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Rafael Endlich Pimentel

TWO BODY FERMION BOUND STATE IN MINKOWSKI SPACE AND THE WICK-CUTKOSKY MODEL

Dissertation approved in its final version by signatories below:

Prof. Dr. Wayne Leonardo Silva de Paula

Advisor

Prof. Dr. Luiz Carlos Sandoval Góes

Prorector of Graduate Studies and Research

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Rafael Endlich Pimentel
Av. Cidade Jardim, 679
12.233-066 – São José dos Campos–SP

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Thesis Committee Composition:

Prof. Dr.	Brett Vern Carlson	President	-	ITA
Prof. Dr.	Wayne Leonardo Silva de Paula	Advisor	-	ITA
Prof. Dr.	Rubens de Melo Marinho Junior	Member	-	ITA
Prof. Dr.	Vladimir Karmanov	External Member	-	Lebedev Phys Inst

ITA

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"If I have seen farther than others,
it is because I stood on the shoulders of giants."
— Sir Isaac Newton

Resumo

Nessa tese foram estudados os estados ligados de duas partículas em uma teoria quântica de campos, utilizando a equação de Bethe-Salpeter com a representação de Nakanishi, e considerando dois diferentes casos. O primeiro trata-se do estado de dois bósons distintos ligados por um bóson sem massa, também chamado de modelo de Wick-Cutkosky. Para resolver esse modelo, desenvolveu-se novos métodos utilizando integrações por partes na representação de Nakanishi; um desses consiste em transformar uma das integrações em uma série e outro permite que se obtenha uma inédita equação diferencial no espaço dos parâmetros da transformada. O segundo problema atacado consiste no estado ligado de dois férmions, ligados por um bóson massivo. Nele, utilizou-se da chamada projeção na frente de luz para se obter um kernel contínuo para o operador, visando atacar o problema numericamente. Entretanto, a projeção na frente de luz de uma amplitude pode gerar termos singulares, conhecidos como contribuições de ponto final. Em (CARBONELL; KARMANOV, 2010), esses termos singulares foram tratados utilizando-se de regularizadores e fatores de forma. Em contrapartida, nessa tese os termos singulares foram tratados analiticamente, escrevendo as integrais da frente de luz em termos de distribuições para enfim obter os termos singulares que contribuem para a equação de Bethe-Salpeter para o estado ligado de dois férmions.

Abstract

In this thesis, two particle bound states in quantum field theories were studied using the Bethe-Salpeter equation with Nakanishi representation, in two different cases. The first deal with two distinct bosons bounded by another massless boson, the so called Wick-Cutkosky model. To solve it, two new methods were developed using integration by parts in the Nakanishi representation; one of these transforms a integration into a series and the other obtain a new differential equation in the space of the Nakanishi transform parameters. The second problem deals with the bound state of two fermions, bounded by a massive boson. In it, a Light-Front projection was used to obtain a continuous kernel for the operator in order to solve the problem numerically. However, the Light-Front projection of an amplitude can give singular terms, known as endpoint contributions. In (CARBONELL; KARMANOV, 2010), these singular terms were dealt with using regularizators and form factors. However, in this thesis the singular terms were developed analytically, writing the Light Front integrals as distributions in order to obtain the singular terms that contribute to the Bethe-Salpeter equation for the two fermion bound state.

List of Tables

List of Abbreviations and Acronyms

PTIR	Perturbative Theoretical Integral Equation
BSA	Bethe-Salpeter Amplitude
BSE	Bethe-Salpeter Equation
QFT	Quantum Field Theory
QM	Quantum Mechanics

List of Symbols

α	Eigenvalue
g	Coupling constant
m	Particle mass
μ	Interaction Particle mass
M	Bound state mass

Contents

1 Introduction

1.1 Objective

This thesis aims to develop the framework of the light-front (DIRAC, 1949) projected Bethe-Salpeter equation (BSE) (BETHE, 1951) in Minkowski space with the Nakanishi representation. Specifically, this is done through two main fronts: the Wick-Cutkosky model (WICK, 1954; CUTKOSKY, 1954) and the fermionic massive BSE .

The Wick-Cutkosky is one of the first relativistics bound state models studied, and was published only 3 years after the Bethe-Salpeter equation. It describes two bosons bounded by another massless boson in the ladder approximation and has a very simple equation describing it. Due to this simplicity, it will be the problem chosen to develop new approaches to solve the scalar BSE, based on the uniqueness of the Nakanishi Perturbative Integral Representation and integration by parts of the weight function.

The fermionic BSE describes two fermions bounded by a scalar massive particle in the ladder approximation. This problem is more complex than the scalar one but it is needed to describe real physical bound states. Followed the steps in (CARBONELL; KARMANOV, 2010) this system is studied with the Nakanishi representation projected onto the Light Front. However, this projection often results in singular terms called endpoint contributions .That article dealt with it these singular contributions using a form factor and a regularizer, but in this thesis the these singular terms will be calculated by performing the integrals and dealing with the resulting distributions directly.

1.2 Motivation

The solution of the bound state problem in Minkowski space has been historically avoided since relativistic propagators have poles and thus are difficult to deal with numerically. So, usually the problem is attacked using the Wick rotation (WICK, 1954), which transforms a relativistic quantum field theory (QFT) into an euclidean QFT, whose propagators are of the form $\frac{-i}{k_E^2+m^2}$, which is well behaved numerically, since it is bounded and smooth. However, the main problem with this method is that it is enough to obtain the mass of the bound state (DORKIN *et al.*, 2011) but not the dynamical content, such as form factors (KARMANOV *et al.*, 2008), which depends on the Bethe-Salpeter amplitude (BSA) itself. Thus new frameworks to solve the BSE in the Minkowski space should be investigated.

In the sixties, N. Nakanishi developed an parametric integral representation, very similar of the type used by Wick, for Feynman Diagrams and scattering amplitudes (NAKANISHI, 1971)

$$\phi(k) = \int_0^\infty d\gamma' \int_0^1 \prod_h dz'_h \delta(\sum_h z'_h - 1) \frac{g^{(n)}(\gamma', z')}{(\gamma' - \sum_h z'_h s_h - i\epsilon)^n}, \tag{1.1}$$

where s_h are scalar products of the amplitude external momenta. He analyzed this representation and demonstrated important properties such as analicity and uniqueness. Although Nakanishi studied this formula for perturbative purposes, the initial ideia of Wick was to use it to solve the nonperturbative BS equation. So, although this formula is build constructively by analysing each Feynman diagram, the final result can be used as an Ansatz for a nonperturbative amplitude.

In the nineties, Kusaka and Williams (KUSAKA; WILLIAMS, 1995; KUSAKA *et al.*, 1997) observed that the Nakanishi representation could be used to solve the massive interaction ladder BS equation in Minkowski space , but the method was still not very practical for numerical purposes. Thus, in 2006 Karmanov and Carbonell (KARMANOV, 2006) proposed a new technique based on the application of the Nakanishi representation to the BS equation and then projecting it onto the Light Front. This resulted in an generalized integral equation with smooth kernels. Then Frederico, Salmè and Viviani (VIVIANI, 2012) extended the light-front projected BS equation for scattering states and also developed a new method based on the uniqueness of the Nakanishi representation (FREDERICO *et al.*, 2014).

Previously, such framework has been applied to study the massive scalar ladder BSE in 3+1 (KARMANOV, 2006), ladder + Crossed-Box BSE (IANNONE, 2013) 2+1 dimensions (PIMENTEL, 2013; PAIVA, 2014), scattering states (VIVIANI, 2012) and the fermionic BSE in 3+1 dimensions (CARBONELL; KARMANOV, 2010). However, altough scalar 2+1 theories can help predict qualitative properties of bidimensional materials, the fermionic 2+1 massless bound state in Minkowski space is needed in order to study materials such as graphene. Thus, this motivated some results here developed, such as new methods to obtain the Wick-Cutkosky model from the 3+1 massless scalar bound state, and correction terms for the light-front projected fermionic 3+1 BSE. Altough these new models and methods still can't investigate graphene properties directly they are the necessary first steps towards a full solution.

1.3 Organization

This thesis can be divided in a introductory theory presented in Chapters 2 and 3, and the results presented in Chapters 4 and 5.

Chapter 2 aims to give a very brief heuristic discussion of the Bethe-Salpeter equation used to model bound states in quantum field theory. However, for a complete discussion of the subject one should look at the references. Chapter 3 develops the Nakanishi Perturbative Theoretical Integral Representation (PTIR), for a given Feynman diagram, discuss its integral by parts results, and defines the light-front (LF) variables used in the thesis. Chapter 4 discuss two new methods to solve the well known Wick-Cutkosky model: the first expands the LF integral into a sum using integration by parts, and the second uses the Nakanishi PTIR uniqueness theorem, without the LF projection, to obtain a partial differential equation in the Nakanishi parameters. And Chapter 5 calculates the singular contributions that arise from the end point singularities after the LF projection of a fermionic BSE. Finally, conclusions and futures directions are discussed.

2 The Bethe-Salpeter Equation

2.1 Introduction

This Chapter intends to briefly explain how to treat a two body bound state in a quantum field theory, using the Bethe-Salpeter equation, and how it differs from the non-relativistic hamiltonian treatment. Usually, the study of quantum field theory is heavily directed to scattering processes, due to its application in particle accelerators and pedagogical value. So, standard techniques use the perturbative framework, which takes a given n point green function of the theory and expands it for a weak coupling constant g near the free theory. That produces results that can be seen as Taylor expansions of the observables in the bare coupling constant g, and renormalization corrections .

However, this strategy fails to describe bound states, even in non-relativistic theories, because they appear as poles in the green function, thus are inherently non-analytic. Therefore, to analyze them one should use nonperturbative methods, such as integral equations or the euclidean path integral formulation on the lattice. The lattice approach is the standard method to obtain the masses of complex bound states such as the proton but lacks the ability to obtain dynamical information due to its euclidean nature. Thus, integral methods in Minkowski space based on Bethe-Salpeter and/or Schwinger-Dyson equations are in demand to fully solve the bound state problem.

2.2 Relativistic and non-relativistic bound states

Altough QFT courses usually focuses on the scattering processes, the QM courses spend a good amount of time studying the single particle bound state problem. One reason one may argue is that non-relativistic bound states are commonly treated using the time independent hamiltonian framework

$$H |\Psi\rangle = E |\Psi\rangle \,, \tag{2.1}$$

which ammounts to obtain the spectrum of the hamiltonian H, a partial differential operator. Fundamentally, we are able to use this strategy for a nonrelativistic system because at this scale interactions can be treated as an instantaneous action at distance. Thus, it is possible to define vector states made of hyperplanes of the wave-function Ψ, restricted to the same time t_0, and the Hamiltonian can be seen as a Lie Algebra which evolves the state vector from the time t_0 to $t_0 + dt$. Additionally, since the interaction is instantaneous there is no need for interacting fields.

In QFT however, every interaction is local in time and space, and cannot travel faster than the speed of light. So, the strategy of taking slices of $t = t_0$ hyperplanes as state vectors are not so efficient in a relativistic environment, because a time evolution of dt cannot obtain information form the entire hyperplane, at arbitrary distances. Of course is still possible to use a Hamiltonian framework in QFT if one expands the matter and interaction fields in the Fock space, but is not a practical or "natural" way to compute relativistic observables. Therefore, in a QFT it is better to study the field correlations between n given *spacetime points* (t_i, x_i) , in spite of two hyperplanes $\Psi(t_1)$ and $\Psi(t_2)$, which are called the *n-point green functions* of the theory. Thus, almost every calculation in QFT is made using derived objects from the green functions in momentum space, called *Feynman propagators*.

2.3 The scalar Bethe-Salpeter equation

The differential equation (2.1) can be inverted and transformed into an integral equation. First, separate the free hamiltonian and the potential

$$H = H_0 + V. \tag{2.2}$$

Then one can rewrite (2.1) and invert it

$$\begin{aligned} H |\Psi\rangle &= E |\Psi\rangle & (2.3)\\ (H_0 + V) |\Psi\rangle &= E |\Psi\rangle & (2.4)\\ (H_0 - E) |\Psi\rangle &= -V |\Psi\rangle & (2.5)\\ |\Psi\rangle &= -\frac{1}{(H_0 - E)} V |\Psi\rangle , & (2.6) \end{aligned}$$

with $\frac{1}{(H_0-E)}$ as an integral operator. Therefore, as shown by (2.6), it is possible to treat a bound state with integral equations even in nonrelativistic systems; it is just not as popular as the differential approach. In a scalar quantum field theory, it is also possible to treat bound states as integral equations, with the Bethe-Salpeter equation

$$\phi(k,p) = \frac{1}{(\frac{p}{2}-k)^2 - m^2}\frac{1}{(\frac{p}{2}+k)^2 - m^2} ig^2 \int \frac{d^4k'}{(2\pi)^4}\frac{\phi(k',p)}{(k-k')^2 - \mu^2 + i\epsilon}, \tag{2.7}$$

where g is the coupling constant, m the matter particle mass and μ the interaction particle mass. It is an integral equation that uses as building blocks relativistic propagators of the form

$$\frac{i}{k^2 - m^2} \tag{2.8}$$

3 The Nakanishi Perturbative Integral Representation and the Light-Front Projection

3.1 Introduction

This Chapter introduces an integral representation formula for quantum field theory amplitudes. It was extensively studied in the sixties by N. Nakanishi, whose goal was to provide a unified way to write any scattering amplitude. The idea was to generalize the Feynman parametrization method, but using a weight function in the numerator, and the same denominator for a given set of external particles. Therefore, he was able to sum all the Feynman diagrams for a given process, which are perturbative objects, and obtain the a formula for the full scattering amplitude, which is nonperturbative. However, it must be noted that this type of analytical formula with a weight function was already use by Wick in his seminal paper, in order to obtain the ground state for the Wick-Cutkosky model. Therefore, it is only natural to try to use the Nakanishi perturbative-theory integral representation (PTIR) as an Ansatz for the nonperturbative BSE.

The last section also gives a brief introduction to the Light-Front framework, which is used as an intermediate step in the solution of the BSE, in order to get smooth integrals and avoid poles in the denominators.

3.2 Feynman parametric integral representation

Let G be a connected Feynman diagram with N external momenta p_i, $i \in \{1, \cdots, N\}$, n internal propagators with momenta l_j and masses m_j, $j \in \{, \cdots, n\}$ and k loops. Note that the p_i must satisfy the four-momentum conservation $\sum_{i=1}^{N} p_i = 0$. Each internal momentum can be written as external momenta and loop momenta of the form

$$l_j = \sum_{r=1}^{k} b_{jr} q_r + \sum_{i=1}^{N} c_{ji} p_i, \tag{3.1}$$

where q_r is the loop momentum associated with the loop number r, and $b_{jr}, c_{ji} \in \{-1, 0, 1\}$. Within this framework, it is possible to write the integral $f_G(p_i)$ associated with a given Feynman diagram G, aside from multiplicative factors, as

$$f_G(p_i) = \prod_{r=1}^{k} \int d^4 q_r \frac{1}{(l_1^2 - m_1^2 + i\epsilon) \cdots (l_n^2 - m_n^2 + i\epsilon)}, \tag{3.2}$$

Where factors such as the coupling constants for each vertex were ignored. To calculate (3.2), it is useful to use the Feynman parametrization,

$$\frac{1}{A_1 \cdots A_n} = \prod_{i=1}^{n} \int_0^1 d\alpha_i \frac{\delta(\sum \alpha_i - 1)}{(\sum_{i=1}^{n} \alpha_i A_i)^n} \tag{3.3}$$

in order to join the denominators as

$$f_G(p_i) = \prod_{r=1}^{k} \int d^4 q_r \prod_{i=1}^{n} \int_0^1 d\alpha_i \frac{\delta(\sum \alpha_i - 1)}{(\sum_{j=1}^{n} \alpha_j (l_j^2 - m_j^2) + i\epsilon)^n}. \tag{3.4}$$

To evaluate the integral in $d^4 q_r$, the first step is to expand the l_j^2 as

$$l_j^2 = \sum_{rr'} b_{jr} b_{jr'} q_r q_r' + 2 \sum_{r,i} b_{jr} c_{ji} q_r p_i + \sum_{i,i'} c_{ji} c_{ji'} p_i p_{i'}. \tag{3.5}$$

However, the crossed terms $q_r q_r'$ are undesirable, because they make it harder to perform the $d^4 q_i$ integration. So, it is useful to see l_j^2 as a quadratic form in the vector $\{q_r\}$ and diagonalize or triangularize it, so that it is possible to write

$$\sum_j \alpha_j l_j^2 = \sum_r d_r Q_r^2 + \sum_{i,i'} e_{ii'} p_i p_{i'}, \tag{3.6}$$

where $Q_r = q_r + \sum_{i>r} \gamma_{ri} q_i + \sum_{j=1}^{N} \beta_{rj} p_j$ are obtained completing the squares for each q_i such that there are no more crossed terms, and $e_{ii'}$ is a function of the α_j. Thus, it is possible to write the Feynman integral, shifting the integration variable from q_r to Q_r as

$$f_G(p_i) = \prod_{r=1}^{k} \int d^4 Q_r \prod_{i=1}^{n} \int_0^1 d\alpha_i \frac{\delta(\sum \alpha_i - 1)}{((\sum_{r=1}^{k} d_r Q_r^2 - \sum_{j=1}^{n} \alpha_j m_j^2) + \sum_{ii'} e_{ii'} p_i p_i' + i\epsilon)^n}. \tag{3.7}$$

Now, it is possible to perform the d^4q_r integral, using the formula

$$\begin{aligned}\int d^4Q_r\frac{1}{(d_rQ_r^2+A_r+i\epsilon)^n}&=\frac{1}{d_r^n}\int d^4Q_r\frac{1}{(Q_r^2+\frac{A_r+i\epsilon}{d_r})^n}\\&=\frac{1}{d_r^n}i\pi^2\frac{1}{n-1}\frac{1}{n-2}\frac{1}{(\frac{A_r+i\epsilon}{d_r})^{n-2}}\\&=i\pi^2\frac{1}{n-1}\frac{1}{n-2}\frac{1}{d_r^2}\frac{1}{(A_r+i\epsilon)^{n-2}}.\end{aligned}\tag{3.8}$$

So, performing all the k d^4Q_r integrals, we are left with

$$f_G(p_i)=\frac{(i\pi)^k(n-2k-1)!}{(n-1)!}\prod_{i=1}^{n}\int_0^1 d\alpha_i\frac{\delta(\sum\alpha_i-1)}{U^2(\sum_{ii'}e_{ii'}p_ip_i'-\sum_{i=1}^n\alpha_im_j^2+i\epsilon)^{n-2k}},\tag{3.9}$$

where $U=d_1\cdots d_k$. This formula already hints at the general structure of a feynman integral: an integral representation where the denominator of the integrand is a linear combination of the scalar products of external momenta and particles masses. However, in this representation the $e_{ii'}$ coefficients of the scalar products, are only indirect functions of the integration variables α_i and are unique to each diagram. Moreover, the exponent $n-2k$ depends on the number of loops of a given Feynman diagram. In order to overcome this issues and facilitate the procedure of summing Feynman diagrams for a given amplitude, Nakanishi intended to build a framework where each integral would be given in the same form, with the same exponent and the same denominator, and identify each of them using a weight function in the numerator.

