \oint A(x_{CM} + x(\tau)) \cdot dx = \sum_{i=0}^{N_{ppl}} \int_{x_i}^{xi+1} A(x_{CM} + x(\tau)) \cdot dx \,,$$

where the last integral is a line integral over the curve $\rfloor_i$, the straight line connecting x_i and x_{i+1}.

Substituting the fixed gauge:

$$\int_{x_i}^{xi+1} A(x_{CM} + x(\tau)) \cdot dx = \frac{B}{2}(x_i(1)x_{i+1}(2) - x_i(2)x_{i+1}(1)), \tag{4.1}$$

and therefore:

$$\int_0^T A(x_{CM} + x(\tau)) \cdot \dot{x} d\tau = \sum_{i=0}^{N_{ppl}} \frac{B}{2}(x_i(1)x_{i+1}(2) - x_i(2)x_{i+1}(1)). \tag{4.2}$$

That expression is the exponent of the contribution of each loop of our ensemble, and therefore encodes all information needed to compute the average.

Next, we discuss the obtained numerical results:

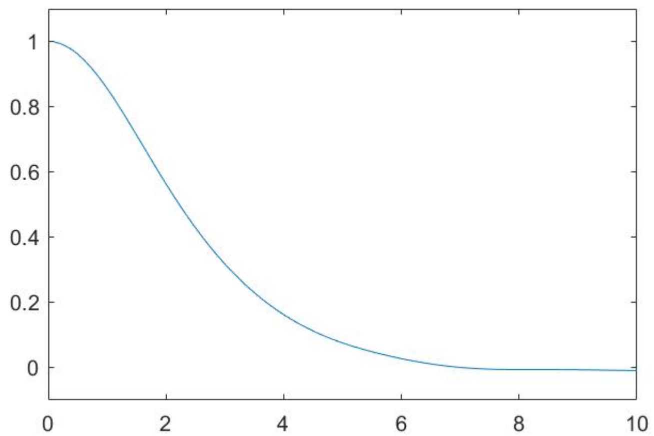

FIGURE 4.4 – Wilson Loops expected value (vertical axis) as a function of BT (horizontal axis)

The numerical result in Fig. 4.4 was obtained using $N_{ppl} = 1024$ and a ensemble of $N_l = 2000$ loops. The above graph is generated by 100 points evenly spaced in the interval $[0, 10]$.

We can benchmark our numerical results with the theoretical result, known for this field configuration, and see how they relate for different parametrizations of the algorithm.

Namely,

$$\left\langle e^{i\int_0^T A(x_{CM}+x(\tau))\cdot\dot{x}d\tau}\right\rangle = \frac{BT}{\sinh BT}\,. \tag{4.3}$$

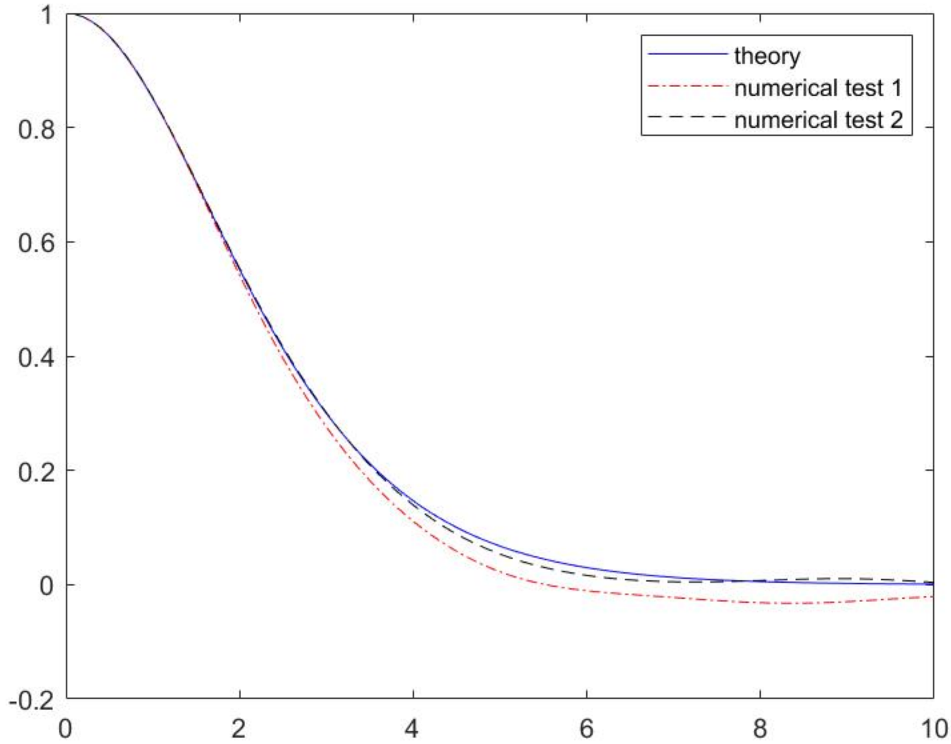

FIGURE 4.5 – Theoretical curve plotted against the numerical found curves for an ensemble of 1000 loops created with 1024 points each, test 1, and an ensemble of 4000 loops created with 2048 points each, test 2. The plots are BT (horizontal axis) versus Loop ensemble average (vertical axis)