3.3 Off-shell integral representation

Now that the Feynman parametric representation was obtained, it should be used to derive the Nakanishi off-shell integral representation. First, one need to refer to Nakanishi's work and note that the denominator of (3.9) can be written as, in the modified cut-set representation

$$\sum_{ii'}e_{ii'}p_ip_i'-\sum_{i=1}^n\alpha_im_j^2=\sum_h\eta_hs_h-\sum_{j=1}^n\alpha_jm_j^2,\tag{3.10}$$

where s_h are the so called invariant squares, and are terms of the form $s_h=(\sum_j p_j)^2$, in order to avoid crossed terms such as $p_ip_{i'}$ with $i\neq i'$. It is important to represent the denominator as a linear combination of s_h because it makes all the $\eta_h>0$. In order to obtain the Nakanishi representation, one first need to normalize the η_h as $\frac{\eta_h}{\sum_h \eta_h}$:

$$f_G(p_i) = \frac{(i\pi)^k(n-2k-1)!}{(n-1)!} \prod_{i=1}^{n} \int_0^1 d\alpha_i \frac{\delta(\sum \alpha_i - 1)}{U^2(\sum_j \eta_j)^{n-2k}(\sum_h \frac{\eta_h}{\sum_j \eta_j} s_h - \frac{\sum_{i=1}^n \alpha_i m_j^2}{\sum_j \eta_j} + i\epsilon)^{n-2k}}. \tag{3.11}$$

And now it is possible to define new integration variables $\chi = \frac{\sum_{i=1}^n \alpha_i m_j^2}{\sum_j \eta_j} \in [0, \infty)$, $z_h = \frac{\eta_h}{\sum_j \eta_j} \in [0,1]$ with $\sum_h z_h = 1$. To do it, we multiply the integrand of (3.11) by the identity relation

$$1 = \prod_h \int_0^1 dz_h \delta\left(z_h - \frac{\eta_h}{\sum_j \eta_j}\right) \int_{0^-}^{\infty} d\chi \delta\left(\chi - \frac{\sum_l \alpha_l m_l^2}{\sum_j \eta_j}\right), \tag{3.12}$$

where the 0^- integration limit was used just in case there is a term proportional to $\delta(\chi)$ or another distribution with support on $\chi = 0$, for instance if all $m_l = 0$. Finally, the Eq. (3.12) is inserted in the integrand of (3.11)

$$\begin{aligned} f_G(p_i) &= \frac{(i\pi)^k(n-2k-1)!}{(n-1)!} \prod_{i=1}^{n} \int_0^1 d\alpha_i \frac{\delta(\sum \alpha_i - 1)}{U^2(\sum_j \eta_j)^{n-2k}(\sum_h \frac{\eta_h}{\sum_j \eta_j} s_h - \frac{\sum_{i=1}^n \alpha_i m_j^2}{\sum_j \eta_j} + i\epsilon)^{n-2k}} \\ &\times \prod_h \int_0^1 dz_h \delta\left(z_h - \frac{\eta_h}{\sum_j \eta_j}\right) \int_{0^-}^{\infty} d\chi \delta\left(\chi - \frac{\sum_l \alpha_l m_l^2}{\sum_j \eta_j}\right) \\ &= \prod_h \int_0^1 dz_h \delta(\sum_h z_h - 1) \int_{0^-}^{\infty} d\chi \frac{\phi_G^{(n-2k)}(\chi, z_h)}{(\sum_h z_h s_h - \chi + i\epsilon)^{n-2k}}, \end{aligned} \tag{3.13}$$

with

$$\begin{aligned} &\delta(\sum_h z_h - 1)\phi_G^{(n-2k)}(\chi, z_h) \\ &= \frac{(i\pi)^k(n-2k-1)!}{(n-1)!} \prod_{i=1}^{n} \int d\alpha_i \frac{\delta(\sum \alpha_i - 1)(\sum_j \eta_j)^{n-2k}}{U^2(\sum_j \eta_j)^{n-2k}} \prod_h \delta\left(z_h - \frac{\eta_h}{\sum_j \eta_j}\right) \delta\left(\chi - \frac{\sum_l \alpha_l m_l^2}{\sum_j \eta_j}\right), \end{aligned} \tag{3.14}$$

where $\phi_G^{(n-2k)}(\chi, z_h)$ is the so called off-shell weight function.

First, it is important to know that although called a function, it is generally a distribution, which is expected after so many delta functions. Also, certain highly symmetrical Feynman diagrams, such as the box diagram, can make some scalar products in the denominator lineary dependent and thus may be necessary a delta function in the numerator asserting this dependency relation. Second, Nakanishi's original idea was to sum all the diagram to obtain a scattering amplitude, so all of them must be made to exponent one in the denominator, using sucessive integrations by parts, such as

$$\begin{aligned} f_G(p_i) &= \prod_h \int_0^1 dz_h \delta(\sum_h z_h - 1) \int_{0^-}^{\infty} d\chi \frac{\phi_G^{(n-2k)}(\chi, z_h)}{(\sum_h z_h s_h - \chi + i\epsilon)^{n-2k}} \\ &= \prod_h \int_0^1 dz_h \delta(\sum_h z_h - 1) \int_{0^-}^{\infty} d\chi \frac{\phi_G^{(1)}(\chi, z_h)}{(\sum_h z_h s_h - \chi + i\epsilon)}, \end{aligned} \tag{3.15}$$

where

$$\phi_G^{(1)}(\chi, z_h) = (-1)^{n-2k-1} \frac{\partial^{n-2k-1}}{\partial \chi^{n-2k-1}} \phi_G^{(n-2k)}(\chi, z_h). \tag{3.16}$$

The type of quantum amplitude explored in this thesis is a vertex amplitude, i.e., one with three external momenta p_1, p_2, p_3. Let $f_3(p_i)$ be this amplitude, it can be written as

$$f_3(p_i) = \prod_h \int_0^1 dz_h \delta(\sum_h z_h - 1) \int_{0^-}^{\infty} d\chi \frac{\phi_3^{(1)}(\chi, z_h)}{(\sum_h z_h p_h^2 - \chi + i\epsilon)}. \tag{3.17}$$

3.4 Half on-shell integral representation

In order to use the vertex off-shell representation for the BSA, we must put one of the momenta on-shell, since the bound state P is on-shell. First, since using the momentum conservation one may write :

$$p_1 = \left(\frac{p}{2} + k\right) \tag{3.18}$$

$$p_2 = \left(\frac{p}{2} - k\right) \tag{3.19}$$

$$p_3 = -(p) \tag{3.20}$$

Which satifies $\sum_h p_h = 0$ implicitly. Substituting (3.20) in (3.17), and using $p^2 = M^2$, (3.17) can be rewritten as

$$f_3(p_i) = \prod_h \int_0^1 dz_h \delta(\sum_h z_h - 1) \int_{0^-}^{\infty} d\chi \frac{\phi_3^{(1)}(\chi, z_h)/(z_1 + z_2)}{(k^2 + p \cdot k \frac{(z_1 - z_2)}{(z_1 + z_2)} + \frac{\frac{M^2}{4}(z_1 + z_2 + 4z_3) - \chi}{(z_1 + z_2)} + i\epsilon)}. \tag{3.21}$$

Note that the term $\frac{\frac{M^2}{4}(z_1+z_2+4z_3)-\chi}{(z_1+z_2)}$ does not depends on any external momenta, so one should transform it in a new parameter $-\gamma'$ for the representation. Also, the term $\frac{(z_1-z_2)}{(z_1+z_2)}$ multiplying $p \cdot k$ will be changed to $z' \in [-1, 1]$. Inserting in (3.21) the identity relations

$$1 = \int d\gamma' \delta(\gamma' + \left(\frac{\frac{M^2}{4}(z_1+z_2+4z_3)-\chi}{(z_1+z_2)} \right)), \tag{3.22}$$

and

$$1 = \int_{-1}^{1} dz' \delta(z' - \left(\frac{z_1 - z_2}{z_1 + z_2} \right)), \tag{3.23}$$

f_3 can be written as

$$f_3(p,k) = \int d\gamma' \int_{-1}^{1} dz' \frac{g^{(1)}(\gamma', z')}{k^2 + z' p \cdot k - \gamma' + i\epsilon}, \tag{3.24}$$

where

$$g^{(1)}(\gamma', z') = \prod_h \int_0^1 dz_h \delta(\sum_h z_h - 1) \int_{0^-}^{\infty} d\chi$$
$$\times \frac{\phi_3^{(1)}(\chi, z_h)}{(z_1+z_2)} \delta(z' - \left(\frac{z_1 - z_2}{z_1 + z_2} \right)) \delta(\gamma' + \left(\frac{\frac{M^2}{4}(z_1+z_2+4z_3)-\chi}{(z_1+z_2)} \right)). \tag{3.25}$$

Here the integral in γ' did not have any boundary in order to avoid discuss the complicated problem of the Nakanishi representation support. However, if one knows that the support is, for instance $\gamma' > -\gamma_0 \rightarrow \gamma' + \gamma_0 > 0$, Eq. (3.24) can be written as

$$f_3(p,k) = \int_0^{\infty} d\gamma' \int_{-1}^{1} dz' \frac{g^{(1)}(\gamma', z')}{k^2 + z' p \cdot k - \gamma' - \gamma_0 + i\epsilon}. \tag{3.26}$$

Finally, a very important mathematical property of the PTIR is the uniqueness of their weight function (NAKANISHI, 1971). Specifically, if two quantum amplitudes are equal, then its weight functions must also be equal

$$\int_0^{\infty} d\gamma' \int_{-1}^{1} dz' \frac{g'^{(1)}(\gamma', z')}{k^2 + z' p \cdot k - \gamma' - \gamma_0 + i\epsilon} = \int_0^{\infty} d\gamma' \int_{-1}^{1} dz' \frac{g^{(1)}(\gamma', z')}{k^2 + z' p \cdot k - \gamma' - \gamma_0 + i\epsilon}$$
$$\Rightarrow g'^{(1)}(\gamma', z') = g^{(1)}(\gamma', z'). \tag{3.27}$$

In his book, Nakanishi aimed to have the PTIR with exponent $n = 1$ always, such that the uniqueness theorem was proven for $n = 1$ only. However, with the BSE it is useful to be able to choose what exponent n in the PTIR better fits a given problem. So, note that the uniqueness property can be extended to different orders n because the weight functions are related by integration by parts

$$\int_0^\infty d\gamma' \int_{-1}^1 dz' \frac{g'^{(1)}(\gamma',z')}{k^2+z'p\cdot k-\gamma'-\gamma_0+i\epsilon} = \int_0^\infty d\gamma' \int_{-1}^1 dz' \frac{g^{(1)}(\gamma',z')}{k^2+z'p\cdot k-\gamma'-\gamma_0+i\epsilon}$$
$$\int_0^\infty d\gamma' \int_{-1}^1 dz' \frac{g'^{(n)}(\gamma',z')}{(k^2+z'p\cdot k-\gamma'-\gamma_0+i\epsilon)^n} = \int_0^\infty d\gamma' \int_{-1}^1 dz' \frac{g^{(n)}(\gamma',z')}{(k^2+z'p\cdot k-\gamma'-\gamma_0+i\epsilon)^n}$$
$$\Rightarrow g'^{(n)}(\gamma',z') = g^{(n)}(\gamma',z'). \tag{3.28}$$

Naturally, when one applies a LF projection on both sides of (3.28), its uniqueness property is still valid. Thus, it still can be used in the framework of the light-front projected Nakanishi representation. This theorem is an important tool to solve problems by removing completely the dependency of the equation towards the momentum variables k and p; and will be used in this thesis similarly to how it was already used in (FREDERICO *et al.*, 2014).

3.5 Integration-by-parts relations

The three leg half-off-shell amplitude BS amplitude can be written as

$$\phi(k,p) = \int_0^\infty d\gamma' \int_{-1}^1 dz' \frac{g^{(n)}(\gamma',z')}{(k^2+p\cdot kz'-\gamma'-\kappa^2+i\epsilon)^n}, \tag{3.29}$$

where n is a variable to be chosen and $g^{(n)}(\gamma',z')$ is the corresponding weight-function. A natural question that appears is how to relate $g^{(n)}(\gamma',z')$ with $g^{(m)}(\gamma',z')$ for given n and m. The way to investigative these relations is to explore the integration by parts in the parameters γ' and z'.

3.5.1 Integration by parts in the γ' parameter

To perform the integration by parts, one can choose to differentiate the weight-function and integrate the denominator or the opposite. We advocate here to integrate the weight-function and differentiate the denominator, because in this way we can control better the boundary term in the integration by parts and isolate the relationship between $g^{(n)}(\gamma',z')$ and $g^{(n+1)}(\gamma',z')$. More details follows below:

$$
\begin{aligned}
&\int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{g^{(n)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^n} \\
&+ n\int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{\int_{\gamma_0}^{\gamma'} d\gamma'' g^{(n)}(\gamma'', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^{n+1}} \\
&= \left(\int_{-1}^{1} dz' \frac{1}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^n} \int_{\gamma_0}^{\gamma'} d\gamma'' g^{(n)}(\gamma'', z') \right) \Bigg|_{\gamma'=0}^{\gamma'=\infty}
\end{aligned} \tag{3.30}
$$

The γ_0 is a free parameter to define the primitive of $g^{(n)}(\gamma', z')$ and we wish to use this degree of freedom to disappear with the boundary term in the integration by parts. We claim that setting $\gamma_0 = 0$ is the correct value. To verify, let's substitute it in (3.30):

$$
\begin{aligned}
&\left(\int_{-1}^{1} dz' \frac{1}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^n} \int_{0}^{\gamma'} d\gamma'' g^{(n)}(\gamma'', z') \right) \Bigg|_{\gamma'=0}^{\gamma'=\infty} \\
&= \left(\int_{-1}^{1} dz' \frac{1}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^n} \int_{0}^{\gamma'} d\gamma'' g^{(n)}(\gamma'', z') \right) \Bigg|_{\gamma'=\infty}
\end{aligned} \tag{3.31}
$$

In fact, any $\gamma_0 < 0$ would work because the support of $g^{(n)}(\gamma', z')$ is $\gamma' \geq 0$, but $\gamma_0 = 0$ is enough. This last term will be zero if the integral $\int_0^{\gamma'} d\gamma'' g^{(n)}(\gamma'', z')$ grows slower than the decay of the denominator $\frac{1}{(\gamma' + \kappa^2 - k^2 - p\cdot kz' - i\epsilon)^n}$. Explicitly, this is true if $g^{(n)}(\gamma', z') = \mathcal{O}(\gamma'^n)$. Using induction, it is easy to see that we only need that $g^{(1)}(\gamma', z') = \mathcal{O}(\gamma')$. This is reasonable to expect because if $g^{(1)}$ increases faster than γ' we would have an amplitude which increases at least logarithmically in the momenta, and that wouldn't be acceptable physically. Finally, we remove the boundary term and obtain

$$
\begin{aligned}
\int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{g^{(n)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^n} &= -n\int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{\int_{0}^{\gamma'} d\gamma'' g^{(n)}(\gamma'', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^{n+1}} \\
g^{(n+1)}(\gamma', z') &= -n\int_{0}^{\gamma'} d\gamma'' g^{(n)}(\gamma'', z'), \\
\frac{\partial}{\partial\gamma'} g^{(n+1)}(\gamma', z') &= -n(g^{(n)}(\gamma', z')).
\end{aligned} \tag{3.32}
$$

This shows us that $g^{(n)}$ is smoother as n grows since it is given by integrals. Otherwise, trying to make n smaller can also make it very singular, sometimes even an distribution when you need to differentiate a discontinuous function. For instance, if we take $g^{(2)}(\gamma', z') = \Theta(\gamma')f(z')$ we would need $g^{(1)}(\gamma', z') = \delta(\gamma')f(z')$. This shows that the original strategy pursued by Nakanishi, which consist in transforming every Feynman diagram to weight function $n = 1$ may result in a very singular weight-function for the scattering amplitude. So, we can conclude that in for each problem we should select n that facilitates the solution.

3.5.2 Integration by parts in the z' parameter

It is also possible to integrate by parts in the z' parameter. When we did that with the γ', the useful effect was to change the order of the denominator. However, since z' multiplies $p \cdot k$, this term will be put in evidence when we differentiate the denominator by z', and this can be useful in different types of equations. Moreover, the main method will be the opposite: differentiate the weight-function while integrating the denominator.

$$\begin{aligned}&\int_0^\infty d\gamma' \int_{-1}^1 dz' \frac{g^{(n)}(\gamma',z')}{(\gamma'+\kappa^2-k^2-p\cdot kz'-i\epsilon)^n}\\&+\frac{1}{(p\cdot k)(n-1)}\int_0^\infty d\gamma' \int_{-1}^1 dz' \frac{\partial_{z'}g^{(n)}(\gamma',z')}{(\gamma'+\kappa^2-k^2-p\cdot kz'-i\epsilon)^{n-1}}\\&=\frac{1}{(p\cdot k)(n-1)}\left(\int_0^\infty d\gamma' \frac{1}{(\gamma'+\kappa^2-k^2-p\cdot kz'-i\epsilon)^{n-1}}g^{(n)}(\gamma',z')\right)\Bigg|_{z'=-1}^{z'=1}\end{aligned} \tag{3.33}$$

If $g^{(n)}(\gamma',1) = g^{(n)}(\gamma',-1) = 0$ then the boundary term disappears. However if that is not the case,we can also move this term to under the integral sign in z' with δ functions and rewrite:

$$\begin{aligned}&\int_0^\infty d\gamma' \int_{-1}^1 dz' \frac{g^{(n)}(\gamma',z')}{(\gamma'+\kappa^2-k^2-p\cdot kz'-i\epsilon)^n}\\&=\frac{1}{(p\cdot k)(n-1)}\int_0^\infty d\gamma' \int_{-1}^1 dz' \frac{-\partial_{z'}g^{(n)}(\gamma',z')+g^{(n)}(\gamma',z')(\delta(z'-1)-\delta(z'+1))}{(\gamma'+\kappa^2-k^2-p\cdot kz'-i\epsilon)^{n-1}}.\end{aligned} \tag{3.34}$$

Now, multiplying both sides by $p \cdot k$ we have

$$
\begin{aligned}
&(p\cdot k)\int_0^\infty d\gamma'\int_{-1}^1 dz'\frac{g^{(n)}(\gamma',z')}{(\gamma'+\kappa^2-k^2-p\cdot kz'-i\epsilon)^n}\\
&=\frac{1}{(n-1)}\int_0^\infty d\gamma'\int_{-1}^1 dz'\frac{-\partial_{z'}g^{(n)}(\gamma',z')+g^{(n)}(\gamma',z')(\delta(z'-1)-\delta(z'+1))}{(\gamma'+\kappa^2-k^2-p\cdot kz'-i\epsilon)^{n-1}}\\
&=\int_0^\infty d\gamma'\int_{-1}^1 dz'\frac{\int_0^{\gamma'}d\gamma''(-\partial_{z'}g^{(n)}(\gamma'',z')+g^{(n)}(\gamma'',z')(\delta(z'-1)-\delta(z'+1)))}{(\gamma'+\kappa^2-k^2-p\cdot kz'-i\epsilon)^n}\\
&=n\int_0^\infty d\gamma'\int_{-1}^1 dz'\frac{-\partial_{z'}g^{(n+1)}(\gamma',z')+g^{(n+1)}(\gamma',z')(\delta(z'-1)-\delta(z'+1))}{(\gamma'+\kappa^2-k^2-p\cdot kz'-i\epsilon)^n}.
\end{aligned} \tag{3.35}
$$

We now understand what we gain by studying the integration by parts in z': we learn how to deal with a multiplication by $p\cdot k$ outside the sign of the integral. Now, whenever we have an $p\cdot k$ multiplying a Nakanishi PITR, we know that it acts on the weight-function as $g^{(n)}(\gamma',z')\to\int_0^{\gamma'}d\gamma''(-\partial_{z'}g^{(n)}(\gamma'',z')+g^{(n)}(\gamma'',z')(\delta(z'-1)-\delta(z'+1)))$. This will be useful to deal with the BS equation directly with the uniqueness, without using the light-front projection.

3.5.3 Multiplication by k^2

Let's use what we learned in the previous sections to deal with the problem of multiplying by k^2 a Nakanishi PITR

$$
k^2\int_0^\infty d\gamma'\int_{-1}^1 dz'\frac{g^{(n)}(\gamma',z')}{(\gamma'+\kappa^2-k^2-p\cdot kz'-i\epsilon)^n}. \tag{3.36}
$$

To solve it, we must first rewrite $k^2=-(\gamma'+\kappa^2-k^2-p\cdot kz')+\gamma'+\kappa^2-z'p\cdot k$, substitute it in (3.36), and use the previous results on multiplication by $p\cdot k$

$$
\begin{aligned}
&k^2 \int_0^\infty d\gamma' \int_{-1}^{1} dz' \frac{g^{(n)}(\gamma', z')}{(\gamma' + \kappa^2 - k^2 - p \cdot kz' - i\epsilon)^n} \\
&= \int_0^\infty d\gamma' \int_{-1}^{1} dz' \left(\frac{-1}{(\gamma' + \kappa^2 - k^2 - p \cdot kz' - i\epsilon)^{n-1}} + \frac{\gamma' + \kappa^2 - z'p \cdot k}{(\gamma' + \kappa^2 - k^2 - p \cdot kz' - i\epsilon)^n} \right) g^{(n)}(\gamma', z') \\
&= \int_0^\infty d\gamma' \int_{-1}^{1} dz' (\frac{-(n-1)\int_0^{\gamma'} d\gamma'' g^{(n)}(\gamma'', z') + (\gamma' + \kappa^2) g^{(n)}(\gamma', z')}{(\gamma' + \kappa^2 - k^2 - p \cdot kz' - i\epsilon)^n} \\
&+ \frac{-z'p \cdot k(g^{(n)}(\gamma', z'))}{(\gamma' + \kappa^2 - k^2 - p \cdot kz' - i\epsilon)^n}) \\
&= \int_0^\infty d\gamma' \int_{-1}^{1} dz' (\frac{-(n-1)\int_0^{\gamma'} d\gamma'' g^{(n)}(\gamma'', z') + (\gamma' + \kappa^2) g^{(n)}(\gamma', z')}{(\gamma' + \kappa^2 - k^2 - p \cdot kz' - i\epsilon)^n} \\
&+ \frac{(\partial_{z'} - \delta(z-1) + \delta(z+1)) \int_0^{\gamma'} d\gamma'' (z' g^{(n)}(\gamma'', z'))}{(\gamma' + \kappa^2 - k^2 - p \cdot kz' - i\epsilon)^n}).
\end{aligned}
\tag{3.37}
$$

Finally, to simplify the notation let us define the operator $\int^{\gamma'}$ that acts on a weight-function as $(\int^{\gamma'}) g^n(\gamma', z') = \int_0^{\gamma'} d\gamma'' g^n(\gamma'', z')$. Now, we can write the final answer

$$
\begin{aligned}
&k^2 \int_0^\infty d\gamma' \int_{-1}^{1} dz' \frac{g^{(n)}(\gamma', z')}{(\gamma' + \kappa^2 - k^2 - p \cdot kz' - i\epsilon)^n} \\
&= \int_0^\infty d\gamma' \int_{-1}^{1} dz' \left(\frac{(\gamma' + \kappa^2 - ((n-1) + \partial_{z'} + \delta(z-1) - \delta(z+1)) z' \int^{\gamma'}) g^{(n)}(\gamma', z')}{(\gamma' + \kappa^2 - k^2 - p \cdot kz' - i\epsilon)^n} \right).
\end{aligned}
\tag{3.38}
$$

So, we discover that multiplying by k^2 outside the integral sign is equivalent to act on the weight-function with the operator $(\gamma' + \kappa^2 - ((n-1) + \partial_{z'} + \delta(z-1) - \delta(z+1)) z' \int^{\gamma'})$ under the integral sign.

3.6 Light Front Variables

The Nakanishi PTIR gives a compact way to represent an amplitude, however there still is the problem of how to deal numerically with the poles in the denominator. One possible approach described in this thesis is to extract the denominator from both sides of the equation using the uniqueness theorem, and arrive at an equation with only the Nakanishi variables γ' and z'. Another strategy is to use the light-front projection, i.e., the integral $\int dk^-$, because it eliminates the pole, arriving at a strictly positive denominator.

The Light-Front framework was introduced by Dirac and consists in a new set of coordinates to express four-vectors. For instance, the momentum coordinates are

$$k^- = k_0 - k_3, \tag{3.39}$$

$$k^+ = k_0 + k_3, \tag{3.40}$$

$$k_\perp = (k_1, k_2). \tag{3.41}$$

And after the $\int dk^-$ is performed, we are left only with k^+ and $k_\perp$. Moreover, since they always appear only indirectly as scalar products, let us calculate those with LF variables, in the bound state center of mass frame

$$k^2 = k^- k^+ - k_\perp^2, \tag{3.42}$$

$$p \cdot k = \frac{M}{2}\left(k^- + k^+\right). \tag{3.43}$$

In order to make the final result algebraically simpler, we define the new set of variables $\gamma \in [0,\infty)$,$z \in [-1,1]$ such that

$$k^+ = -z\frac{M}{2}, \tag{3.44}$$

$$k_\perp^2 = \gamma, \tag{3.45}$$

which enable us to write the scalar products as

$$k^2 = \left(-z\frac{M}{2}\right) k^- - \gamma, \tag{3.46}$$

$$p \cdot k = \frac{M}{2}\left(k^-\right) - z\frac{M^2}{4}. \tag{3.47}$$

And with these new variables we can demonstrate explicitly the effect of the LF projection on the Nakanishi PTIR

$$\int dk^- \int d\gamma' \int_{-1}^{1} dz' \frac{g_i^{(4)}(\gamma', z')}{(k^2 + p \cdot kz' - \gamma' - \kappa^2 + i\epsilon)^4}$$
$$= -\frac{2\pi i}{3M} \int d\gamma' \frac{g_i^{(4)}(\gamma', z)}{(\gamma + \gamma' + \kappa^2 + \frac{M^2}{4} z^2)^3}. \tag{3.48}$$

As we can see, the LF projection transforms a denominator with poles to one strictly positive. So, now it is possible to tackle the BS integral equation with ordinary methods, such as basis expansion and numerical quadrature.

4 Wick-Cutkosky Model Revisited

4.1 Introduction

The Bethe-Salpeter (BS) equation was proposed to treat the problem of the bound state in a relativistic framework. However, since its kernel has poles, the numerical solution can not be obtained using standard numerical algorithms for integral equations such as quadrature methods used for euclidean amplitudes. In order to solve this difficulty, in 1954 Wick and Cutkosky published articles proposing new methods (WICK, 1954; CUTKOSKY, 1954) . In particular, the Wick rotation avoids the poles and, since then, this has been the canonical way to solve the BS equation. Nevertheless, Wick also proposed the strategy of using a particular type of integral representation for the BS amplitude for a massless interaction, and using a Dirac delta ansatz for his separable weight function he was able to obtain a simpler equation for the massless case. Moreover, Cutkosky extended Wick's work to the full spectrum of the massless ladder interaction case containing all possible angular momentum values. This model has been known as the Wick-Cutkosky model, specially important as a toy model since it gives a simple integral equation on only one parameter.

In this chapter we present new methods to solve this model using the Nakanishi PTIR. Firstly, a new strategy is presented based on transforming the γ' integral into a sum, using integration by parts, which was published in (PIMENTEL; PAULA, 2016). Secondly, two new approaches are developed using the uniqueness theorem of the Nakanishi PTIR and integration by parts, which gives both an integral and a differential equation for the full s-wave spectrum, purely in the space of the Nakanishi parameters γ' and z'.

4.2 Wick-Cutkosky Model with the Nakanishi Representation

The starting point is the scalar BS equation with a ladder massless interaction, given by

$$\phi(k,p) = \frac{1}{m^2 - (\frac{p}{2}-k)^2}\frac{1}{m^2 - (\frac{p}{2}+k)^2} ig^2 \int \frac{d^4k'}{(2\pi)^4}\frac{\phi(k',p)}{(k-k')^2 + i\epsilon}, \tag{4.1}$$

where g is the coupling constant, p is the total momentum of the bound state and k is the relative momentum.

The s-wave BS amplitude $\phi(k,p)$ can be written as a three leg Nakanishi representation with one momentum on-shell (p) and two off-shell ($\frac{p}{2}-k$ and $\frac{p}{2}+k$),

$$\phi(k,p) = \frac{-i}{4\pi}\int_0^\infty d\gamma' \int_{-1}^1 dz' \frac{g(\gamma',z')}{(\gamma' + \kappa^2 - k^2 - p.kz' - i\epsilon)^3}. \tag{4.2}$$

Note that the symmetry $\phi(k_0,\vec{k},p) = \phi(-k_0,\vec{k},p)$, such as for an s-wave, is translated as $g(\gamma',z') = g(\gamma',-z')$ for the weight function in this representation. Substituting this representation in the previous BS equation and projecting the equation in the light-front by integrating in k^- we obtain from (KARMANOV, 2006)

$$\int_0^\infty d\gamma' \frac{g(\gamma',z)}{\left(\gamma' + \gamma + z^2m^2 + (1-z^2)\kappa^2\right)^2} = \int_0^\infty d\gamma' \int_{-1}^1 dz'\ V(\gamma,z;\gamma',z')g(\gamma',z'). \tag{4.3}$$

The Kernel of the integral equation is

$$\begin{aligned} V(\gamma,z;\gamma',z') = {} & \frac{\alpha m^2}{2\pi}\frac{1}{(\gamma + z^2m^2 + (1-z^2)\kappa^2)}\frac{1}{(\gamma' + z'^2m^2 + (1-z'^2)\kappa^2)} \\ & \times \left(\frac{\theta(z-z')}{\left(\gamma + \gamma'\frac{1-z}{1-z'} + z^2m^2 + (1-z^2)\kappa^2\right)}\frac{1-z}{1-z'} + \frac{\theta(z'-z)}{\left(\gamma + \gamma'\frac{1+z}{1+z'} + z^2m^2 + (1-z^2)\kappa^2\right)}\frac{1+z}{1+z'} \right), \end{aligned} \tag{4.4}$$

where we defined $\alpha = g^2/4\pi$. The Bethe-Salpeter equation can be rewritten in a suitable way as

$$\int_0^\infty d\gamma' \frac{g(\gamma',z)}{\left(\gamma' + \gamma + z^2m^2 + (1-z^2)\kappa^2\right)^2} = \frac{\alpha m^2}{2\pi}\frac{1}{d_0(z)}\int_{-1}^1 dz' \left[\omega(z,z')\theta(z-z') + \omega(-z,-z')\theta(z'-z)\right], \tag{4.5}$$

where we introduced the auxiliary functions

$$\omega(z,z') = \int_0^\infty d\gamma' \frac{g(\gamma',z')}{(\gamma'+a(z'))(\gamma'+c(z,z'))}; \quad d_0(z) = \gamma + z^2m^2 + (1-z^2)\kappa^2;$$
$$a(z') = z'^2m^2 + (1-z'^2)\kappa^2; \quad c(z,z') = \frac{1-z'}{1-z}d_0(z). \tag{4.6}$$

Now, we intend to transform the integration in γ' into a sum, using integration by parts. To do that, we first define the sequence $G^{(n)}(\gamma',z)$ as

$$G^{(0)}(\gamma',z) = g(\gamma',z); \quad G^{(n+1)}(\gamma',z) = -\int_{\gamma'}^\infty d\gamma''\, G^{(n)}(\gamma'',z); \tag{4.7}$$
$$G^{(n)}(\gamma',z) = \frac{\partial}{\partial\gamma'}G^{(n+1)}(\gamma',z). \tag{4.8}$$

Please note that the index n in $G^{(n)}(\gamma',z)$ is only used to index the sequence and is unrelated to the n used to denote the order of the weight-function $g^{(n)}(\gamma',z')$ in the previous chapter.

The strategy to be developed here is to expand de $d\gamma'$ integral as a sum. So, we perform an integration by parts in γ' in the LHS of eq. (4.3)

$$\int_0^\infty d\gamma' \frac{g(\gamma',z)}{(\gamma'+\gamma+z^2m^2+(1-z^2)\kappa^2)^2} = \int_0^\infty d\gamma' \frac{g(\gamma',z)}{(\gamma'+d_0(z))^2} \tag{4.9}$$
$$= \int_0^\infty d\gamma' \left(\frac{\partial}{\partial\gamma'}G^{(1)}(\gamma',z)\right)\frac{1}{(\gamma'+d_0(z))^2}$$
$$= G^{(1)}(0,z)\frac{1}{d_0(z)^2} \tag{4.10}$$
$$- \int_0^\infty d\gamma'\, G^{(1)}(\gamma',z)\frac{\partial}{\partial\gamma'}\left(\frac{1}{(\gamma'+d_0(z))^2}\right). \tag{4.11}$$

Each integration by parts generates a new term for the sum. In general, the relationship that generates the n-th term of the sum is

$$\int_0^\infty d\gamma'\, G^{(n)}(\gamma',z)\frac{\partial^{(n)}}{\partial\gamma'^{(n)}}\left(\frac{1}{(\gamma'+d_0(z))^2}\right) = \int_0^\infty d\gamma' \left(\frac{\partial}{\partial\gamma'}G^{(n+1)}(\gamma',z)\right)\frac{\partial^{(n)}}{\partial\gamma'^{(n)}}\frac{1}{(\gamma'+d_0(z))^2}$$
$$= \left(G^{(n+1)}(\gamma',z)\frac{\partial^{(n)}}{\partial\gamma'^{(n)}}\left(\frac{1}{(\gamma'+d_0(z))^2}\right)\right)\Bigg|_0^\infty - \int_0^\infty d\gamma'\, G^{(n+1)}(\gamma',z)\frac{\partial^{(n+1)}}{\partial\gamma'^{(n+1)}}\left(\frac{1}{(\gamma'+d_0(z))^2}\right). \tag{4.12}$$

Thus, after performing n integrations by parts we have

$$\int_0^\infty d\gamma' \frac{g(\gamma',z)}{\left(\gamma'+\gamma+z^2m^2+(1-z^2)\kappa^2\right)^2} =$$
$$= \sum_{i=1}^{n}(-1)^{i-1}\left(G^{(i)}(\gamma',z)\frac{\partial^{(i-1)}}{\partial\gamma'^{(i-1)}}\left(\frac{1}{\left(\gamma'+d_0(z)\right)^2}\right)\right)\Big|_0^\infty \tag{4.13}$$
$$+ \quad (-1)^n \int_0^\infty d\gamma'\ G^{(n)}(\gamma',z)\frac{\partial^{(n)}}{\partial\gamma'^{(n)}}\left(\frac{1}{\left(\gamma'+d_0(z)\right)^2}\right). \tag{4.14}$$

The remaining term in the limit of large n vanishes and each term of the sum is proportional to $G^{(i)}(0,z)$. So, we define this term as a sequence of functions in the variable z

$$b_i(z) = G^{(i)}(0,z). \tag{4.15}$$

Finally, in the limit of $n \to \infty$

$$\int_0^\infty d\gamma' \frac{g(\gamma',z)}{\left(\gamma'+\gamma+z^2m^2+(1-z^2)\kappa^2\right)^2} = \sum_{i=1}^{\infty} i!\frac{b_i(z)}{d_0(z)^{i+1}}, \tag{4.16}$$

which eliminates the $d\gamma'$ integral and transforms the LHS of (4.3) into a sum. We now intend to do the same for the RHS:

$$
\begin{aligned}
&\int_0^\infty d\gamma' \frac{g(\gamma', z')}{(\gamma' + a(z'))(\gamma' + c(z))} \\
= &\sum_{j=1}^{\infty}(-1)^{j-1}\left(G^{(j)}(\gamma', z')\frac{\partial^{(j-1)}}{\partial\gamma'^{(j-1)}}\left(\frac{1}{(\gamma' + a(z'))\,(\gamma' + c(z, z'))}\right)\right)\Bigg|_0^\infty \\
= &\sum_{j=1}^{\infty}(-1)^{j-1}b_j(z')\sum_{n=0}^{j-1}\binom{j-1}{n}\frac{\partial^{(j-n-1)}}{\partial\gamma'^{(j-n-1)}}\left(\frac{1}{\gamma' + a(z')}\right)\frac{\partial^{(n)}}{\partial\gamma'^{(n)}}\left(\frac{1}{\gamma' + c(z, z')}\right) \\
= &\sum_{j=1}^{\infty}b_j(z')\sum_{n=0}^{j-1}\frac{(j-1)!}{n!(j-n-1)!}\frac{(j-n-1)!}{a(z')^{j-n}}\frac{n!}{c(z, z')^{n+1}} \\
= &\sum_{j=1}^{\infty}b_j(z')\sum_{n=0}^{j-1}\frac{(j-1)!}{a(z')^{j-n}c(z, z')^{n+1}} \\
= &\sum_{i=1}^{\infty}\frac{1}{c(z)^i}\sum_{j=i}^{\infty}\frac{(j-1)!b_j(z')}{a(z')^{j-i+1}} \\
= &\sum_{i=1}^{\infty}\left(\frac{1-z}{(1-z')d_0(z)}\right)^i\sum_{j=i}^{\infty}\frac{(j-1)!b_j(z')}{a(z')^{j-i+1}}. \qquad (4.17)
\end{aligned}
$$

Collecting the terms, the BS equation for the Wick-Cutkosky model can be expressed as

$$
\begin{aligned}
\sum_{i=1}^{\infty} i!\frac{b_i(z)}{d_0(z)^{i+1}} &= \frac{\alpha m^2}{2\pi}\frac{1}{d_0(z)}\int_{-1}^{1}dz'\left(\sum_{i=1}^{\infty}\frac{1-z}{(1-z')d_0(z)}\right)^i\sum_{j=i}^{\infty}\frac{(j-1)!b_j(z')}{a(z')^{j-i+1}}\theta(z-z') \\
&+ \quad z\to -z \text{ and } z'\to -z'. \qquad (4.18)
\end{aligned}
$$

Note that for $\gamma \to \infty \Rightarrow d_0 \to \infty$ and therefore the small i terms dominates the series. Then we can match the $\frac{1}{d_0^i}$ terms of the series and we have, since $a(z)$ and $b_i(z)$ are even:

$$
b_i(z) = \frac{\alpha m^2}{2\pi}\int_{-1}^{1}dz'\frac{1}{i!}\left(\left(\frac{1-z}{1-z'}\right)^i\theta(z-z') + \left(\frac{1+z}{1+z'}\right)^i\theta(z'-z)\right)\sum_{j=i}^{\infty}\frac{(j-1)!b_j(z')}{a(z')^{j-i+1}}. \qquad (4.19)
$$

To solve the eigenequation we search for solutions where $b_i(z) = 0$ if $i > i_{max}$ for some i_{max}. Also, since the resulting matrix is triangular by blocks, the eigenvalues are solely determined by the $i = j = i_{max}$ equations in the diagonal. Therefore, they determined by solving the equations

$$
b_i(z) = \frac{\alpha m^2}{2\pi}\int_{-1}^{1}dz'\frac{1}{i}\left(\left(\frac{1-z}{1-z'}\right)^i\theta(z-z') + \left(\frac{1+z}{1+z'}\right)^i\theta(z'-z)\right)\frac{b_i(z')}{a(z')}, \qquad (4.20)
$$

which is exactly the Wick-Cutkosky equation originally obtained. Moreover, to reconstruct the eigenfunction one needs the full set of equations for $i \leq i_{max}$.

Previously it was used the Ansatz $g(\gamma', z') = \delta(\gamma')f(z')$ (KARMANOV, 2006; WICK, 1954), which is equivalent to set $b_i(z') = 0$ for $i > 1$ in the present formalism. Therefore, in order to extend this idea for the full s-wave spectrum, and be able to reconstruct $g(\gamma', z')$ from $b_i(z')$ we propose the following Ansatz

$$g(\gamma', z') = \sum_{i=1}^{i_{max}} b_i(z')\delta^{(i-1)}(\gamma'), \tag{4.21}$$

where the $\delta^{(i-1)}(\gamma') = \frac{\partial^{i-1}}{\partial \gamma'^{i-1}}\delta(\gamma')$.

This distribution has support only in $\gamma' = 0$ and reproduces equations (4.16) and (4.17), because integrating a derivate of a delta gives the derivative of the integrand. Therefore, this reproduces the same effect as the derivatives performed in the integration by parts. So, given the uniqueness theorem this must be the solution for the Nakanishi weight function. Also, note that the support of the distribution is the same as the boundary of the integral ($\gamma' = 0$), which is ill-defined. Therefore, if one wants to be mathematically precise, the γ' integral should start at $-\epsilon$ for any $\epsilon > 0$.

Finally, in order to reconstruct the Bethe-Salpeter amplitude we should use the Nakanishi Perturbative Integral Representation (4.2) and insert the Ansatz (4.21) to give

$$\phi(k, p) = \frac{-i}{4\pi} \sum_{n=1}^{i_{max}} \frac{(n+1)!}{3} \int_{-1}^{1} dz' \frac{b_n(z')}{(\kappa^2 - k^2 - p.kz' - i\epsilon)^{n+2}}. \tag{4.22}$$

It is intesting to note that (4.22) is in fact the initial Ansatz used by Cutkosky in (CUTKOSKY, 1954). Moreover, the light-front wave function can also be obtained from (4.22) using the relation (KARMANOV, 2006)

$$\psi(\mathbf{k}_\perp, z) = \frac{(\omega.k_1)(\omega.k_2)}{\pi(\omega.p)} \int_{-\infty}^{\infty} \phi(k + \beta\omega, p)d\beta, \tag{4.23}$$

which gives

$$\psi(\mathbf{k}_\perp, z) = \frac{1 - z^2}{8\sqrt{\pi}} \sum_{i=1}^{i_{max}} i! \frac{b_i(z)}{(\mathbf{k}_\perp^2 + m^2 - (1 - z^2)\frac{M^2}{4})^{i+1}}. \tag{4.24}$$

4.3 Numerical Results

In this section we compare the numerical values of the first four eigenvalues by solving the Bethe-Salpeter equation for zero mass exchange (4.3) and the ones obtained from eq.

(4.20), obtained using the method described in the last section.

First, we solve the light-front projected BS equation (4.3) using a Laguerre basis in γ and γ', and a Legendre basis in z , with 5 basis functions in γ and 5 in z, and obtain the eigenvalues of the ground state ($\alpha_1^{(\gamma',z')}$) and the first excited state ($\alpha_2^{(\gamma',z')}$) for various bounding energies B/m. Then, we solve eq. (4.20) using a Legendre basis in z obtaining the eigenvalues of the ground state ($\alpha_1^{(z')}$) and the first excited state ($\alpha_2^{(z')}$). The results are presented in table 4.1.

TABLE 4.1 – Values of $\alpha = g^2/4\pi$ as a function of binding energy B/m for a massless scalar exchange for the ground state and the first excited state. The first and third column shows the results for eq. (4.3). The second and fourth column shows the results for eq. (4.20).

B/m	$\alpha_1^{(\gamma',z')}$	$\alpha_1^{(z')}$	$\alpha_2^{(\gamma',z')}$	$\alpha_2^{(z')}$
0.1	1.11	1.12	2.90	2.93
0.2	1.78	1.79	4.90	4.85
0.3	2.34	2.35	6.58	6.53
0.4	2.84	2.84	8.09	8.05
0.5	3.29	3.30	9.44	9.42

For the ground state and the first excited state the numerical results has a margin of error of about 1%, which supports the method presented in this work.