In Fig. 4.5 we show an agreement between the curves for the given parametrizations and the theoretical one. That being said, a great improvement is observed in test 2, in which we have a better loop emsemble, in the large BT regime.

4.3 Proper-time Integration

With a good numerical estimator of the integrand, to recover the effective Lagrangian density we are left with the proper-time integration step. This can be done using different numerical integration routines. For our calculation purposes, we implemented a Gauss-Laguerre integration scheme.

The Gauss-Laguerre integration is a quadrature method used to solve integrals of the following type:

$$\int_0^\infty e^{-x} f(x) dx, \tag{4.4}$$

which is the exact format of our integrand up to a scaling factor.

Instead of naively integrating using the numerical method we should first analyse the integrand in question.

We know the integrand theoretical behavior is

$$\frac{e^{-m^2T}}{T^{D/2+1}} \left[\frac{BT}{\sinh BT} - 1 \right] . \tag{4.5}$$

The $T \to \infty$ limit is well controlled, even if we consider the massless case. Our concern is with the $T \to 0$ behavior, as depending on the spacetime dimension D singularities might arise.

Expanding the $\frac{BT}{\sinh BT}$ in a power series we get:

$$\frac{BT}{\sinh BT} = 1 - \frac{1}{6} B^2 T^2 + \mathcal{O}(T^4) \, .$$

Let's think, for example, in the case where $D = 3$. In this case, the small T behavior of the integrand is, after some rescaling,

$$I(T) \approx -\frac{1}{6} \frac{e^{-T}}{\sqrt{T}} \, ,$$

and therefore has a singularity in the origin. It's easy to see that this singularity is integrable, so that analytically speaking we are in good terms.

However, our numerical algorithm does not behave well in the presence of the singularity and we have to deal with it in order to get a stable convergence. For that, we consider the following regularization procedure:

$$I = \int_0^\infty \frac{dT}{T^{5/2}} e^{-m^2T} \left[\left\langle e^{i \int_0^T A(x_{CM} + x(\tau)) \cdot \dot{x} d\tau} \right\rangle - 1 \right] \tag{4.6}$$

$$\begin{aligned} I = \int_0^\infty dT \left[\frac{1}{T^{5/2}} e^{-m^2T} \left[\left\langle e^{i \int_0^T A(x_{CM} + x(\tau)) \cdot \dot{x} d\tau} \right\rangle - 1 \right] + \left(\frac{B}{m^2} \right)^{1/2} \frac{1}{6} \frac{e^{-m^2T}}{\sqrt{T}} \right] , \\ - \left(\frac{B}{m^2} \right)^{1/2} \int_0^\infty dT \frac{1}{6} \frac{e^{-m^2T}}{\sqrt{T}} \end{aligned} \tag{4.7}$$

In 4.7, the first integral now has no longer a singular behavior in the origin and can be directly calculated using the Gauss-Laguerre quadrature. The second one can be solved analytically using standard methods. Therefore, we can perform the numerical integration of the regularized integrand and after that just subtract the known counter-term.

The results obtained for the effective Lagrangian follow next.