4.4 Wick-Cutkosky Model with the Uniqueness Method onto the Light Front

The Wick-Cutkosky Model equations can also be obtained with the so called uniqueness method, which consists in a type of analytic inversion of the operator in the LHS of the Nakanishi BSE. Consequentially, the momentum variables disappears from the equation and we are left with a simple eigenequation, when we were dealing previously with a generalized eigenequation relating only the Nakanishi PTIR parameters γ' and z'. The basic idea is to factorize the same operator in the left and right handed sides, use the uniqueness theorem and remove it from the equation. Summarizing

$$\begin{aligned} \int_0^\infty d\gamma' \frac{g_1(\gamma',z')}{(\gamma+\gamma'+z^2m^2+(1-z^2)\kappa^2)^2} &= \int_0^\infty d\gamma' \frac{g_2(\gamma',z')}{(\gamma+\gamma'+z^2m^2+(1-z^2)\kappa^2)^2} \\ \Rightarrow g_1(\gamma',z') &= g_2(\gamma',z'). \end{aligned} \tag{4.25}$$

Let's restart the calculation from the following massless BSE with the Nakanishi Representation projected in the Light-Front

$$\int_0^\infty d\gamma' \frac{g(\gamma',z)}{(\gamma'+\gamma+z^2m^2+(1-z^2)\kappa^2)^2} = \int_0^\infty d\gamma' \int_{-1}^{1} dz' V(\gamma,z;\gamma',z')g(\gamma',z'), \quad (4.26)$$

in which the kernel $V(\gamma,z,\gamma',z')$ is

$$V(\gamma,z,\gamma',z') = \frac{\alpha m^2}{2\pi}\frac{1}{(\gamma+z^2m^2+(1-z^2)\kappa^2)}\frac{1}{(\gamma'+z'^2m^2+(1-z'^2)\kappa^2)}$$
$$\times\left(\frac{\theta(z-z')}{\left(\gamma+\gamma'\frac{1-z}{1-z'}+z^2m^2+(1-z^2)\kappa^2\right)}\frac{1-z}{1-z'}+\frac{\theta(z'-z)}{\left(\gamma+\gamma'\frac{1+z}{1+z'}+z^2m^2+(1-z^2)\kappa^2\right)}\frac{1+z}{1+z'}\right), \quad (4.27)$$

where we defined $\alpha = g^2/4\pi$.

Note that the kernel V has, in fact, two different denominators $(\gamma+\gamma'\frac{1-z}{1-z'}+z^2m^2+(1-z^2)\kappa^2)$ and $(\gamma+\gamma'\frac{1+z}{1+z'}+z^2m^2+(1-z^2)\kappa^2)$, and we need to reduce the γ dependence into only one denominator in order to apply the uniqueness method. Also, it is necessary to eliminate the z' dependence in the denominator so that we can take it out of the dz' integration. To facilitate it, we can rewrite (4.27) as

$$V(\gamma,z,\gamma',z') = W(\gamma,z,\gamma',z')\Theta(z-z') + W(\gamma,-z,\gamma',-z')\Theta(z'-z), \quad (4.28)$$

where,

$$W(\gamma,z,\gamma',z') = \frac{\alpha m^2}{2\pi}\frac{1}{(\gamma+z^2m^2+(1-z^2)\kappa^2)}\frac{1}{(\gamma'+z'^2m^2+(1-z'^2)\kappa^2)}$$
$$\times\left(\frac{\theta(z-z')}{\left(\gamma+\gamma'\frac{1-z}{1-z'}+z^2m^2+(1-z^2)\kappa^2\right)}\frac{1-z}{1-z'}\right) \quad (4.29)$$

Let's take the first kernel term $W(\gamma,z,\gamma',z')\Theta(z-z')$ and change the integration variable $\gamma'' \to \gamma' \frac{1-z}{1-z'}$ such that

$$\int_0^\infty d\gamma' \int_{-1}^1 dz' W(\gamma, z, \gamma', z')\Theta(z-z')g(\gamma', z') = \int_0^\infty d\gamma'' \int_{-1}^1 dz' \frac{V'(\gamma, z, \gamma'')\Theta(z-z')g(\gamma'' \frac{1-z'}{1-z}, z')}{(\gamma'' \frac{1-z'}{1-z} + z'^2 m^2 + (1-z'^2)\kappa^2)}, \tag{4.30}$$

where

$$V'(\gamma, z, \gamma') = \frac{\alpha m^2}{2\pi} \frac{1}{\gamma + z^2 m^2 + (1-z^2)\kappa^2} \frac{1}{\gamma + \gamma' + z^2 m^2 + (1-z^2)\kappa^2}. \tag{4.31}$$

The important thing to note here is that the $V'(\gamma, z, \gamma'')$ does depends on z', thus it can be taken out of the dz' integral. We can now do the same for the $W(\gamma, -z, \gamma', z')\Theta(z'-z)$ term and make the change of integration variable $\gamma'' \to \gamma' \frac{1+z}{1+z'}$ which enables us to rewrite, noting that $V'(\gamma, z, \gamma') = V'(\gamma, -z, \gamma')$

$$\begin{aligned} &\int_0^\infty d\gamma' \int_{-1}^1 dz' W(\gamma, -z, \gamma', -z')\Theta(z'-z)g(\gamma', z') \\ = &\int_0^\infty d\gamma'' \int_{-1}^1 dz' \frac{V'(\gamma, z, \gamma'')\Theta(z'-z)g(\gamma'' \frac{1+z'}{1+z}, z')}{(\gamma'' \frac{1+z'}{1+z} + z'^2 m^2 + (1-z'^2)\kappa^2)}. \end{aligned} \tag{4.32}$$

Summarizing, we can rewrite,

$$\begin{aligned} RHS &= \int_0^\infty d\gamma' \int_{-1}^1 dz' V(\gamma, z; \gamma', z') g(\gamma', z') \\ &= \int_0^\infty d\gamma'' V'(\gamma, z, \gamma'') \int_{-1}^1 dz' \Bigg(\frac{\Theta(z-z')g(\gamma'' \frac{1-z'}{1-z}, z')}{\gamma'' \frac{1-z'}{1-z} + z'^2 m^2 + (1-z'^2)\kappa^2} \\ &+ \frac{\Theta(z'-z)g(\gamma'' \frac{1+z'}{1+z}, z')}{\gamma'' \frac{1+z'}{1+z} + z'^2 m^2 + (1-z'^2)\kappa^2} \Bigg). \end{aligned} \tag{4.33}$$

Now that we extracted V' to outside the dz' integral, we must combine the denominators in V' using a Feynman parametrization

$$V'(\gamma, z, \gamma') = \frac{\alpha m^2}{2\pi} \int_0^1 \frac{d\xi}{(\gamma + z^2 m^2 + (1-z^2)\kappa^2 + \gamma'' \xi)^2}. \tag{4.34}$$

Finally, the last step is to remove the ξ out of the denominator. To do that, we make a final variable transformation $\gamma''' \to \gamma'' \xi$ and now we are able to isolate the desired denominator to apply the uniqueness method

$$
\begin{aligned}
RHS = & \int_0^\infty \frac{d\gamma'''}{(\gamma + \gamma''' + z^2m^2 + (1-z^2)m^2)^2} \\
& \times \int_{-1}^1 dz' \int_0^1 d\xi \left(\frac{\alpha m^2}{2\pi} \frac{1}{\xi} \right) \\
& \times \left(\frac{\Theta(z-z')g(\frac{\gamma'''(1-z')}{\xi(1-z)}, z')}{\frac{\gamma'''(1-z')}{\xi(1-z)} + z'^2m^2 + (1-z'^2)\kappa^2} + \frac{\Theta(z'-z)g(\frac{\gamma'''(1+z')}{\xi(1+z)}, z')}{\frac{\gamma'''(1+z')}{\xi(1+z)} + z'^2m^2 + (1-z'^2)\kappa^2} \right). \qquad (4.35)
\end{aligned}
$$

Now we can write $LHS = RHS$ and eliminate the common operator. With this we obtain the final massless Bethe-Salpeter equation with the uniqueness method

$$
\begin{aligned}
g(\gamma''', z) = & \int_{-1}^1 dz' \int_0^1 d\xi \left(\frac{\alpha m^2}{2\pi} \frac{1}{\xi} \right) \\
& \times \left(\frac{\Theta(z-z')g(\frac{\gamma'''(1-z')}{\xi(1-z)}, z')}{\frac{\gamma'''(1-z')}{\xi(1-z)} + z'^2m^2 + (1-z'^2)\kappa^2} + \frac{\Theta(z'-z)g(\frac{\gamma'''(1+z')}{\xi(1+z)}, z')}{\frac{\gamma'''(1+z')}{\xi(1+z)} + z'^2m^2 + (1-z'^2)\kappa^2} \right). \qquad (4.36)
\end{aligned}
$$

To check the consistency of this equation towards the previous integration by parts method, we substitute the known ground state solution $g(\gamma', z') = \delta(\gamma')b_1(z')$. Consequentially, we substitute $g(\frac{\gamma'''(1-z')}{\xi(1-z)}, z') = \frac{\delta(\gamma''')\xi(1-z)}{1-z'}b_1(z')$, $g(\frac{\gamma'''(1+z')}{\xi(1+z)}, z') = \frac{\delta(\gamma''')\xi(1+z)}{1+z'}b_1(z')$, thus

$$
\begin{aligned}
\delta(\gamma''')b_1(z) = & \int_{-1}^1 dz' \int_0^1 d\xi \left(\frac{\alpha m^2}{2\pi} \right) \delta(\gamma''')b_1(z') \\
& \times \left(\frac{\Theta(z-z')(1-z)}{(z'^2m^2 + (1-z'^2)\kappa^2)(1-z')} + \frac{\Theta(z'-z)(1+z)}{(z'^2m^2 + (1-z'^2)\kappa^2)(1+z')} \right), \qquad (4.37)
\end{aligned}
$$

and eliminating $\delta(\gamma''')$ from both sides gives

$$
\begin{aligned}
b_1(z) \;=\; & \left(\frac{\alpha m^2}{2\pi} \right) \int_{-1}^1 dz' b_1(z') \\
\times \; & \left(\frac{\Theta(z-z')(1-z)}{(z'^2m^2 + (1-z'^2)\kappa^2)(1-z')} + \frac{\Theta(z'-z)(1+z)}{(z'^2m^2 + (1-z'^2)\kappa^2)(1+z')} \right), \qquad (4.38)
\end{aligned}
$$

which is the correct equation for the fundamental state of the Wick-Cutkosky model. If one prefers to isolate a kernel for the uniqueness Wick-Cutkosky model, we should make the additional integral variable transformations. For instance, if we make $y \to \frac{\gamma'''(1-z')}{\xi(1-z)}$ the first term in the integral can be changed to

$$
\begin{aligned}
&\int_0^1 d\xi \left(\frac{1}{\xi}\right)\left(\frac{\Theta(z-z')g(\frac{\gamma'''(1-z')}{\xi(1-z)},z')}{\frac{\gamma'''(1-z')}{\xi(1-z)}+z'^2m^2+(1-z'^2)\kappa^2)}\right) \\
= &\int_{-\infty}^{\infty}\frac{dy}{y}\frac{\Theta(y-\frac{\gamma'''(1-z')}{(1-z)})\Theta(z-z')}{y+z'^2m^2+(1-z'^2)\kappa^2}g(y).
\end{aligned}
\tag{4.39}
$$

On the other hand, if we make $y \to \frac{\gamma'''(1+z')}{\xi(1+z)}$ the second term in the integral can be changed to

$$
\begin{aligned}
&\int_0^1 d\xi \left(\frac{1}{\xi}\right)\left(\frac{\Theta(z'-z)g(\frac{\gamma'''(1+z')}{\xi(1+z)},z')}{\frac{\gamma'''(1+z')}{\xi(1+z)}+z'^2m^2+(1-z'^2)\kappa^2)}\right) \\
= &\int_{-\infty}^{\infty}\frac{dy}{y}\frac{\Theta(y-\frac{\gamma'''(1+z')}{(1+z)})\Theta(z'-z)}{y+z'^2m^2+(1-z'^2)\kappa^2}g(y).
\end{aligned}
\tag{4.40}
$$

Finally, we can rewrite the uniqueness Wick-Cutkosky equation as

$$
g(\gamma''',z)=\int_{-1}^{1}dz'\int_{-\infty}^{\infty}dyV^{UNIQ}(\gamma''',z,y,z')g(y,z'),
\tag{4.41}
$$

where

$$
V^{UNIQ}(\gamma''',z,y,z')=\frac{\alpha m^2}{2\pi}\frac{\Theta(y-\frac{\gamma'''(1-z')}{1-z})\Theta(z-z')+\Theta(y-\frac{\gamma'''(1+z')}{1+z})\Theta(z'-z)}{y(y+z'^2m^2+(1-z'^2)\kappa^2)}.
\tag{4.42}
$$

Note that the approach present in this section is similar to one one employed in for the massive case since both of them are based on the uniqueness theorem. The main difference here is that the massless case can be dealt with using only integration by parts, while in that reference the strategy was to employ Feynamn parametrization and delta functions as an intermediate step in order to obtain the same denominator at the right and left hand sides. The main advantage of the method here presented is a better analytical control, since it depends only on integration by parts and not on implicit parametrizations and distributions; while its disavantage is that it is restricted to the massless case.

4.5 A new differential approach to the uniqueness method

In the previous sections, the uniqueness method was used after a light-front projection, integrating the BSE in the variable k^-. However, this was only appropriate because we already started from the light-front projected BSE, otherwise the projection would only be an unnecessary step in the calculation. In this section let us show that it is possible to use the uniqueness method without the LF projection using only integration by parts relations.

Starting from the scalar massless BSE

$$\phi(k,p) = \frac{1}{(\frac{p}{2}-k)^2 - m^2}\frac{1}{(\frac{p}{2}+k)^2 - m^2} ig^2 \int \frac{d^4k'}{(2\pi)^4}\frac{\phi(k',p)}{(k-k')^2 + i\epsilon}, \tag{4.43}$$

We substitute in it the Nakanishi PTIR for the s-wave amplitude $\phi(k,p)$

$$\phi(k,p) = \int d\gamma' \int_{-1}^{1} dz' \frac{g^{(3)}(\gamma',z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^3}, \tag{4.44}$$

and multiply both sides of the BSE by the inverse of the free propagators $(\frac{p}{2} \pm k)^2 - m^2 = k^2 \pm p\cdot k - \kappa^2$, to arrive at a equation of the form

$$\begin{aligned} &(k^2 + p\cdot k - \kappa^2)(k^2 - p\cdot k - \kappa^2) \int d\gamma' \int_{-1}^{1} dz' \frac{g^{(3)}(\gamma',z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^3} \\ &= ig^2 \int \frac{d^4k'}{(2\pi)^4}\frac{1}{(k-k')^2 + i\epsilon} \int d\gamma' \int_{-1}^{1} dz' \frac{g^{(3)}(\gamma',z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^3}. \end{aligned} \tag{4.45}$$

Now we would like to absorb the effect of multiplying by the inverse of the free propagators in the left hand side as an operator acting directly on the weight function. To do that we bring the inverse propagators terms inside the integral:

$$\begin{aligned} &(k^2 + p\cdot k - \kappa^2)(k^2 - p\cdot k - \kappa^2) \int d\gamma' \int_{-1}^{1} dz' \frac{g^{(3)}(\gamma',z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^3} \\ &= \int d\gamma' \int_{-1}^{1} dz' g^{(3)}(\gamma',z') \times \Big(\frac{1}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)} + \\ &+ \frac{2(\gamma' - p\cdot kz')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^2} + \frac{\gamma'^2 - 2\gamma'(p\cdot k)z' - (1-z'^2)(p\cdot k)^2}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^3}\Big). \end{aligned} \tag{4.46}$$

The crucial step now is to use integration by parts in the Nakanishi PTIR to avoid the

$(p \cdot k)$ terms in the numerator and to use only one common denominator $\frac{1}{(k^2+p\cdot kz'-\gamma'-\kappa^2+i\epsilon)}$ instead of three denominators of exponents one, two and three. There are two useful formulas that come from the integration by parts when the boundary term are null:

$$\int d\gamma' \frac{g^{(n)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^n} = \int d\gamma' \frac{-\partial_{\gamma'} g^{(n)}(\gamma', z')/(n-1)}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^{n-1}}, \tag{4.47}$$

$$(p\cdot k)\int_{-1}^{1} dz' \frac{g^{(n)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^n} = \int_{-1}^{1} dz' \frac{\partial_{z'} g^{(n)}(\gamma', z')/(n-1)}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^{n-1}}. \tag{4.48}$$

For the γ' integration by parts, the condition to have a boundary term null is that $\frac{g^{(n)}(\gamma',z')}{(k^2+p\cdot kz'-\gamma'-\kappa^2+i\epsilon)^{n-1}} \to 0$ when $\gamma' \to 0$, and for the z' integration by parts we must have $g^{(n)}(\gamma', 1) = g^{(n)}(\gamma', -1) = 0$. It is important to note that there is a $(p\cdot k)^2$ term in Eq. (4.46), which will need two integration by parts in z'. However the boundary term is zero both times because $(1 - z^2)g^{(3)}(\gamma', z')$ gives a second order zero at $z' = \pm 1$.

Finally, we apply the integration by parts formulas (4.48) in the Eq. (4.46) and have, for the exponents 2 and 3

$$\int d\gamma' \int_{-1}^{1} dz' \frac{2(\gamma' - p\cdot kz')g^{(3)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^2} = \int d\gamma' \int_{-1}^{1} dz' \frac{-2(\partial_{\gamma'}\gamma' + \partial_{z'}z')g^{(3)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)}, \tag{4.49}$$

$$\begin{aligned} &\int d\gamma' \int_{-1}^{1} dz' \frac{(\gamma'^2 - 2\gamma'(p\cdot k)z' - (1 - z'^2)(p\cdot k)^2)g^{(3)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^3} \\ &= \int d\gamma' \int_{-1}^{1} dz' \frac{\frac{1}{2}(\partial^2_{\gamma'}\gamma'^2 + 2\partial_{\gamma'}\partial_{z'}\gamma' z' - \partial^2_{z'}(1 - z'^2))g^{(3)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)}, \end{aligned} \tag{4.50}$$

where the terms in parentheses are the final differential operators. For instance, $(\partial_{z'}z')g^{(3)}(\gamma', z')$ should be read as $\partial_{z'}(z'(g^{(3)}))(\gamma', z')$; the operator "multiply by z' " followed by the operator $\partial_{z'}$. Summing all the terms, we have, for the left hand side of the BSE

$$
\begin{aligned}
&(k^2 + p\cdot k - \kappa^2)(k^2 - p\cdot k - \kappa^2)\int d\gamma' \int_{-1}^{1} dz' \frac{g^{(3)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^3} \\
&= \int d\gamma' \int_{-1}^{1} dz' \frac{(\frac{1}{2}(\partial^2_{\gamma'}\gamma'^2 + 2\partial_{\gamma'}\partial_{z'}\gamma' z' - \partial^2_{z'}(1-z'^2)) - 2(\partial_{\gamma'}\gamma' + \partial_{z'}z') + 1)g^{(3)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)} \\
&= \int d\gamma' \int_{-1}^{1} dz' \frac{(\frac{1}{2}\gamma'^2\partial^2_{\gamma'} + (z'\partial_{z'} + 1)(\gamma'\partial_{\gamma'} + 1) - (1-z'^2)\partial^2_{z'} - 1)g^{(3)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)}. \qquad (4.51)
\end{aligned}
$$

To solve the right hand side, we use the loop integral $L(a, b, \mu, n) = \int \frac{d^4k'}{((k-k')^2 - \mu^2 + i\epsilon)(k^2 + a\cdot k + b)^n}$, where $\frac{1}{(k^2+a\cdot k+b)^n}$ is a generic test function. Using $\mu = 0, n = 3, a = pz', b = -\kappa^2 - \gamma'$, we have

$$
\begin{aligned}
&ig^2 \int \frac{d^4k'}{(2\pi)^4} \frac{1}{(k-k')^2 + i\epsilon} \int d\gamma' \int_{-1}^{1} dz' \frac{g^{(3)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^3} \\
&= \frac{ig^2}{(2\pi)^4} \int d\gamma' \int_{-1}^{1} dz' g^{(3)}(\gamma', z') \int \frac{d^4k'}{(k-k')^2 + i\epsilon} \frac{1}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^3} \\
&= \frac{ig^2}{(2\pi)^4} \int d\gamma' \int_{-1}^{1} dz' g^{(3)}(\gamma', z') L(pz', -\kappa^2 - \gamma', 0, 3) \\
&= \frac{ig^2}{(2\pi)^4} \int d\gamma' \int_{-1}^{1} dz' g^{(3)}(\gamma', z') \frac{i\pi^2}{2} \frac{1}{(-\kappa^2 - \gamma' - z'^2 M^2/4)(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)} \\
&= \frac{g^2}{32\pi^2} \int d\gamma' \int_{-1}^{1} dz' \frac{1}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)} \frac{g^{(3)}(\gamma', z')}{(\gamma' + m^2 + (z'^2 - 1)M^2/4)}. \qquad (4.52)
\end{aligned}
$$

Now we can identify the left hand side and the right hand side to obtain the equation

$$
\begin{aligned}
&\int d\gamma' \int_{-1}^{1} dz' \frac{(\frac{1}{2}\gamma'^2\partial^2_{\gamma'} + (z'\partial_{z'} + 1)(\gamma'\partial_{\gamma'} + 1) - (1-z'^2)\partial^2_{z'} - 1)g^{(3)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)} \\
&= \frac{g^2}{32\pi^2} \int d\gamma' \int_{-1}^{1} dz' \frac{1}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)} \frac{g^{(3)}(\gamma', z')}{(\gamma' + m^2 + (z'^2 - 1)M^2/4)} \qquad (4.53)
\end{aligned}
$$

Note that both the left hand side and the right hand side have as a prefactor the operator $\int d\gamma' \int_{-1}^{1} dz' \frac{1}{k^2 + p\cdot kz' - \gamma' - \kappa^2}$. So, we can use the uniqueness theorem to extract this operator from both sides and we arrive at the differential equation for the s-wave

Wick-Cutkosky model

$$(\frac{1}{2}\gamma'^2\partial^2_{\gamma'}+(z'\partial_{z'}+1)(\gamma'\partial_{\gamma'}+1)-(1-z'^2)\partial^2_{z'}-1)g^{(3)}(\gamma',z')=\frac{g^2}{32\pi^2}\frac{g^{(3)}(\gamma',z')}{(\gamma'+m^2+(z'^2-1)M^2/4)},\tag{4.54}$$

which can be simplified multiplying both sides by $(\gamma'+m^2+(z'^2-1)M^2/4)$. Thus, we write it as

$$Dg^{(3)}(\gamma',z')=\frac{g^2}{32\pi^2}g^{(3)}(\gamma',z'),\tag{4.55}$$

where D is the differential operator

$$D=(\gamma'+m^2+(z'^2-1)M^2/4)(\frac{1}{2}\gamma'^2\partial^2_{\gamma'}+(z'\partial_{z'}+1)(\gamma'\partial_{\gamma'}+1)-(1-z'^2)\partial^2_{z'}-1).\tag{4.56}$$

Since this is a equation for the Wick-Cutkosky model, it should be able to reproduce the previous known differential equation in z'. So, as a test, we substitute the known ground state solution $g^{(3)}(\gamma',z')=\delta(\gamma')b(z')$. Each differential term acts on the delta function as $\frac{1}{2}\gamma'^2\partial^2_{\gamma'}\delta(\gamma')=\delta(\gamma')$ and $\gamma'\partial_{\gamma'}\delta(\gamma')=-\delta(\gamma')$. Therefore, the differential operator D acts on $g^{(3)}(\gamma',z')$ as

$$\begin{aligned}
&Dg^{(3)}(\gamma',z')=\\
&=(\gamma'+m^2+(z'^2-1)M^2/4)(\frac{1}{2}\gamma'^2\partial^2_{\gamma'}+(z'\partial_{z'}+1)(\gamma'\partial_{\gamma'}+1)-(1-z'^2)\partial^2_{z'}-1)\delta(\gamma')b(z')\\
&=(\gamma'+m^2+(z'^2-1)M^2/4)(1+(z'\partial_{z'}+1)(-1+1)-(1-z'^2)\partial^2_{z'}-1)\delta(\gamma')b(z')\\
&=(\gamma'+m^2+(z'^2-1)M^2/4)(-(1-z'^2)\partial^2_{z'})\delta(\gamma')b(z')\\
&=-\delta(\gamma')(m^2+(z'^2-1)M^2/4)(1-z'^2)\partial^2_{z'}b(z').
\end{aligned}\tag{4.57}$$

Writing the left hand side equals the right hand side, and extracting the common term $\delta(\gamma')$ we arrive at the equation

$$(m^2+(z'^2-1)M^2/4)(1-z'^2)\partial^2_{z'}b(z')=\frac{g^2}{32\pi^2}b(z'),\tag{4.58}$$

which is the differential equation for the Wick-Cutkosky model, originally obtained by Wick (WICK, 1954; HWANG; KARMANOV, 2004).

The main innovation of this method is to obtain one single differential equation for every s-wave solution instead of an enumerable set of differential equations, indexed by the principal quantum number, as obtained by Wick and Cutkosky. Also, this method can be partly extended to the massive case, since the main idea is to move the free propagators to the left hand side of the BSE. However, a massive interaction loop integral gives logarithms, so the right hand side would need be calculated numerically.

5 Fermion Bound State with a massive scalar interaction

5.1 Introduction

The BS equation is used to solve the bound state problem in a relativistic field theory, thus it is naturally formulated in Minkowski space. Consequently, to solve it one has to deal with poles arising from relativistic propagator. In the past decades since its formulation, the popular approach to solve the BS equation was to use a Wick rotation; a contour deforming procedure in the k_0 complex plane, which translates the lorentz metric into a euclidean one. This is equivalent to transforming a relativistic field theory into an euclidean field theory, which does not have poles in its propagators, and thus are better suited for a numerical treatment.

In recent years an alternative approach has been tried using the so callend Nakanishi PTIR, an integral representation for field theory amplitudes. That, together with a LF projection to eliminate the poles in the propagators, has been the strategy behind some recent papers in the field. However, projecting a quantum amplitude onto the light front can result in singular contributions, due to endpoint singularities (HAUTMANN, 2007; MELIKHOV; SIMULA, 2003). Thus these singularities must be dealt with somehow, either using regulators or facing the distributions that arise.

This chapter will follow from J. Carbonell and V. A. Karmanov work on the fermion bound state problem with the Nakanishi PTIR projected onto the light front (CARBONELL; KARMANOV, 2010). In that paper, the authors used a form factor $F(q) = \frac{\mu^2-\Lambda^2}{q^2-\Lambda^2+i\epsilon}$, and a numerical regulator $\eta(k,p) = \frac{(m^2-L^2)}{(\frac{p}{2}+k)^2-L^2+i\epsilon}\frac{(m^2-L^2)}{(\frac{p}{2}-k)^2-L^2+i\epsilon}$, in order to get smoother kernels and make the LF projection integral convergent. Explicitly, the fermionic BSE is expanded in a base S_i and can be written as

$$\phi_i(k,p) = \frac{i}{(p/2+k)^2 - m^2 + i\epsilon} \frac{i}{(p/2-k)^2 - m^2 + i\epsilon} \int \frac{d^4k'}{(2\pi)^4} \frac{(-ig^2)\sum_{i'=1}^{4} c_{ii'}\phi_{i'}(k',p)}{(k-k')^2 - \mu^2 + i\epsilon}, \tag{5.1}$$

then they modified (5.30) inserting $\eta(k,p)$ and $F(q)$ such that

$$\begin{aligned} \eta(k,p)\phi_i(k,p) = \eta(k,p) &\frac{i}{(p/2+k)^2 - m^2 + i\epsilon} \frac{i}{(p/2-k)^2 - m^2 + i\epsilon} \\ \times &\int \frac{d^4k'}{(2\pi)^4} \frac{(-ig^2)F(q)^2\sum_{i'=1}^{4} c_{ii'}\phi_{i'}(k',p)}{(k-k')^2 - \mu^2 + i\epsilon}. \end{aligned} \tag{5.2}$$

However, the authors also performed a calculation using only the $F(q)$ and not also the $\eta(k,p)$, whose results are missing some terms due to cumbersome distributional corrections of the LF integral given by endpoint contributions. The alternative approach explored in this thesis is to avoid these regulators and instead use a Nakanishi PTIR of order 4 to make the LF projection convergent, and explore what are these distributional corrections.

5.2 Fermionic BS equation

In this section, we present the fermionic BS equation with an scalar interaction following the derivation presented in the previous mentioned work. The BS equation for a two fermions bound state, mediated by a massive scalar, with a ladder kernel, is given by

$$\phi(k,p) = \frac{i(m + \not{p}/2 + \not{k})}{(p/2+k)^2 - m^2 + i\epsilon} \int \frac{d^4k'}{(2\pi)^4} \phi(k',p) \frac{(-ig^2)}{(k-k')^2 - \mu^2 + i\epsilon} \frac{i(m - \not{p}/2 + \not{k})}{(p/2-k)^2 - m^2 + i\epsilon}, \tag{5.3}$$

where μ is the scalar interaction mass, m is the fermions masses and g the coupling constant. For a $J^\pi = 0^+$ state, one may write the BS amplitude as

$$\phi(k,p) = S_1\phi_1 + S_2\phi_2 + S_3\phi_3 + S_4\phi_4, \tag{5.4}$$

where the S_i form a basis of spin structures and the ϕ_i are scalar functions of k^2 and $p \cdot k$. The chosen basis was

$$S_1 = \gamma_5, \tag{5.5}$$

$$S_2 = \frac{\not p}{M}\gamma_5, \tag{5.6}$$

$$S_3 = \frac{k \cdot p}{M^3}\not p\gamma_5 - \frac{1}{M}\not k\gamma_5, \tag{5.7}$$

$$S_4 = \frac{i}{M^2}\sigma_{\mu\nu}p_\mu k_\nu\gamma_5, \tag{5.8}$$

with $\sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$. Substituting (5.4) in (5.9), and using the trace properties of the basis, we can write the following equation for each scalar ϕ_i

$$\phi_i(k,p) = \frac{i}{(p/2+k)^2 - m^2 + i\epsilon}\frac{i}{(p/2-k)^2 - m^2 + i\epsilon}\int \frac{d^4k'}{(2\pi)^4}\frac{(-ig^2)\sum_{i'=1}^{4} c_{ii'}\phi_{i'}(k',p)}{(k-k')^2 - \mu^2 + i\epsilon}, \tag{5.9}$$

where the $c_{ii'}$ are defined by the trace relation

$$c_{ii'} = \frac{Tr\left[S_i(\not p/2 + \not k + m)(ig)S'_{i'}(ig)(\not p/2 - \not k + m)\right]}{Tr\left[S_i^2\right]}, \tag{5.10}$$

and S'_i denotes the S_i with the exchange $k \to k'$.

Using this formula (5.10), we arrive at the individual expressions for each $c_{ii'}$

$$c_{11} = m^2 + \frac{M^2}{4} - k^2, \tag{5.11}$$

$$c_{12} = mM, \tag{5.12}$$

$$c_{13} = 0, \tag{5.13}$$

$$c_{14} = -b' M^2, \tag{5.14}$$

$$c_{21} = mM, \tag{5.15}$$

$$c_{22} = m^2 + \frac{M^2}{4} + k^2 - 2\frac{(p\cdot k)^2}{M^2}, \tag{5.16}$$

$$c_{23} = -2b'(p\cdot k), \tag{5.17}$$

$$c_{24} = -2b' mM, \tag{5.18}$$

$$c_{31} = 0, \tag{5.19}$$

$$c_{32} = 2(p\cdot k), \tag{5.20}$$

$$c_{33} = \frac{b'}{b}\left(m^2 - \frac{M^2}{4} + 2\frac{(p\cdot k)^2}{M^2} - k^2\right), \tag{5.21}$$

$$c_{34} = 2\frac{b'}{b}\frac{m}{M}(p\cdot k), \tag{5.22}$$

$$c_{41} = M^2, \tag{5.23}$$

$$c_{42} = 2mM, \tag{5.24}$$

$$c_{43} = 2\frac{b'}{b}\frac{m}{M}(p\cdot k), \tag{5.25}$$

$$c_{44} = -\frac{b'}{b}\left(\frac{M^2}{4} - m^2 - k^2\right), \tag{5.26}$$

with b and b' defined by

$$b = \frac{1}{M^4}((p\cdot k)^2 - M^2 k^2) = \frac{\vec{k}^2}{M^2}, \tag{5.27}$$

$$b' = \frac{1}{M^4}((p\cdot k)(p\cdot k') - M^2(k\cdot k')) = \frac{\vec{k}\cdot\vec{k'}}{M^2}. \tag{5.28}$$

Now, we write each ϕ_i as a Nakanishi representation of order 4

$$\phi_i(k,p) = \int d\gamma' \int_{-1}^{1} dz' \frac{g_i^{(4)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^4}, \tag{5.29}$$

and substitute it in (5.9) to arrive at

$$
\begin{aligned}
&\int d\gamma' \int_{-1}^{1} dz' \frac{g_i^{(4)}(\gamma', z')}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^4} = \\
&\int d\gamma' \int_{-1}^{1} dz' \frac{1}{((p/2+k)^2 - m^2 + i\epsilon)\,((p/2-k)^2 - m^2 + i\epsilon)} \frac{ig^2}{(2\pi)^4} \\
&\times \sum_{i'=1}^{4} g_{i'}^{(4)}(\gamma', z') \int \frac{d^4k'}{(k-k')^2 - \mu^2 + i\epsilon} \frac{c_{ii'}}{(k^2 + p\cdot kz' - \gamma' - \kappa^2 + i\epsilon)^4}. \qquad (5.30)
\end{aligned}
$$

The next step is to perform the loop integral on the right hand side. In order to do that, we must observe that each $c_{ii'}$ have one of three general structures: it is a polynomial in $\{ k^2, p\cdot k \}$ or, a polynomial in $\{ k^2, p\cdot k \}$ times b'/b or a polynomial in $\{ k^2, p\cdot k \}$ times b' :

$$
c_{ii'} = d_{ii'}(k^2, p\cdot k)(\delta_{ii'\in\mathcal{A}} + \frac{b'}{b}\delta_{ii'\in\mathcal{B}} + b'\delta_{ii'\in\mathcal{C}}). \qquad (5.31)
$$

Where $d_{ii'}$ is the polynomial, and all the possible (i, i') are divided in three sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ corresponding to each of these cases

$$
\begin{aligned}
\mathcal{A} &= \{ (i, i') \mid i' \in \{ 1, 2 \} \}, &(5.32)\\
\mathcal{B} &= \{ (i, i') \mid i \in \{ 3, 4 \}, i' \in \{ 3, 4 \} \}, &(5.33)\\
\mathcal{C} &= \{ (i, i') \mid i \in \{ 1, 2 \}, i' \in \{ 3, 4 \} \}. &(5.34)
\end{aligned}
$$

Now we can use the loop integral formulas calculated in the appendix:

$$
\begin{aligned}
L(\alpha, \beta, \mu, n) &= \int d^4k' \frac{1}{(k-k')^2 - \mu^2} \frac{1}{(k'^2 + \alpha\cdot k' + \beta)^n} \\
&= i\pi^2 \frac{1}{n-1} \int_0^1 dv \frac{v^{n-1}}{(v(1-v)(k+\alpha/2)^2 + v(\beta - \alpha^2/4) + (1-v)\mu^2)^{n-1}}, \qquad (5.35)
\end{aligned}
$$

and

$$L^{(F)}(\alpha,\beta,\mu,n) = \int d^4k' \frac{b'}{(k-k')^2-\mu^2} \frac{1}{(k'^2+\alpha\cdot k'+\beta)^n}$$
$$= i\pi^2 \frac{b}{n-1}\int_0^1 dv \frac{(1-v)v^{n-1}}{(v(1-v)(k+\alpha/2)^2+v(\beta-\alpha^2/4)+(1-v)\mu^2)^{n-1}}, \tag{5.36}$$

and use them, with $n=4$, $\alpha = pz'$, $\beta = -\gamma' - \kappa^2$, to obtain the loop integral for each $c_{ii'}$ on the right hand side of (5.9)

$$\int \frac{d^4k'}{(k-k')^2-\mu^2+i\epsilon} \frac{c_{ii'}}{(k^2+p\cdot kz'-\gamma'-\kappa^2+i\epsilon)^4}$$
$$= \int \frac{d^4k'}{(k-k')^2-\mu^2+i\epsilon} \frac{d_{ii'}(k^2,p\cdot k)(\delta_{ii'\in\mathcal{A}}+\frac{b'}{b}\delta_{ii'\in\mathcal{B}}+b'\delta_{ii'\in\mathcal{C}})}{(k^2+p\cdot kz'-\gamma'-\kappa^2+i\epsilon)^4}$$
$$= d_{ii'}(k^2,p\cdot k)(\delta_{ii'\in\mathcal{A}}L(\alpha,\beta,\mu,n) + \left(\frac{\delta_{ii'\in\mathcal{B}}}{b}+\delta_{ii'\in\mathcal{C}}\right) L^{(F)}(\alpha,\beta,\mu,n))$$
$$= d_{ii'}(k^2,p\cdot k)$$
$$\times \left(\delta_{ii'\in\mathcal{A}} i\pi^2 \frac{1}{n-1}\int_0^1 dv \frac{v^{n-1}}{(v(1-v)(k+\alpha/2)^2+v(\beta-\alpha^2/4)+(1-v)\mu^2)^{n-1}}\right.$$
$$\left. + \left(\frac{\delta_{ii'\in\mathcal{B}}}{b}+\delta_{ii'\in\mathcal{C}}\right) i\pi^2 \frac{b}{n-1}\int_0^1 dv \frac{(1-v)v^{n-1}}{(v(1-v)(k+\alpha/2)^2+v(\beta-\alpha^2/4)+(1-v)\mu^2)^{n-1}}\right)$$
$$= d_{ii'}(k^2,p\cdot k)(\delta_{ii'\in\mathcal{A}}+(\delta_{ii'\in\mathcal{B}}+\delta_{ii'\in\mathcal{C}}b)(1-v))$$
$$\times \left(i\pi^2\frac{1}{3}\int_0^1 dv \frac{v^3}{(v(1-v)(k+z'p/2)^2-v(\gamma'+\kappa^2+z'^2M^2/4)+(1-v)\mu^2)^3}\right). \tag{5.37}$$

Now we can substitute it in (5.30) and get

$$\int d\gamma' \int_{-1}^1 dz' \frac{g_i^{(4)}(\gamma',z')}{(k^2+p\cdot kz'-\gamma'-\kappa^2+i\epsilon)^4} =$$
$$-\frac{g^2}{48\pi^2}\int d\gamma' \int_{-1}^1 dz' \frac{1}{((p/2+k)^2-m^2+i\epsilon)((p/2-k)^2-m^2+i\epsilon)}$$
$$\times \left(\int_0^1 dv \frac{v^3}{(v(1-v)(k+z'p/2)^2-v(\gamma'+\kappa^2+z'^2M^2/4)+(1-v)\mu^2)^3}\right)$$
$$\times \sum_{i'=1}^4 g_{i'}^{(4)}(\gamma',z')d_{ii'}(k^2,p\cdot k)(\delta_{ii'\in\mathcal{A}}+(\delta_{ii'\in\mathcal{B}}+\delta_{ii'\in\mathcal{C}}b)(1-v)). \tag{5.38}$$

5.3 The singular contributions

The next step to solve the fermionic BSE is to project the equation onto the light front, i.e. perform the integral $\int dk^-$. First, for the left hand side we will use the LF integral calculated in the appendix

$$I_{LF}(\alpha,\beta,m,n) = \int \frac{dk^-(k^-)^m}{(\alpha k^- + \beta)^n} \tag{5.39}$$

With $\alpha = \frac{M}{2}(z'-z)$, $\beta = -\gamma - \gamma' - \kappa^2 - \frac{M^2}{4}zz'$, $m=0$ and $n=4$. Thus, we have

$$\begin{aligned}
&\int dk^- \int d\gamma' \int_{-1}^{1} dz' \frac{g_i^{(4)}(\gamma',z')}{(k^2+p\cdot kz'-\gamma'-\kappa^2+i\epsilon)^4} &(5.40)\\
&= \int d\gamma' \int_{-1}^{1} dz' g_i^{(4)}(\gamma',z') \int dk^- \frac{1}{(k^2+p\cdot kz'-\gamma'-\kappa^2+i\epsilon)^4}\\
&= \int d\gamma' \int_{-1}^{1} dz' g_i^{(4)}(\gamma',z') I_{LF}(\frac{M}{2}(z'-z), -\gamma-\gamma'-\kappa^2-\frac{M^2}{4}zz', 0, 4)\\
&= \int d\gamma' \int_{-1}^{1} dz' g_i^{(4)}(\gamma',z') \left(-\frac{\pi i}{3}\frac{\delta(\frac{M}{2}(z'-z))}{(-\gamma-\gamma'-\kappa^2-\frac{M^2}{4}zz')^3}\right)\\
&= -\frac{2\pi i}{3M}\int d\gamma' \frac{g_i^{(4)}(\gamma',z)}{(\gamma+\gamma'+\kappa^2+\frac{M^2}{4}z^2)^3}. &(5.41)
\end{aligned}$$

Now, we must apply the LF projection to the right hand side. However, the goal of this chapter is to obtain the additional contributions only, that appears as distributions,and there are only these contributions when there are at least $(k^-)^2$ or $(k^-)^3$ in the numerator. In order to see that, note that the RHS of the fermionic BSE follows the general structure

$$I'_{LF}(\alpha,\beta,m,n) = \int \frac{dk^-(k^-)^m}{(k^2+p\cdot k-\kappa^2+i\epsilon)(k^2-p\cdot k-\kappa^2+i\epsilon)(\alpha k^-+\beta+i\epsilon)^n}, \tag{5.42}$$

which converges for n big enough. However, if we set $\alpha = 0$, this integral can be rewriteen as

$$I'_{LF}(0,\beta,m,n) = \int \frac{dk^-(k^-)^m}{(k^2+p\cdot k-\kappa^2+i\epsilon)(k^2-p\cdot k-\kappa^2+i\epsilon)(\beta+i\epsilon)^n}, \tag{5.43}$$

and now the denominator under n does not contribute to its convergence anymore. Since we can write $k^2 = k^+k^- - k_\perp^2$, the free propagators only contribute with $\frac{1}{(k^-)^2}$. Thus, if $m = 2$ or $m = 3$ the integral diverges. Finally, this divergence indicates that the integral (5.42) results in distributions with support $\alpha = 0$ if $m \geq 2$, for instance one could have a term proportional to $\delta(\alpha)$.