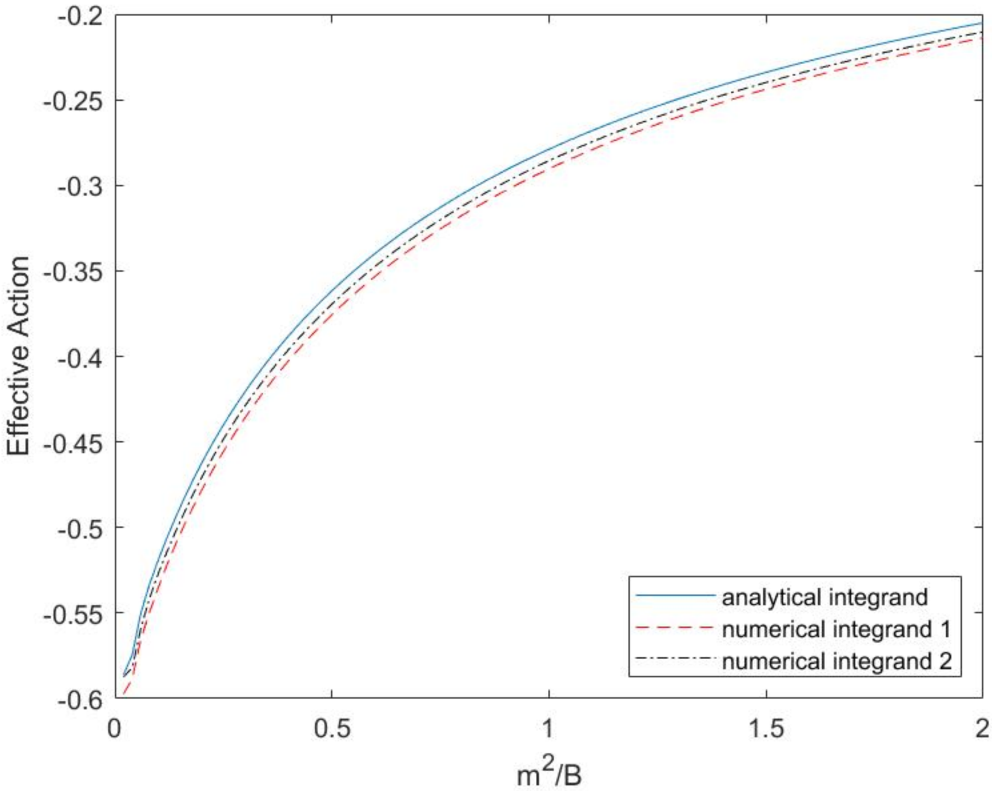

FIGURE 4.6 – Effective Lagragian in units of $\left(\frac{B}{4\pi}\right)^{D/2}$ versus m^2/B

The results presented in Fig. 4.6 are in agreement with the theoretical curve as in (GIES; LANGFELD, 2001). For the spacetime integral it can be computed by a simple integration, however, for this case of study, as we can see there is no dependence on the center of mass. This is expected since our field is homogeneous and consequently our Lagrangian should be invariant under translations. Therefore, the spacetime integral only adds a volume factor to the Lagrangian Density and is not very interesting.

We can proceed with $D = 4$ dimensions and the discussion is similar. In this specific case, to obtain a finite result, we again should analyse which terms in the Taylor expansion lead to diverging contributions. In the three dimensional case we had a singular integrand, but whose analytical contribution was finite indeed. Since our numerical integration scheme had difficulties dealing with it, we regularized the numerical integrand and as a counter-term added the contribution due to the singular term. The difference here, though, is that now the analytical contribution of this same term in the series expansion of the integrand is infinite. To solve this, we subtract it from the integrand using a renormalization scheme, in which we consider the action including the bare Maxwell action term and impose a renormalization condition as in (LANGFELD *et al.*, 2002). The net result is that we just subtract this term contribution and end up with a finite result.

The numerical results for the action follows in Fig. 4.7.