In order to identify which are the singular ones, note that the only terms that cointains k^- are the

$$d_{ii'}(k^2, p \cdot k)(\delta_{ii' \in \mathcal{A}} + (\delta_{ii' \in \mathcal{B}} + \delta_{ii' \in \mathcal{C}} b)(1 - v)) = \sum_{j=0}^{3} \eta_{ii'}^{(j)} (k^-)^j, \tag{5.44}$$

where $\eta_{ii'}^{(j)}$ is a polynomial in the LF variables γ and z. So, if we expand all scalar terms in LF variables of the form

$$k^2 = k^+k^- - k_\perp^2 = \left(-z\frac{M}{2}\right) k^- - \gamma, \tag{5.45}$$

$$(p \cdot k) = \frac{M}{2}\left(k^- + k^+\right) = \left(\frac{M}{2}\right) k^- - z\frac{M^2}{4}, \tag{5.46}$$

$$\begin{aligned} b &= (\frac{1}{M^4}(p \cdot k)^2 - M^2k^2) \\ &= \frac{1}{M^4}((\left(\frac{M}{2}\right) k^- - z\frac{M^2}{4})^2 - M^2(\left(-z\frac{M}{2}\right) k^- - \gamma)) \\ &= \frac{1}{M^4}\left(\left(\left(\frac{M}{2}\right)^2 (k^-)^2 - z\frac{M^3}{4}k^- + (z\frac{M^2}{4})^2\right) + \left(z\frac{M^3}{2}\right) k^- + \gamma M^2\right) \\ &= \frac{(k^-)^2}{4M^2} + z\frac{k^-}{4M} + \frac{z^2}{16} + \frac{\gamma}{M^2} \end{aligned} \tag{5.47}$$

We conclude, combining (5.47) with (5.44) , that the only terms (i, i') that have $(k^-)^2$ in the numerador are the (1,4), (2,2) , (2,3) , (2,4), (3,3), and the only that have $(k^-)^3$ is (2,3). Therefore, these $\eta_{ii'}^{(j)}$, with $j \geq 2$ are

$$\eta_{14}^{(2)} = -\frac{1}{4}(1 - v), \tag{5.48}$$

$$\eta_{22}^{(2)} = -\frac{1}{2}, \tag{5.49}$$

$$\eta_{23}^{(2)} = -\frac{z}{8}(1 - v), \tag{5.50}$$

$$\eta_{24}^{(2)} = -\frac{m}{2M}(1 - v), \tag{5.51}$$

$$\eta_{33}^{(2)} = \frac{1}{2}(1 - v), \tag{5.52}$$

$$\eta_{23}^{(3)} = -\frac{1}{4M}(1 - v). \tag{5.53}$$

As one can check, these coefficients are directly proportional to the $F_{j;ii'}$ presented in appendix B, obtained with a form factor $F(q)$ and Nakanishi exponenent $n = 3$.

Now we can obtain the LF projected BS equation

$$\begin{aligned}
&\int d\gamma' \frac{g_i^{(4)}(\gamma', z)}{(\gamma + \gamma' + \kappa^2 + \frac{M^2}{4}z^2)^3} \\
&= \frac{3M}{2\pi i}\frac{g^2}{48\pi^2}\int dk^- \\
&\times \int d\gamma' \int_{-1}^{1} dz' \frac{1}{((p/2+k)^2 - m^2 + i\epsilon)\,((p/2-k)^2 - m^2 + i\epsilon)} \\
&\times \left(\int_0^1 dv \frac{v^3}{(v(1-v)(k+z'p/2)^2 - v(\gamma' + \kappa^2 + z'^2M^2/4) + (1-v)\mu^2)^3} \right) \\
&\times \sum_{i'=1}^{4} g_{i'}^{(4)}(\gamma', z') \sum_{j=0}^{3} \eta_{ii'}^{(j)} (k^-)^j \\
&= \frac{3M}{2\pi i}\frac{g^2}{48\pi^2}\int d\gamma' \int_{-1}^{1} dz' \int_0^1 dv \sum_{i'=1}^{4} g_{i'}^{(4)}(\gamma', z') \sum_{j=0}^{3} \eta_{ii'}^{(j)} (k^-)^j \\
&\times \int dk^- \frac{1}{((p/2+k)^2 - m^2 + i\epsilon)\,((p/2-k)^2 - m^2 + i\epsilon)} \\
&\times \left(\frac{v^3}{(v(1-v)(k+z'p/2)^2 - v(\gamma' + \kappa^2 + z'^2M^2/4) + (1-v)\mu^2)^3} \right).
\end{aligned} \tag{5.54}$$

To perform the $\int dk^-$, we must use the LF integral calculated in the appendix

$$I'_{LF}(\alpha, \beta, m, n) = \int \frac{dk^-(k^-)^m}{(k^2 + p\cdot k - \kappa^2 + i\epsilon)(k^2 - p\cdot k - \kappa^2 + i\epsilon)(\alpha k^- + \beta + i\epsilon)^n}, \tag{5.55}$$

and use it with the following variables

$$\alpha = v(1-v)\frac{M}{2}(z' - z), \tag{5.56}$$

$$\beta = -v\left(+\gamma' + \kappa^2 + \frac{1}{4}M^2 z'^2\right) - (1-v)v\left(-\frac{1}{4}M^2 z'\,(z'-z) + \gamma\right) + \mu^2(1-v), \tag{5.57}$$

for the cases $m = 2$, $n = 3$ and $m = 3$, $n = 3$ to arrive at

$$
\begin{aligned}
I'_{LF}(\alpha,\beta,2,3) &= Regular \\
&+ \left(\frac{-8\pi i \delta(z'-z)}{v(1-v)M^3(1-z^2)\left(v\left(\gamma'+\kappa^2-\frac{1}{4}M^2z^2\right)+\gamma(1-v)v-\mu^2(1-v)\right)^2} \right),
\end{aligned}
\tag{5.58}
$$

$$
\begin{aligned}
I'_{LF}(\alpha,\beta,3,3) &= Regular \\
&+ \left(\frac{-8\pi i \delta(z'-z)}{v(1-v)M^3(1-z^2)\left(v\left(\gamma'+\kappa^2-\frac{1}{4}M^2z^2\right)+\gamma(1-v)v-\mu^2(1-v)\right)} \right) \\
&+ \left(\frac{-\pi i 64 z(\gamma+m^2)\delta'(z'-z)}{v^2(1-v)^2M^5(1-z^2)^2\left(v\left(\gamma'+\kappa^2-\frac{1}{4}M^2z^2\right)+\gamma(1-v)v-\mu^2(1-v)\right)^2} \right).
\end{aligned}
\tag{5.59}
$$

where the so called *regular* terms of the integral are the ones that does not contain distributions, only ordinary functions.

Finally, we substitute these results in the BSE to obtain the new distributional corrections

$$
\begin{aligned}
&\int d\gamma' \frac{g_i^{(4)}(\gamma',z)}{(\gamma+\gamma'+\kappa^2+\frac{M^2}{4}z^2)^3} \\
&= \frac{3M}{2\pi i}\frac{g^2}{48\pi^2}\int d\gamma' \int_{-1}^{1} dz' \int_0^1 dv v^3 \sum_{i'=1}^{4} g_{i'}^{(4)}(\gamma',z') \sum_{j=0}^{3} \eta_{ii'}^{(j)} I'_{LF}(\alpha,\beta,j,3) \\
&= Regular \\
&+ \frac{g^2}{4\pi^2}\int d\gamma' \int_0^1 dv v^3 \\
&\times \Bigg(\left(-\frac{\sum_{i'=1}^{4} g_{i'}^{(4)}(\gamma',z)\eta_{ii'}^{(2)}(\delta_{1i}\delta_{4i'}+\delta_{2i}\delta_{2i'}+\delta_{2i}\delta_{3i'}+\delta_{2i}\delta_{4i'}+\delta_{3i}\delta_{3i'})}{v(1-v)M^3(1-z^2)\left(v\left(\gamma'+\kappa^2-\frac{1}{4}M^2z^2\right)+\gamma(1-v)v-\mu^2(1-v)\right)^2} \right) \\
&+ \left(\frac{g_3^{(4)}(\gamma',z)\delta_{i2}/4}{vM^3(1-z^2)\left(v\left(\gamma'+\kappa^2-\frac{1}{4}M^2z^2\right)+\gamma(1-v)v-\mu^2(1-v)\right)} \right) \\
&+ \left(\frac{-2z(\gamma+m^2)\partial_{z'} g_3^{(4)}(\gamma',z)\delta_{i2}}{v^2(1-v)M^5(1-z^2)^2\left(v\left(\gamma'+\kappa^2-\frac{1}{4}M^2z^2\right)+\gamma(1-v)v-\mu^2(1-v)\right)^2} \right) \Bigg)
\end{aligned}
\tag{5.60}
$$

It is importante to note that there are two terms that have a pole in $v=1$, making the dv integral diverge logarithmically: the $i=2, i'=2, j=2$ and the $i=2, i'=3, j=3$, proportional to the delta derivative. Since the divergence is slow, it should not give numerical problems. However if one wants to have a complete analytical analysis, these terms must be somehow renormalized or cancel each other divergences.

5.4 Extreme binding energy for a massless vector interaction

The first papers describing the scalar Wick-Cutkosky model explored several particular cases of the equation. One of these is the extreme binding energy case, when the bound state becomes massless ($M = 0$). Since $M = 0$ implies that the propagators can be written as $\frac{i}{(p/2-k)^2-m^2} = \frac{i}{(p/2+k)^2-m^2} = \frac{i}{k^2-m^2}$, the scalar BSE simplifies to the form

$$\phi(k,p) = \frac{i}{k^2-m^2}\frac{i}{k^2-m^2} ig^2 \int \frac{d^4k'}{(2\pi)^4}\frac{\phi(k',p)}{(k-k')^2+i\epsilon}, \tag{5.61}$$

and has $\phi(k,p) = \phi(k,0) = \frac{1}{(k^2-m^2)^3}$ as a solution for the ground state.

It is undeniable the similarity of this solution to the Nakanishi Ansatz used to solve the general Wick-Cutkosky and BSE , thus one may argue that the analysis of the $M = 0$ case can give useful heuristics to attack the $M > 0$ case. Given this motivation, the goal of this chapter is to obtain a similar analysis of the $M = 0$ case, but for the fermionic bound state with a massless vector interaction.

First, let us note that the coefficients of the massless vector and the scalar interactions equations are proportional. In fact, writing the BS amplitude as $\phi(k,p) = \phi_1(k,p)S_1 + \phi_2(k,p)S_2 + \phi_3(k,p)S_3$, where S_i is the same base used for the scalar interaction, we have a new set of coefficients c^V_{ij} related to the scalar ones as $c^V_{ij} = \xi_{ij}c^S_{ij}$ where

$$\xi = \begin{bmatrix} 4 & -2 & 0 & 0 \\ 4 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.62}$$

Since we now know that the coefficients are proportional, let us observe what happens to then as $M \to 0$. Since $c^V_{13} = c^S_{13} = 0$, ϕ_1 is only coupled to ϕ_2. However, $c^S_{12} = mM$, so in the extreme binding energy limit, $c^V_{12} \to 0$ and ϕ_1 decouples from ϕ_2. Thus, we can investigate this limit for a massless vector interaction by analyzing only the decoupled ϕ_1 equation

$$\begin{aligned}\phi_1(k) &= c_{11}^V \frac{i}{k^2-m^2}\frac{i}{k^2-m^2} ig^2 \int \frac{d^4k'}{(2\pi)^4}\frac{\phi_1(k')}{(k-k')^2+i\epsilon} \\ &= 4(m^2-k^2)\frac{-1}{(k^2-m^2)^2} ig^2 \int \frac{d^4k'}{(2\pi)^4}\frac{\phi_1(k')}{(k-k')^2+i\epsilon} \\ &= 4\frac{ig^2}{k^2-m^2}\int \frac{d^4k'}{(2\pi)^4}\frac{\phi_1(k')}{(k-k')^2+i\epsilon}. \end{aligned} \tag{5.63}$$

Now, we multiply both sides by k^2-m^2 and have

$$(k^2-m^2)\phi_1(k) = 4ig^2 \int \frac{d^4k'}{(2\pi)^4}\frac{\phi_1(k')}{(k-k')^2+i\epsilon}. \tag{5.64}$$

The next step is to write ϕ_1 as a parametric representation on γ' of the form

$$\phi_1(k) = \int d\gamma' \frac{p(\gamma')}{(k^2-m^2-\gamma'+i\epsilon)^3}, \tag{5.65}$$

which emerge naturally from the Nakanishi PITR in the limit $M \to 0$

$$\begin{aligned}\phi_1(k) &= \int d\gamma' \int_{-1}^{1} dz' \frac{g^{(3)}(\gamma',z')}{(k^2+p\cdot kz'+M^2/4-m^2-\gamma'+i\epsilon)^3} \\ &= \int \frac{d\gamma'}{(k^2-m^2-\gamma'+i\epsilon)^3}\int_{-1}^{1} dz' g^{(3)}(\gamma',z') \\ &= \int \frac{d\gamma'}{(k^2-m^2-\gamma'+i\epsilon)^3} p(\gamma'), \end{aligned} \tag{5.66}$$

where $p(\gamma') = \int_{-1}^{1} dz' g^{(3)}(\gamma',z')$. Now we can substitute (5.65) in (5.64) and solve it for $p(\gamma')$. First, the left hand side equation gives

$$\begin{aligned}(k^2-m^2)\phi_1(k) &= (k^2-m^2)\int d\gamma' \frac{p(\gamma')}{(k^2-m^2-\gamma'+i\epsilon)^3} \\ &= \int d\gamma' \frac{p(\gamma')}{(k^2-m^2-\gamma'+i\epsilon)^2} + \frac{\gamma' p(\gamma')}{(k^2-m^2-\gamma'+i\epsilon)^3}. \end{aligned} \tag{5.67}$$

Now, we can use integration by parts so that all denominators have exponent 1. Setting the boundary terms to 0,we write

$$
\begin{aligned}
(k^2 - m^2)\phi_1(k) &= \int d\gamma' \frac{p(\gamma')}{(k^2 - m^2 - \gamma' + i\epsilon)^2} + \frac{\gamma' p(\gamma')}{(k^2 - m^2 - \gamma' + i\epsilon)^3} \\
&= \int d\gamma' \frac{-\partial_{\gamma'} p(\gamma')}{(k^2 - m^2 - \gamma' + i\epsilon)} + \frac{\partial^2_{\gamma'}(\gamma' p(\gamma'))/2}{(k^2 - m^2 - \gamma' + i\epsilon)} \\
&= \int d\gamma' \frac{-\partial_{\gamma'} p(\gamma') + \partial^2_{\gamma'}(\gamma' p(\gamma'))/2}{(k^2 - m^2 - \gamma' + i\epsilon)} \\
&= \int d\gamma' \frac{\gamma' \partial^2_{\gamma'} p(\gamma')/2}{(k^2 - m^2 - \gamma' + i\epsilon)}. \qquad (5.68)
\end{aligned}
$$

To compute the right hand side, we use the loop integral $L(a, b, \mu, n)$ presented in the appendix.

$$
\begin{aligned}
4ig^2 \int \frac{d^4k'}{(2\pi)^4} \frac{\phi_1(k')}{(k-k')^2 + i\epsilon} &= 4ig^2 \int \frac{d^4k'}{(2\pi)^4} \frac{1}{(k-k')^2 + i\epsilon} \int d\gamma' \frac{p(\gamma')}{(k^2 - m^2 - \gamma' + i\epsilon)^3} \\
&= \frac{4ig^2}{(2\pi)^4} \int d\gamma' p(\gamma') \int d^4k' \frac{1}{(k-k')^2 + i\epsilon} \frac{1}{(k^2 - m^2 - \gamma' + i\epsilon)^3} \\
&= \frac{4ig^2}{(2\pi)^4} \int d\gamma' p(\gamma') L(0, -m^2 - \gamma', 0, 3) \\
&= \frac{4ig^2}{(2\pi)^4} \int d\gamma' p(\gamma') \frac{i\pi^2}{2} \frac{1}{-m^2 - \gamma'} \frac{1}{k^2 - m^2 - \gamma'} \\
&= \frac{g^2}{2(2\pi)^2} \int d\gamma' \frac{p(\gamma')}{m^2 + \gamma'} \frac{1}{k^2 - m^2 - \gamma'}. \qquad (5.69)
\end{aligned}
$$

Using the uniqueness theorem, we are now able to remove the operator $\frac{1}{2} \int \frac{d\gamma'}{(k^2 - m^2 - \gamma' + i\epsilon)}$ from both sides and have a equation purely in the parameter γ'

$$
\begin{aligned}
\gamma' \partial^2_{\gamma'} p(\gamma') &= \frac{g^2}{(2\pi)^2} \frac{p(\gamma')}{m^2 + \gamma'} \\
(m^2 + \gamma')\gamma' \partial^2_{\gamma'} p(\gamma') &= \frac{g^2}{(2\pi)^2} p(\gamma') \\
D p(\gamma') &= \frac{g^2}{(2\pi)^2} p(\gamma') \qquad (5.70)
\end{aligned}
$$

where D is the differential operator $(m^2 + \gamma')\gamma' \partial^2_{\gamma'}$. We propose a solution of (5.70) expressing $p(\gamma')$ in a basis of the form $|n\rangle = \gamma'^{n+2}$ with $n \in \{0, 1, 2, 3, \cdots\}$, for the region $\gamma' > 0$. It is important to note that there may be additional solutions as distributions located at $\gamma' = 0$, as $p(\gamma') = \delta(\gamma')$ is a solution for the scalar case. So, we should be aware that the present analysis using the proposed basis may not account for the full spectrum of Eq. (5.70). Also, a given $p(\gamma') = \gamma'^n$ may give as an amplitude a divergent integral, so some renormalization procedure may be necessary to recover the BS amplitude.

In this basis, D has matrix elements of the form

$$\langle m|D|n\rangle = (n+2)(n+1)(\delta_{mn} + m^2\delta m(n+1)) \tag{5.71}$$

Since this matrix is triangular, its eigenvalues are the elements of the diagonal $\langle n|D|n\rangle = (n+2)(n+1)$. This gives a spectrum for the possible values of the coupling constant as

$$g^2 = (2\pi)^2(n+2)(n+1) \tag{5.72}$$

With $n \in \{0, 1, 2, 3, \cdots\}$. For the ground state of this model $n = 0$, the eigenvector solution is of the form $p_0(\gamma') = m^2\gamma' + \gamma'^2$.

Finally it is important to remember the limitations of this analysis. First, the decoupled sector of ϕ_2 and ϕ_3 were not analysed and may give additional eigenvectors for the spectrum of the BSE. Second, there may be additional solutions based on distributions such as Dirac deltas and its derivatives. Third, this basis can give divergent integrals for the amplitude, which may require a renormalization procedure.

6 Conclusion

In this thesis, the Wick-Cutkosky model and the fermion bound state problem were investigated in the framework of the Bethe-Salpeter equation in the ladder approximation, in order to obtain better methods to study relativistic bound states in a quantum field theory. It is essential to further develop new methods since the bound state problem in Minkowski space is still an undeveloped research field. Although there are numerous methods to solve it with the euclidean metric, such as Wick rotated BSE and lattice field theory, the Minkowski space BS amplitude is necessary in order to compute some dynamical observables, such as form factors.

The Wick-Cutkosky model was analysed and rederived using two new methods based on the Nakanishi representation. The first expanded one of the integrals into a serie using integration by parts, and the second transformed the integral equation into a differential equation using integration by parts and the uniqueness theorem. These demonstrate that the Wick-Cutkosky is a great toy model that can be used to test new analytical approaches for the BSE.

The fermionic BSE in the ladder approximation was analysed using the Nakanishi representation projected onto the Light Front. The analysis followed from the work developed in (CARBONELL; KARMANOV, 2010), however in this thesis form factors or regulators were not used in order to deal with problematic divergent integrals. Instead, it was proposed to changed the order of the Nakanishi representation and calculate directly integrals which turned out to be Dirac delta functions or its derivatives. Thus, it was possible to present some corrections necessary for the fermionic light-front projected BSE.

Building on the results presented in this Thesis, there are new interesting problems to investigate in order to advance the understanding of relativistic bound states. These may include fermions in $2+1$ dimensions for bidimensional materials, a deeper investigation of the massless fermion bound state since it has a simpler analytic structure and possibly finding a fermionic analogue of the Wick-Cutkosky model, crossed-box contributions for the Wick-Cutkosky model and the fermion bound state and develop a method that does not rely on the LF projection but uses the Nakanishi representation in euclidean space.

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Appendix A - Useful Integrals

In this appendix we derive some useful integrals formulas used through the thesis, in an effort to improve the understandability of the calculations.

A.1 Feynman Parametrization

The Feynman parametrization is a key ingredient in the usual algorithm to perform the loop integrals in quantum field theory. The main idea is to note that the denominator in each propagator is a second order polynomial in the fields momenta, so any linear combination of them is also a second order polynomial. In this section we derive the relation for only two denominators A and B. First, observe the integral formula

$$\int_0^1 dv \frac{1}{(y+vx)^2} = \frac{1}{y}\frac{1}{x+y} \tag{A.1}$$

Now, we can set $y = A$ and $x = B - A$ and we have

$$\int_0^1 dv \frac{1}{(A+v(B-A))^2} = \frac{1}{AB} \tag{A.2}$$

If we want a formula for $\frac{1}{AB^n}$ instead, we need only to differentiate $n-1$ times in B

$$\frac{\partial^{n-1}}{\partial B^{n-1}} \int_0^1 dv \frac{1}{(A+v(B-A))^2} = \frac{\partial^{n-1}}{\partial B^{n-1}} \frac{1}{AB} \tag{A.3}$$

$$(-1)^{n-1}(n)! \int_0^1 dv \frac{v^{n-1}}{(A+v(B-A))^{n+1}} = (-1)^{n-1}(n-1)! \frac{1}{AB^n} \tag{A.4}$$

$$n \int_0^1 dv \frac{v^{n-1}}{(A+v(B-A))^{n+1}} = \frac{1}{AB^n} \tag{A.5}$$

A.2 d^4k Integration

Usually, the second step in a loop integral is to execute a d^4k integral that can be written in the general form

$$\int \frac{d^4k}{(k^2 + b + i\epsilon)^n} \tag{A.6}$$

Here k is a four-vector and b a real number. Since $k^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2$, we can perform (A.6) as a sequence of four unidimensional integrations of the form

$$\int \frac{dk_i}{(k_i^2 + b_i + i\epsilon)^n} \tag{A.7}$$

Where the limits of integration are implicitly from $-\infty$ to ∞. To solve this unidimensional integral, first we start with the case $n = 1$, which has a known primitive

$$\frac{\partial}{\partial k_i} \frac{\tan^{-1}(k_i/\sqrt{b_i})}{\sqrt{b_i}} = \frac{1}{k_i^2 + b_i} \tag{A.8}$$

Which permit us to evaluate the integral (A.7) for $n = 1$

$$\int \frac{dk_i}{(k_i^2 + b_i + i\epsilon)} = \frac{\pi}{\sqrt{b_i}} \tag{A.9}$$

For the case $n > 1$ we differentiate $n - 1$ times in b_i and obtain

$$\int \frac{dk_i}{(k_i^2 + b_i + i\epsilon)^n} = \frac{\pi(1/2)(3/2)\cdots(n-2+1/2)}{(n-1)!b_i^{n-1/2}} \tag{A.10}$$

The problem now is that the denominator has a half integer exponent, so we also need the primitive for this case

$$\frac{\partial}{\partial k_i} \frac{k_i}{b_i\sqrt{k_i^2 + b_i}} = \frac{1}{(k_i^2 + b_i)^{3/2}} \tag{A.11}$$

Which enables us to perform the integral with exponent 3/2

$$\int \frac{dk_i}{(k_i^2 + b_i + i\epsilon)^{3/2}} = \frac{2}{b_i} \tag{A.12}$$

For a higher half integer exponent $n+1/2$ we differentiate $n-1$ times in b_i and have

$$\int \frac{dk_i}{(k_i^2+b_i+i\epsilon)^{n+1/2}} = \frac{(n-1)!}{((n-1)+1/2)\cdots(2+1/2)(1+1/2)}\frac{2}{(b_i)^n} \tag{A.13}$$

Now, we are ready to calculate (A.6), iterating four unidimensional integrations, in this order: k_3, k_2, k_1, k_0.

$$\int \frac{d^4k}{(k^2+b+i\epsilon)^n} = \frac{\pi(1/2)(3/2)\cdots(n-2+1/2)}{(n-1)!} \tag{A.14}$$

$$\times \frac{2(n-2)!}{((n-2)+1/2)\cdots(2+1/2)(1+1/2)} \tag{A.15}$$

$$\times \frac{\pi(1/2)(3/2)\cdots(n-3+1/2)}{(n-2)!} \tag{A.16}$$

$$\times \frac{2(n-3)!}{((n-3)+1/2)\cdots(2+1/2)(1+1/2)} \tag{A.17}$$

$$\times i\frac{1}{b^{n-2}} \tag{A.18}$$

$$= i\pi^2\frac{1}{n-1}\frac{1}{n-2}\frac{1}{b^{n-2}} \tag{A.19}$$

A.3 Loop Integrals

With the tools of the Feynman parametrization and d^4k integral, we are now able to perform a general loop integral. To be concrete, let's define a loop integral with an interaction kernel with mass μ acting on a test function $\frac{1}{(k'^2+ak'+b)^n}$ as

$$L(a,b,\mu,n) = \int d^4k' \frac{1}{(k-k')^2-\mu^2}\frac{1}{(k'^2+ak'+b)^n} \tag{A.20}$$

Where b real number and a is a four-vector proportional to $(1,0,0,0)$. It is interesting to note that there is an useful recursion formula

$$L(a,b,\mu,n+1) = -\frac{1}{n}\frac{\partial}{\partial b}L(a,b,\mu,n) \tag{A.21}$$

Now, we perform a Feynman parametrization to join the denominators

$$L(a,b,\mu,n) = \int d^4k' \int_0^1 dv \frac{nv^{n-1}}{((k'^2 - 2k\cdot k' + k^2) - \mu^2 + v((a+2k)k' + (b+\mu^2 - k^2)))^{n+1}} \tag{A.22}$$

Complete the squares in the denominator of (A.24)

$$L(a,b,\mu,n) = \tag{A.23}$$

$$\int_0^1 dv \int d^4k' \frac{nv^{n-1}}{((k' + (-k+vk+va/2))^2 - (-k+vk+va/2)^2 + k^2 - \mu^2 + v(b+\mu^2-k^2))^{n+1}} \tag{A.24}$$

And shift the integration variable $k'' \to k' + (-k+vk+va/2)$ to perform the d^4k' integral

$$L(a,b,\mu,n) = \tag{A.25}$$

$$i\pi^2 \frac{1}{n-1} \int_0^1 dv \frac{v^{n-1}}{(-(-k+vk+va/2)^2 + k^2 - \mu^2 + v(b+\mu^2-k^2))^{n-1}} \tag{A.26}$$

The next step is to expand the square term as $(-k+vk+va/2)^2 = k^2 - 2vk\cdot(k + a/2) + v^2(k+a/2)^2$ in the denominator and obtain

$$L(a,b,\mu,n) = \tag{A.27}$$

$$= i\pi^2 \frac{1}{n-1} \int_0^1 dv \frac{v^{n-1}}{(-(-2vk\cdot(k+a/2) + v^2(k+a/2)^2) - \mu^2 + v(b+\mu^2-k^2))^{n-1}} \tag{A.28}$$

$$= i\pi^2 \frac{1}{n-1} \int_0^1 dv \frac{v^{n-1}}{(-v^2(k+a/2)^2 - \mu^2 + v(b+\mu^2+k^2+a\cdot k))^{n-1}} \tag{A.29}$$

$$= i\pi^2 \frac{1}{n-1} \int_0^1 dv \frac{v^{n-1}}{(v(1-v)(k+a/2)^2 + v(b-a^2/4) + (1-v)\mu^2)^{n-1}} \tag{A.30}$$

This is the general result, however it is also very important to calculate explicity some common particular n and μ combinations which happens frequently. First, let us set $n = 3$, which is the usual n used to solve the BSE. We have

$$L(a,b,\mu,3) = \tag{A.31}$$

$$= i\pi^2 \frac{1}{2}\int_0^1 dv \frac{v^2}{(v(1-v)(k+a/2)^2 + v(b-a^2/4) + (1-v)\mu^2)^2} \tag{A.32}$$

$$= \frac{i\pi^2}{2} \tag{A.33}$$

$$\times \left(\frac{k^2+ak+b+\mu^2}{(b-a^2/4)\sqrt{(k^2+ak+b+\mu)^2 - 4(b-a^2/4)\mu^4}} \right. \tag{A.34}$$

$$\left. + 2\mu^2 \frac{\log\left(\frac{4(b-a^2/4)\mu^2}{(\sqrt{((k^2+ak+b+\mu^2)^2-(4b-a^2)\mu^2])^2}+(k^2+ak+b+\mu^2))^2}\right)}{((k^2+ak+b+\mu^2)^2 - 4(b-a^2/4)\mu^4)^{3/2}} \right) \tag{A.35}$$

It is very important to note that the presence of a $\log\mu^2$ in the expression above, makes $L(a,b,\mu,3)$ a non-analytic function in μ, so we can't simply expand it in a taylor series in μ. In fact, besides the logarithm term, everything else in (A.35) is analytic in μ^2, which enables us to infer a series expression of the form

$$L(a,b,\mu,3) = c_{00}(a,b,k) + \sum_{i=1}^{\infty}(c_{i0}(a,b,k) + c_{i1}(a,b,k)\log\mu^2)\mu^{2i} \tag{A.36}$$

Also, $L(a,b,\mu,n)$ can be written in a series similar to (A.36), since the recursion (A.21) would preserve the structure in μ^2 and only change de coefficients c_{ij}. Moreover, it is useful to obtain the first order term ($i=1$) explicitly in (A.36) to have a good approximation of $L(a,b,\mu,3)$ in the limit of very small interaction mass μ. Expanding the analytic terms of (A.35) we have

$$L(a,b,\mu,3) = L(a,b,0,3) \tag{A.37}$$

$$+ \frac{i\pi^2}{2}\left(- \frac{1}{(b-a^2/4)(k^2+ak+b)^2} + \frac{4 + 2\log\frac{(b-a^2/4)}{(k^2+ak+b)^2}}{(k^2+ak+b)^3} \right. \tag{A.38}$$

$$\left. + 2\frac{1}{(b-a^2/4)(k^2+ak+b)}\log\mu^2 \right)\mu^2 + \mathcal{O}(\mu^4\log\mu^2) \tag{A.39}$$

Where,

$$L(a,b,0,3) = \frac{i\pi^2}{2}\frac{1}{(b-a^2/4)(k^2+ak+b)} \tag{A.40}$$

Besides the $n=3$ case, it is important to study the massless interaction $\mu=0$ case, which is present, for instance, in the Wick-Cutkosky model and the two fermion bound state with a massless vector interaction. Since we already have $L(a,b,0,3)$ we can obtain $L(a,b,0,n)$ with the recursion relation (A.21) and write, for $n>2$

$$L(a,b,0,n) = -\frac{i\pi^2}{n-1}\frac{\left(b-\frac{a^2}{4}\right)^{2-n} - \left(k^2+ak+b\right)^{2-n}}{(n-2)\left(\frac{a}{2}+k\right)^2} \tag{A.41}$$

$$= \frac{i\pi^2}{n-1}\frac{1}{(n-2)(k^2+ak+b)^{n-2}(b-a^2/4)^{n-2}}\frac{\left(k^2+ak+b\right)^{n-2} - \left(b-\frac{a^2}{4}\right)^{n-2}}{\left(\frac{a}{2}+k\right)^2} \tag{A.42}$$

$$= \frac{i\pi^2}{n-1}\frac{1}{(n-2)(k^2+ak+b)^{n-2}(b-a^2/4)^{n-2}}\sum_{i=0}^{n-3}(k^2+ak+b)^{n-3-i}\left(b-\frac{a^2}{4}\right)^i \tag{A.43}$$

$$= \frac{i\pi^2}{n-1}\sum_{i=0}^{n-3}\frac{1}{(k^2+ak+b)^{i+1}\left(b-\frac{a^2}{4}\right)^{n-2-i}} \tag{A.44}$$

$$\tag{A.45}$$

That is a very important result, because this self-reproducibility of the test function $\frac{1}{(k^2+ak+b)^n}$ when acted by an massless interaction kernel, giving as result a linear combination of $\frac{1}{(k^2+ak+b)^m}$ with $1 \leq m \leq n-1$, is what enabled the discovery of the Wick-Cutkosky model in the first place.

A second type of loop integral appears in calculations involving fermions, due to the presence of numerator terms when the propagators are contracted

$$L^{(F)}(a,b,\mu,n) = \int d^4k'\frac{(p\cdot k)(p\cdot k') - M^2k'^2}{M^4}\frac{1}{(k-k')^2-\mu^2}\frac{1}{(k'^2+ak'+b)^n}$$
$$= \int d^4k'\frac{\vec{k}\cdot\vec{k'}}{M^2}\frac{1}{(k-k')^2-\mu^2}\frac{1}{(k'^2+ak'+b)^n}$$

Where $\vec{k}$ is, in fact, an abuse of notation to write $(0,\vec{k})$. To solve it, one must use the same trick as in Eq. (A.24): a Feynman parametrization followed by a shift in the integration variable. Since the calculation is mostly the same, here we perform it in less details.

$$
\begin{aligned}
&L^{(F)}(a,b,\mu,n) = \\
&= \int d^4k' \frac{\vec{k}\cdot\vec{k'}}{M^2}\int_0^1 dv \frac{nv^{n-1}}{((k'^2-2k\cdot k'+k^2)-\mu^2+v((a+2k)k'+(b+\mu^2-k^2)))^{n+1}} \\
&= \int d^4k' \int_0^1 dv \frac{nv^{n-1}\vec{k}\cdot\vec{k'}}{M^2((k'+(-k+vk+va/2))^2-(-k+vk+va/2)^2+k^2-\mu^2+v(b+\mu^2-k^2)))^{n+1}} \\
&= \int d^4k'' \int_0^1 dv \frac{nv^{n-1}\vec{k}\cdot(\vec{k''}-(-k+vk+va/2))}{M^2((k'')^2-(-k+vk+va/2)^2+k^2-\mu^2+v(b+\mu^2-k^2)))^{n+1}} \\
&= \int d^4k'' \int_0^1 dv \frac{nv^{n-1}(\vec{k}\cdot\vec{k''}+\vec{k}^2(1-v)-\vec{k}\cdot av/2)}{M^2((k'')^2-(-k+vk+va/2)^2+k^2-\mu^2+v(b+\mu^2-k^2)))^{n+1}} \\
&= i\pi^2\frac{1}{n-1}\frac{\vec{k}^2}{M^2}\int_0^1 dv \frac{(1-v)v^{n-1}}{(-(-k+vk+va/2)^2+k^2-\mu^2+v(b+\mu^2-k^2)))^{n+1}} \\
&= i\pi^2\frac{1}{n-1}\frac{\vec{k}^2}{M^2}\int_0^1 dv \frac{(1-v)v^{n-1}}{(v(1-v)(k+a/2)^2+v(b-a^2/4)+(1-v)\mu^2)^{n-1}}
\end{aligned}
$$

Where $a\cdot\vec{k}=0$ and the term $\vec{k}\cdot\vec{k''}$ produces an odd integrand, so the integration vanishes with it, remaining only the $(1-v)\vec{k}^2$ in the numerator. Again, it is important to calculate explicitly the $\mu = 0$ case

$$
\begin{aligned}
L^{(F)}(a,b,\mu,n) &= i\pi^2\frac{1}{n-1}\frac{\vec{k}^2}{M^2}\int_0^1 dv \frac{(1-v)v^{n-1}}{(v(1-v)(k+a/2)^2+v(b-a^2/4))^{n-1}} \\
&= i\pi^2\frac{1}{n-1}\frac{\vec{k}^2}{M^2}\int_0^1 dv \frac{(1-v)}{((1-v)(k+a/2)^2+(b-a^2/4))^{n-1}}
\end{aligned}
$$

Setting $n = 4$ we have

$$
\begin{aligned}
L^{(F)}(a,b,\mu,4) &= i\pi^2\frac{1}{3}\frac{\vec{k}^2}{M^2}\int_0^1 dv \frac{(1-v)}{((1-v)(k+a/2)^2+(b-a^2/4))^3} \\
&= i\pi^2\frac{1}{6}\frac{\vec{k}^2}{M^2}\frac{1}{(b-a^2/4)(k^2+ak+b)^2}
\end{aligned}
$$

And since the recursion relation (A.21) is still valid for $L^{(F)}(a,b,\mu,n)$

$$L^{(F)}(a,b,\mu,n+1) = -\frac{1}{n}\frac{\partial}{\partial b}L^{(F)}(a,b,\mu,n) \tag{A.46}$$

We can iterate it to obtain $L^{(F)}(a,b,\mu,n)$ from $L^{(F)}(a,b,\mu,4)$ for a given $n \geq 4$ obtaining

$$\begin{aligned}
L^{(F)}(a,b,\mu,n) &= (-1)^{n-4}\frac{3!}{(n-1)!}\frac{\partial^{n-4}}{\partial b^{n-4}}L^{(F)}(a,b,\mu,4) \\
&= (-1)^{n}\frac{3!}{(n-1)!}\frac{\partial^{n-4}}{\partial b^{n-4}}i\pi^2\frac{1}{6}\frac{\vec{k}^2}{M^2}\frac{1}{(b-a^2/4)(k^2+ak+b)^2} \\
&= \frac{(-1)^n}{(n-1)!}i\pi^2\frac{\vec{k}^2}{M^2}\frac{\partial^{n-4}}{\partial b^{n-4}}\frac{1}{(b-a^2/4)(k^2+ak+b)^2} \\
&= \frac{(-1)^n}{(n-1)!}i\pi^2\frac{\vec{k}^2}{M^2}\sum_{i=0}^{n-4}\frac{(n-4)!(i+1)(-1)^{n-4}}{(b-a^2/4)^{n-i-3}(k^2+ak+b)^{i+2}} \\
&= \frac{i\pi^2}{(n-1)(n-2)(n-3)}\frac{\vec{k}^2}{M^2}\sum_{i=0}^{n-4}\frac{(i+1)}{(b-a^2/4)^{n-i-3}(k^2+ak+b)^{i+2}}
\end{aligned}$$

This result suggests that, while for $L(a,b,\mu,n)$ the "simplest" n possible is 3, for $L^{(F)}(a,b,\mu,n)$ the "simplest" n is 4, since for $n>4$ the result is a sum. So, it may useful to treat the fermionic BSE with Nakanishi PITR of different orders n, specially when searching for a possible fermionic Wick-Cutkosky model.

A.4 Light-Front Projection

The second tipe of useful integral to be used in this thesis is the Light-Front projection. These are characterized in the momentum space as an integration in the Light-Front variable k^-, and in the position space they project the given physical system in the hyperplane x^-, giving a Light-Front wave function. This projection is also used as a regularizator, so that the once singular BS equation can be treated numerically with the Nakanishi representation.