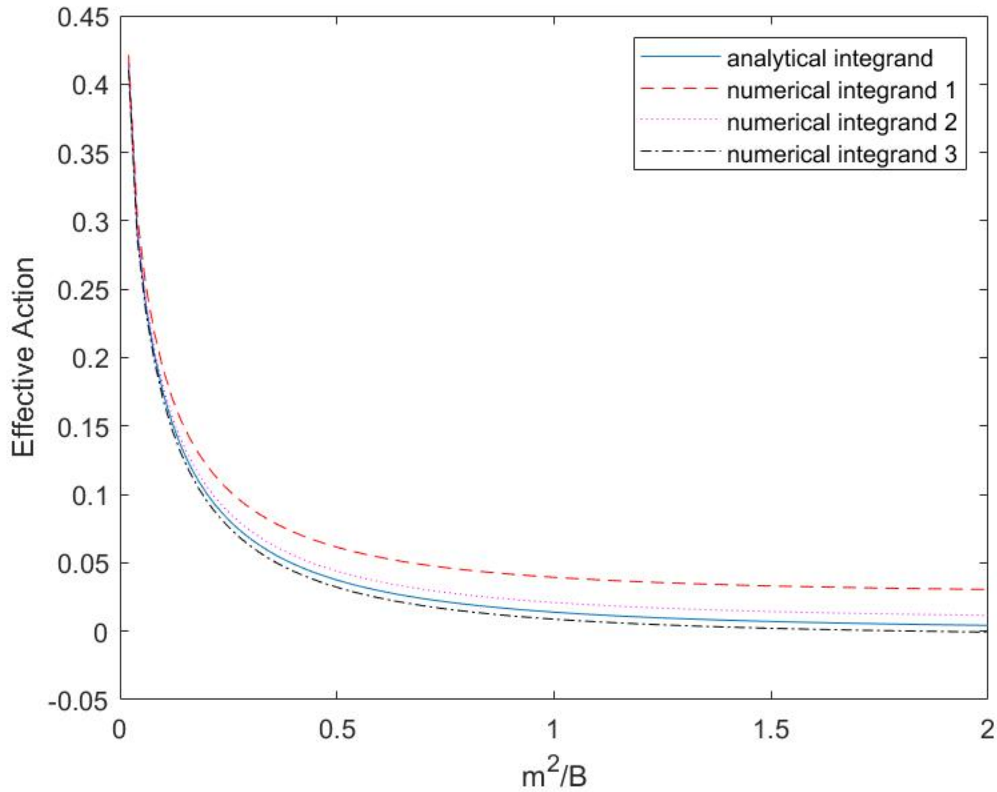

FIGURE 4.7 – Effective Lagragian in units of $\left(\frac{B}{4\pi}\right)^{D/2}$ versus m^2/B for $D = 4$. The numerical integrands were generated with 2000 loops and 1024 points per loop, 4000 loops and 2048 points per loop and 16000 loops and 2048 points per loop, respectively

Once again we can see a good agreement between the worldline numerics algorithm and the result obtained by integrating the analytically known integrand. We can identify that for a fixed parametrization of the algorithm the relative error between the two curves increase as the magnetic field decreases.

We also observe that, again for a fixed parametrization, the errors are bigger now when compared to the displayed results for the three dimensional case. This is expected since we are using a finite ensemble to represent those infinite ensembles of worldline paths. When we increase the spacetime dimension we are increasing the degrees of freedom of the paths, which makes the worldline ensemble to acquire a richer structure. Thus, it requires a higher number of loops to approximate a higher dimensional ensemble of paths with the same accuracy.

Moreover, we see that increasing the number of loops and points per loop the results using the numerical integrands converge towards the one using the analytical one, showing the expected behavior.

5 Conclusion

In this masters thesis we went through the propertime method development and its importance to the Schwinger Effect. We introduced the Worldline Formalism for different background fields and some of the applications of it on the study of physical systems.

Regarding the Worldline methods, we described two methods widely used in the literature. In this context, we explored further the Schwinger effect and calculated pair production rates for the Spinorial QED for the parallel constant fields configuration via Worldline Instantons, extending the analysis of Ref. (GORDON; SEMENOFF, 2015), where it was considered a scalar field in a constant purely electric background.

Then, we reviewed the Worldline Numerics method describing the algorithmic way of generating a finite ensemble of discretized worldlines that were a faithful representation of the infinite ensemble of continuous paths.

Finally, we displayed the results achieved for our implementation of the described method for a field configuration with known analytical results, so that we could benchmark our results and validate the implemented method and our code.

After this benchmarking, due to the good agreement with the theory, the idea for future works is to explore the developed algorithm studying different field configurations not yet explored on the literature. As described in the literature, this algorithm has the potential for multiple applications, as pair creation rates in the Schwinger effect and Casimir energies for complex shaped boundaries.

In addition to that, there are recent advances in the direction of adapting it for the case of curved spacetime. (CORRADINI; MURATORI, 2020) explores non-linear sigma models representing the propagation of a scalar fields in curved space. They treat the space time curvatures as couplings of the scalar field and define potential counter-terms.

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Effective actions via worldline formalism.

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7. INSTITUIÇÃO(ÕES)/ÓRGÃO(S) INTERNO(S)/DIVISÃO(ÕES):

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11. RESUMO:

In this master thesis we study one-loop effective actions obtained via Worldline Formalism for the specific cases of a scalar field coupled to a scalar potential as-well as scalar and spinorial QED. We explore and describe also the Schwinger effect, a direct consequence of a non-vanishing imaginary part in the one-loop effective action. Our study is mainly based on a analytical approach considering a semi-classical regime, where we calculate electrons pair creation rate in the specific case of parallel electric and magnetic fields, an extention of the presented in (Gordon,2015), and our main method of interest, which is a numerical method, the so-called Worldline Numerics, Worldline Monte Carlo or Loop Cloud method, for which we implemented numerically the method and tested it on a constant magnetic background setting. These methods are used throughout the literature to study the physics of the Casimir effect (Gies;Moyaerts,2003), not having to rely always on simple boundary surfaces shapes, and the Schwinger effect, even in the case of dynamical fields (Schutzhold et al,2008), which have the theoretical importance of lowering the Schwinger limit, the theoretical minimum field intensity necessary to observe the phenomena in a experiment.

12. GRAU DE SIGILO:

(X) OSTENSIVO **( ) RESERVADO** **( ) SECRETO**