The simplest of these formulas, and the one from which the general case will be derived, can be written as $\int dk^- \frac{1}{(\alpha k^- + \beta)^2}$. The main source of difficulty in calculating these integrals is that the result is a distribution on the variable α; generally a delta function or its derivatives. Therefore, the strategy to calculate this integral will be to introduce a cutoff $1/\delta$ and then to make $\delta \to 0^+$. Using the scheme, this integral can be written as

$$\int_{-1/\delta}^{1/\delta} dk^- \frac{1}{(\alpha k^- + \beta)^2} = \frac{-2\delta}{\alpha^2 - \beta^2\delta^2} \tag{A.47}$$

Where β is supposed to have a small imaginary part to avoid the pole. To better understand the behavior as δ goes to zero, let us first define the function

$$\eta_\delta(x) = \frac{1}{\pi}\frac{\delta}{\delta^2 + x^2} \tag{A.48}$$

Which is called the poisson kernel and is the fundamental solution of the Laplace equation. One of its important features is that it is normalized to one: $\int dx \eta_\delta(x) = 1$ for any δ. However, note that when $\delta \to 0^+$ $\eta_\delta(x)$ converges pointwise to zero if $x \neq 0$, and converges to ∞ if $x = 0$. Therefore, it exhibit the expected the expected behavior of a Dirac delta function: 0 if $x \neq 0$, ∞ if $x = 0$ and norm $= 1$. In fact, this type of function is what the mathematicians call an aproximation to identity or nascent delta function. Futhermore, if one considers its limit as a distribution (also called weak limit), one gets

$$\lim_{\delta \to 0^+} \eta_\delta(x) = \delta(x) \tag{A.49}$$

And that is the identity to be used so that we can obtain a delta function from integral (A.47), since we can write it as

$$\int_{-1/\delta}^{1/\delta} dk^- \frac{1}{(\alpha k^- + \beta)^2} = \frac{2\pi}{\beta^2}\eta_\delta(i\frac{\alpha}{\beta}) \tag{A.50}$$

And now we are able to make $\delta \to 0^+$ and use (A.49) to obtain

$$\begin{aligned}\int \frac{dk^-}{(\alpha k^- + \beta)^2} &= \frac{2\pi}{\beta^2}\delta(i\frac{\alpha}{\beta}) \\ &= \frac{2\pi}{\beta^2(i/\beta)}\delta(\alpha) \\ &= \frac{-2\pi i}{\beta}\delta(\alpha)\end{aligned}$$

In the general case, we define the integral, with $n \geq m + 2$

$$I_{LF}(\alpha, \beta, m, n) = \int \frac{dk^-(k^-)^m}{(\alpha k^- + \beta)^n} \tag{A.51}$$

To obtain a recursion relation, we differentiate (A.51) in α and β

$$\frac{\partial}{\partial \alpha} \int \frac{dk^-(k^-)^m}{(\alpha k^- + \beta)^n} = -n \int \frac{dk^-(k^-)^{m+1}}{(\alpha k^- + \beta)^{n+1}}$$

$$\frac{\partial}{\partial \beta} \int \frac{dk^-(k^-)^m}{(\alpha k^- + \beta)^n} = -n \int \frac{dk^-(k^-)^m}{(\alpha k^- + \beta)^{n+1}}$$

Which enable us to write the following recursion relations

$$I_{LF}(\alpha, \beta, m+1, n+1) = \frac{-1}{n} \frac{\partial}{\partial \alpha} I_{LF}(\alpha, \beta, m, n)$$

$$I_{LF}(\alpha, \beta, m, n+1) = \frac{-1}{n} \frac{\partial}{\partial \beta} I_{LF}(\alpha, \beta, m, n)$$

Thus, using the base case $n = 2, m = 0$, we obtain a formula for any m, n

$$\begin{aligned}
I_{LF}(\alpha, \beta, m, n) &= \frac{(-1)^{n-2}}{(n-1)!} \frac{\partial^m}{\partial \alpha^m} \frac{\partial^{n-m-2}}{\partial \beta^{n-m-2}} I_{LF}(\alpha, \beta, 0, 2) \\
&= \frac{(-1)^{n-2}}{(n-1)!} \frac{\partial^m}{\partial \alpha^m} \frac{\partial^{n-m-2}}{\partial \beta^{n-m-2}} \left(\frac{-2\pi i}{\beta} \delta(\alpha) \right) \\
&= \frac{(-1)^{n-2}}{(n-1)!} \frac{\partial^{n-m-2}}{\partial \beta^{n-m-2}} \left(\frac{-2\pi i}{\beta} \delta^{(m)}(\alpha) \right) \\
&= \frac{(-1)^{n-2}}{(n-1)!} (-1)^{n-m-2} (n-m-2)! \left(\frac{-2\pi i}{\beta^{n-m-1}} \delta^{(m)}(\alpha) \right) \\
&= (-1)^{m+1} \frac{(n-m-2)!}{(n-1)!} \left(\frac{2\pi i}{\beta^{n-m-1}} \delta^{(m)}(\alpha) \right)
\end{aligned}$$

Where $\delta^{(m)}(\alpha)$ is a shorthand for $\frac{\partial^m}{\partial \alpha^m} \delta(\alpha)$.

A second type of LF projection appears when you have a product of propagators, such as in the right hand side of the BS equation, where there are the free propagators multiplying the loop integral. For this type of this situation, we define the following LF integral formula

$$I'_{LF}(\alpha,\beta,m,n) = \int \frac{dk^-(k^-)^m}{(k^2+p\cdot k-\kappa^2+i\epsilon)(k^2-p\cdot k-\kappa^2+i\epsilon)(\alpha k^-+\beta+i\epsilon)^n} \quad \text{(A.52)}$$

With $n \geq m$. Here we are writting the small imaginary part $i\epsilon$ explicitly because it is more convenient, given the methods used to calculate I'_{LF}. Naturally, in order to integrate in k^-, the free propagators must be written using LF variables. Since $k^2 = k^-k^+ - k_\perp^2$ and $p\cdot k = \frac{M}{2}(k^+ + k^-)$, we may write the propagators as

$$k^2 \pm p\cdot k - \kappa^2 = (k^+ \pm M/2)k^- + (\pm\frac{M}{2}k^+ - \kappa^2 - k_\perp^2) \quad \text{(A.53)}$$

First, we calculate the case $n = 1$, $m = 0$, and use recursion relations to obtain the general case, as was done with the previous integral I_{LF}. To simplify let us first write $k^2 + p\cdot k - \kappa^2 = \alpha' k^- + \beta'$ and $k^2 - p\cdot k - \kappa^2 = \alpha'' k^- + \beta''$, to have a more compact notation. In order to evaluate $I'_{LF}(\alpha,\beta,0,1)$, we will use the residue theorem.Note that $\alpha' = (k^+ + \frac{M}{2}) > 0$ and $\alpha'' = (k^+ - \frac{M}{2}) < 0$, thus if we close the contour in the lower plane, we will not have one of these poles, since they are

1.$k^- = -\frac{\beta''+i\epsilon}{\alpha''}$

2.$k^- = -\frac{\beta'+i\epsilon}{\alpha'}$

3.$k^- = -\frac{\beta+i\epsilon}{\alpha}$

And pole 1 is in the upper plane. Also, pole 3 is in the lower plane iff $\alpha > 0$. Let Res_2 and Res_3 be the residues of I'_{LF} at the poles 2 and 3. Now we can use the residue theorem to evaluate $I'_{LF}(\alpha,\beta,0,1)$ making $\epsilon \to 0$

$$\begin{aligned} I'_{LF}(\alpha,\beta,0,1) &= \int \frac{dk^-}{(\alpha''k^-+\beta'')(\alpha'k^-+\beta')(\alpha k^-+\beta)} \\ &= 2\pi i\left(Res_2 + Res_3\Theta(\alpha)\right) \\ &= 2\pi i\left(\frac{\alpha'}{(\beta''\alpha'-\alpha''\beta')(\beta\alpha'-\alpha\beta')} + \frac{\alpha\Theta(\alpha)}{(\beta''\alpha-\alpha''\beta)(\beta'\alpha-\alpha'\beta)}\right) \end{aligned}$$

Observe that Res_3 has a first-order zero in the variable α, as this is important at delta function manipulations ahead. Using this base integral, we are able to obtain also the needed integrals used in the fermionic BS equation $I'_{LF}(\alpha,\beta,0,3)$, $I'_{LF}(\alpha,\beta,1,3)$, $I'_{LF}(\alpha,\beta,2,3)$, $I'_{LF}(\alpha,\beta,3,3)$, because I'_{LF} obeys the same recursion relations as I'_{LF}

$$I'_{LF}(\alpha,\beta,m+1,n+1) = \frac{-1}{n}\frac{\partial}{\partial\alpha}I'_{LF}(\alpha,\beta,m,n)$$

$$I'_{LF}(\alpha,\beta,m,n+1) = \frac{-1}{n}\frac{\partial}{\partial\beta}I'_{LF}(\alpha,\beta,m,n)$$

Thus, we arrive at

$$\begin{aligned}
I'_{LF}(\alpha,\beta,0,3) &= \frac{(-1)^2}{2}\frac{\partial^2}{\partial\beta^2}I'_{LF}(\alpha,\beta,0,1) \\
&= \pi i\left(\frac{\partial^2}{\partial\beta^2}Res_2 + \Theta(\alpha)\frac{\partial^2}{\partial\beta^2}Res_3\right) \\
&= \pi i\Bigg(\frac{(-1)(-2)\alpha'(-\alpha')^2}{(\beta''\alpha'-\alpha''\beta')(\beta\alpha'-\alpha\beta')^3} + \alpha\Theta(\alpha)\Bigg(\frac{(-1)(-2)(-\alpha'')^2}{(\beta''\alpha-\alpha''\beta)^3(\beta'\alpha-\alpha'\beta)} + \\
&\frac{(-1)^2(-\alpha'')(-\alpha')}{(\beta''\alpha-\alpha''\beta)^2(\beta'\alpha-\alpha'\beta)^2} + \frac{(-1)(-2)(-\alpha')^2}{(\beta''\alpha-\alpha''\beta)(\beta'\alpha-\alpha'\beta)^3}\Bigg)\Bigg) \\
&= \pi i\Bigg(\frac{2\alpha'\alpha'^2}{(\beta''\alpha'-\alpha''\beta')(\beta\alpha'-\alpha\beta')^3} + \alpha\Theta(\alpha)\Bigg(\frac{2\alpha''^2}{(\beta''\alpha-\alpha''\beta)^3(\beta'\alpha-\alpha'\beta)} + \\
&\frac{\alpha''\alpha'}{(\beta''\alpha-\alpha''\beta)^2(\beta'\alpha-\alpha'\beta)^2} + \frac{2\alpha'^2}{(\beta''\alpha-\alpha''\beta)(\beta'\alpha-\alpha'\beta)^3}\Bigg)\Bigg)
\end{aligned}$$

$$\begin{aligned}
I'_{LF}(\alpha,\beta,1,3) &= \frac{(-1)^2}{2}\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}I'_{LF}(\alpha,\beta,0,1) \\
&= \pi i\left(\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}Res_2 + \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\left(\frac{Res3}{\alpha}(\alpha\Theta(\alpha))\right)\right) \\
&= \pi i\left(\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}(Res_2) + \Theta(\alpha)\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}(Res_3) + \frac{\partial}{\partial\beta}\left(\frac{Res3}{\alpha}(\alpha\delta(\alpha))\right)\right) \\
&= \pi i\left(\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}(Res_2) + \Theta(\alpha)\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}(Res_3)\right)
\end{aligned}$$

$$
\begin{aligned}
I'_{LF}(\alpha,\beta,2,3) &= \frac{(-1)^2}{2}\frac{\partial^2}{\partial\alpha^2}I'_{LF}(\alpha,\beta,0,1)\\
&= \pi i\left(\frac{\partial^2}{\partial\alpha^2}Res_2 + \frac{\partial^2}{\partial\alpha^2}\left(\frac{Res3}{\alpha}(\alpha\Theta(\alpha))\right)\right)\\
&= \pi i\left(\frac{\partial^2}{\partial\alpha^2}(Res_2) + \frac{\partial}{\partial\alpha}\left(\Theta(\alpha)\frac{\partial}{\partial\alpha}(Res_3) + \frac{Res3}{\alpha}(\alpha\delta(\alpha))\right)\right)\\
&= \pi i\left(\frac{\partial^2}{\partial\alpha^2}(Res_2) + \frac{\partial}{\partial\alpha}\left(\Theta(\alpha)\frac{\partial}{\partial\alpha}(Res_3)\right)\right)\\
&= \pi i\left(\frac{\partial^2}{\partial\alpha^2}(Res_2) + \Theta(\alpha)\frac{\partial^2}{\partial\alpha^2}(Res_3) + \delta(\alpha)\frac{\partial}{\partial\alpha}Res_3\right)
\end{aligned}
$$

These calculations shows that distributional terms only appears for $m \geq 2$, mainly due to the first-order zero present at Res_3, as the identity $\alpha\delta(\alpha) = 0$ cancels this term in the case $m = 1$. To obtain the last integral, $I'_{LF}(\alpha,\beta,3,3)$, we must use the reverse of the previous recursion relations

$$
\begin{aligned}
\frac{-1}{3}\frac{\partial}{\partial\beta}I'_{LF}(\alpha,\beta,3,3) &= I'_{LF}(\alpha,\beta,3,4)\\
I'_{LF}(\alpha,\beta,3,3) &= -3\int_{-\infty}^{\beta} d\beta' I'_{LF}(\alpha,\beta',3,4)\\
&= \int_{-\infty}^{\beta} d\beta'\frac{\partial}{\partial\alpha}I'_{LF}(\alpha,\beta',2,3)
\end{aligned}
$$

Where was used the following abuse of notation for the integration:$\int_{-\infty}^{\beta} d\beta f(\beta) = \int_{-\infty}^{\beta} d\beta' f(\beta')$. And since we already calculated $I'_{LF}(\alpha,\beta',2,3)$, we can finally have $I'_{LF}(\alpha,\beta',3,3)$

$$
\begin{aligned}
I'_{LF}(\alpha,\beta,3,3) &= \int_{-\infty}^{\beta} d\beta\frac{\partial}{\partial\alpha}I'_{LF}(\alpha,\beta,2,3)\\
&= \int_{-\infty}^{\beta} d\beta\frac{\partial}{\partial\alpha}\pi i\left(\frac{\partial^2}{\partial\alpha^2}(Res_2) + \Theta(\alpha)\frac{\partial^2}{\partial\alpha^2}(Res_3) + \delta(\alpha)\frac{\partial}{\partial\alpha}Res_3\right)\\
&= \pi i\int_{-\infty}^{\beta} d\beta\left(\frac{\partial^3}{\partial\alpha^3}(Res_2) + \Theta(\alpha)\frac{\partial^3}{\partial\alpha^3}(Res_3) + \delta(\alpha)2\frac{\partial^2}{\partial\alpha^2}Res_3 + \delta'(\alpha)\frac{\partial}{\partial\alpha}Res_3\right)
\end{aligned}
$$

.These results were always written as derivatives of the residues, in order to have a shorter expression. However, they will be used in this thesis to obtain the distributional parts of these integrals, that were not explicitly calculated in the literature. Thus, let us write $I'_{LF}(\alpha, \beta, m, n) = Regular + Singular$, where the singular terms are the ones containing distributions and the regular terms contain ordinary functions. We now want to obtain an expanded formula for the singular terms of $I'_{LF}(\alpha, \beta, 2, 3)$ and $I'_{LF}(\alpha, \beta, 3, 3)$

$$
\begin{aligned}
I'_{LF}(\alpha, \beta, 2, 3) &= Regular + \pi i \delta(\alpha) \left(\frac{\partial}{\partial \alpha} Res_3 \right) \\
&= Regular + \delta(\alpha) \pi i \left(\frac{\beta^2 \alpha' \alpha'' - \alpha^2 \beta' \beta''}{(\beta\alpha' - \alpha\beta')^2 (\beta\alpha'' - \alpha\beta'')^2} \bigg|_{\alpha=0} \right) \\
&= Regular + \delta(\alpha) \pi i \left(\frac{1}{\beta^2 \alpha' \alpha''} \right)
\end{aligned}
$$

$$
\begin{aligned}
I'_{LF}(\alpha, \beta, 3, 3) &= Regular + \pi i \int_{-\infty}^{\beta} d\beta \left(\delta(\alpha) 2 \frac{\partial^2}{\partial \alpha^2} Res_3 + \delta'(\alpha) \frac{\partial}{\partial \alpha} Res_3 \right) \\
&= Regular \\
&+ \delta(\alpha) \left(2\pi i \int_{-\infty}^{\beta} d\beta \frac{\partial^2}{\partial \alpha^2} Res_3 \right) \\
&+ \delta'(\alpha) \left(\pi i \int_{-\infty}^{\beta} d\beta \frac{\partial}{\partial \alpha} Res_3 \right) \\
&= Regular \\
&+ \delta(\alpha) \left(-2\pi i \frac{\beta}{(\beta\alpha' - \alpha\beta')(\beta\alpha'' - \alpha\beta'')} \right) \\
&+ \delta'(\alpha) \left(-\pi i \frac{\beta \left(\beta(\alpha''\beta' + \alpha'\beta'') - 2\alpha\beta'\beta'' \right)}{(\beta\alpha' - \alpha\beta')^2 (\beta\alpha'' - \alpha\beta'')^2} \right) \\
&= Regular \\
&+ \delta(\alpha) \left(-\pi i \frac{\beta}{(\beta\alpha' - \alpha\beta')(\beta\alpha'' - \alpha\beta'')} \bigg|_{\alpha=0} \right) \\
&+ \delta'(\alpha) \left(-\pi i \frac{\beta \left(\beta(\alpha''\beta' + \alpha'\beta'') - 2\alpha\beta'\beta'' \right)}{(\beta\alpha' - \alpha\beta')^2 (\beta\alpha'' - \alpha\beta'')^2} \bigg|_{\alpha=0} \right) \\
&= Regular \\
&+ \delta(\alpha) \left(-\pi i \frac{1}{\beta\alpha'\alpha''} \right) \\
&+ \delta'(\alpha) \left(-\pi i \frac{(\alpha''\beta' + \alpha'\beta'')}{(\beta\alpha'\alpha'')^2} \right)
\end{aligned}
$$

As a last step, we must substitute the $\alpha', \alpha'', \beta', \beta''$ by the corresponding expressions in term of the LF variables such as γ and z. Also, it is better to assume already that $\alpha k^- + \beta$ corresponds to ones used in this theses, given by the resulting loop integral, since it enables us to simplify some expressions. So, the variables can be expressed as

$$
\begin{aligned}
\alpha' &= -\frac{M}{2}(-1+z) \\
\alpha'' &= -\frac{M}{2}(1+z) \\
\beta' &= -(\gamma + m^2 - (1-z)\frac{M^2}{4}) \\
\beta'' &= -(\gamma + m^2 - (1+z)\frac{M^2}{4}) \\
\alpha &= v(1-v)\frac{M}{2}(z'-z) \\
\beta &= -v\left(+\gamma' + \kappa^2 + \frac{1}{4}M^2 z'^2\right) - (1-v)v\left(-\frac{1}{4}M^2 z'\,(z'-z) + \gamma\right) + \mu^2(1-v)
\end{aligned}
$$

Note that, since the distributions have support only on $\alpha = 0$, we can substitute z' by z in all of the expressions. This makes the singular terms algebrically simpler than the regular ones. Thus, we may finally write the final expressions for the singular terms of $I'_{LF}(\alpha, \beta, 2, 3)$ and $I'_{LF}(\alpha, \beta, 3, 3)$

$$I'_{LF}(\alpha,\beta,2,3) = Regular + \left(\frac{-\pi i\delta(v(1-v)\frac{M}{2}(z'-z))}{\frac{M^2}{4}(1-z^2)\left(v\left(\gamma'+\kappa^2-\frac{1}{4}M^2z^2\right)+\gamma(1-v)v-\mu^2(1-v)\right)^2}\right)$$
$$= Regular + \left(\frac{-8\pi i\delta(z'-z)}{v(1-v)M^3(1-z^2)\left(v\left(\gamma'+\kappa^2-\frac{1}{4}M^2z^2\right)+\gamma(1-v)v-\mu^2(1-v)\right)^2}\right)$$

$$I'_{LF}(\alpha,\beta,3,3) = Regular + \pi i\int_{-\infty}^{\beta} d\beta\left(\delta(\alpha)2\frac{\partial^2}{\partial\alpha^2}Res_3+\delta'(\alpha)\frac{\partial}{\partial\alpha}Res_3\right)$$
$$= Regular$$
$$+\left(\frac{-\pi i\delta(v(1-v)\frac{M}{2}(z'-z))}{\frac{M^2}{4}(1-z^2)\left(v\left(\gamma'+\kappa^2-\frac{1}{4}M^2z^2\right)+\gamma(1-v)v-\mu^2(1-v)\right)}\right)$$
$$+\left(-\pi i\frac{16z(\gamma+m^2)\delta'(v(1-v)\frac{M}{2}(z'-z))}{M^3(1-z^2)^2\left(v\left(\gamma'+\kappa^2-\frac{1}{4}M^2z^2\right)+\gamma(1-v)v-\mu^2(1-v)\right)^2}\right)$$
$$= Regular$$
$$+\left(\frac{-8\pi i\delta(z'-z)}{v(1-v)M^3(1-z^2)\left(v\left(\gamma'+\kappa^2-\frac{1}{4}M^2z^2\right)+\gamma(1-v)v-\mu^2(1-v)\right)}\right)$$
$$+\left(\frac{-\pi i64z(\gamma+m^2)\delta'(z'-z)}{v^2(1-v)^2M^5(1-z^2)^2\left(v\left(\gamma'+\kappa^2-\frac{1}{4}M^2z^2\right)+\gamma(1-v)v-\mu^2(1-v)\right)^2}\right)$$

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5. TÍTULO E SUBTÍTULO:

Two Body Fermion Bound State in Minkowski Space and the Wick-Cutkosky Model

6. AUTOR(ES):

Rafael Endlich Pimentel

7. INSTITUIÇÃO(ÕES)/ÓRGÃO(S) INTERNO(S)/DIVISÃO(ÕES):

Instituto Tecnológico de Aeronáutica – Divisão de Ciências Fundamentais – ITA/IEF

8. PALAVRAS-CHAVE SUGERIDAS PELO AUTOR:

Equação de Bethe-Salpeter; Férmions; Grafeno

9.PALAVRAS-CHAVE RESULTANTES DE INDEXAÇÃO:

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11. RESUMO:

In this thesis, two particle bound states in quantum field theories were studied using the Bethe-Salpeter equation with Nakanishi representation, in two different cases. The first deal with two distinct bosons bounded by another massless boson, the so called Wick-Cutkosky model. To solve it, two new methods were developed using integration by parts in the Nakanishi representation; one of these transforms a integration into a series and the other obtain a new differential equation in the space of the Nakanishi transform parameters. The second problem deals with the bound state of two fermions, bounded by a massive boson. In it, a Light-Front projection was used to obtain a continuous kernel for the operator in order to solve the problem numerically. However, the Light-Front projection of an amplitude can give singular terms, known as endpoint contributions. In (CARBONELL; KARMANOV, 2010), these singular terms were dealt with using regu- larizators and form factors. However, in this thesis the singular terms were developed analytically, writing the Light Front integrals as distributions in order to obtain the singular terms that contribute to the Bethe-Salpeter equation for the two fermion bound state.

12. GRAU DE SIGILO:

(X) OSTENSIVO **( ) RESERVADO** **( ) SECRETO**